Haar integrals on finite and compact quantum groups

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These notes were written for a short graduate course given at the University of Caen in September 2010. It turned out that it was necessary to include some introductory material about tensor products and group algebras. Then we tried to present some nice features of the elementary theory of Hopf algebras, both in the algebraic and the C^*-algebraic settings. Our main goal was the existence and unicity of "Haar integrals", whose proofs are quite different in both settings, and the presentation of some "real-life" examples where computation skills can be trained.

We apologize in advance for all errors that could remain in this text. Updates and corrections will be made available at `http://www.math.unicaen.fr/~vergnioux/`.

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1 Introduction

1.1 Tensor products

1.1.1 We fix a base field k. If E, F are sets, we denote by $\mathscr{F}(E,F)$ the set of functions from E to F. We simply denote $\mathscr{F}(E)$ the space $\mathscr{F}(E,k)$. We will often write the composition of functions as a product, i.e. $fg = f \circ g$ for $f \in \mathscr{F}(F,G)$, $g \in \mathscr{F}(E,F)$. If V, W are vector spaces we denote by $L(V,W)$ the space of linear maps from V to W.

Definition 1.1 *Let V, W be two vector spaces. There exists a vector space $V\otimes W$ together with a bilinear map $i : V \times W \to V\otimes W$ satisfying the following universal property : for every bilinear map $\phi : V \times W \to X$, there exists a unique linear map $\Phi : V\otimes W \to X$ such that $\Phi \circ i = \phi$. Moreover $V\otimes W$ is unique up to an isomorphism exchanging the maps i. For every $\zeta \in V$, $\xi \in W$, the vector $i(\zeta,\xi)$ is denoted $\zeta\otimes\xi \in V\otimes W$.*

The use of the *tensor product* $V\otimes W$ allows to replace bilinear maps on $V \times W$ by linear maps. The vectors $\zeta\otimes\xi$ are called *elementary tensors* in $V\otimes W$, they form a cone in $V\otimes W$ and by definition they obey the following relations :

$$(\zeta + \lambda\zeta')\otimes\xi = (\zeta\otimes\xi) + \lambda(\zeta'\otimes\xi), \qquad \zeta\otimes(\xi + \lambda\xi') = (\zeta\otimes\xi) + \lambda(\zeta\otimes\xi').$$

Moreover elementary tensors span the vector space $V\otimes W$: a general vector $\nu \in V\otimes W$ can be written as a finite sum $\nu = \sum_i \nu_{1,i}\otimes\nu_{2,i}$. It is sometimes convenient to drop the indices i and write $\nu = \sum \nu_{(1)}\otimes\nu_{(2)}$. Note that this decomposition into elementary tensors is not unique.

On the other hand one can show that for any bases $(\zeta_i)_{i\in I}$, $(\xi_j)_{j\in J}$ of V and W respectively, the elementary tensors $(\zeta_i\otimes\xi_j)_{(i,j)\in I\times J}$ form a basis of $V\otimes W$. In particular $\dim(V\otimes W) = \dim V \times \dim W$. If V (or W) is 1-dimensional, any identification $V \simeq k$ yields an isomorphism $V\otimes W \simeq W$ given by $\lambda\otimes\xi = \lambda\xi$. The following results is sometimes usefull: if $(\zeta_i)_{i\in I}$ is a linearly independent family in V, then $(\zeta_i\otimes\xi_i)_{i\in I}$ is linearly independent in $V\otimes W$ for any non-zero vectors $\xi_i \in W$. In particular if two non-zero elementary tensors $\zeta\otimes\xi$, $\zeta'\otimes\xi'$ are equal then ζ, ζ' are colinear as well as ξ, ξ', and if $\zeta\otimes\xi = 0$ then ζ or ξ is the zero vector.

EXAMPLE 1.2 When $V = k^{(E)}$, $W = k^{(F)}$ are spaces of functions with finite support, we will take as tensor product the space $V\otimes W = k^{(E\times F)}$, with the canonical map i given by $i(f,g) = ((x,y) \mapsto f(x)g(y))$. Then, if $\phi : V \times W \to k$ is a bievaluation map $(f,g) \mapsto f(x)g(y)$, its extension Φ as in the Definition is simply the linear evaluation map $h \mapsto h(x,y)$. □

1.1.2 Consider now linear maps $\phi : V \to V'$, $\psi : W \to W'$. It is easy to check that the map $(\zeta,\xi) \mapsto \phi(\zeta)\otimes\psi(\xi)$ is bilinear, and hence it defines a linear map denoted $\phi\otimes\psi : V\otimes W \to V'\otimes W'$. Using the notation above, is it given by $(\phi\otimes\psi)(\nu) = \sum \phi(\nu_{(1)})\otimes\psi(\nu_{(2)})$.

Observing that the element $\phi\otimes\psi$ of $L(V\otimes W, V'\otimes W')$ is bilinear in ϕ, ψ we get moreover a linear map $L(V,V')\otimes L(W,W') \to L(V\otimes W, V'\otimes W')$. This map is injective, and hence an isomorphism when all spaces are finite-dimensional. As an interesting particular case, we obtain an inclusion $W\otimes V^* = L(k,W)\otimes L(V,k) \subset L(V,W)$, given by the formula $(\xi\otimes\phi)(\zeta) = \xi\langle\phi,\zeta\rangle$, which is an insomorphism when V and W are finite-dimensional.

Given any tensor product space $V\otimes W$, a particularly usefull linear map is the *flip map* $\Sigma : V\otimes W \to W\otimes V$ given by $\Sigma(\zeta\otimes\xi) = \xi\otimes\zeta$. When $V = W$ and char $k \neq 2$, the tensor product $V\otimes V$ decomposes into the direct sum of the subspace of *symetric tensors* ν, such that $\Sigma(\nu) = \nu$, and *antisymetric tensors* ν, such that $\Sigma(\nu) = -\nu$. Note that these subspaces are not spanned by elementary tensors.

Exercise 1. Describe the kernel of $\phi\otimes\psi$ for $\phi \in L(V,V')$, $\psi \in L(W,W')$. If V', W' are quotients of V, W, describe $V'\otimes W'$ as a quotient of $V\otimes W$.

Exercise 2. Show that the canonical map $L(V,V')\otimes L(W,W') \to L(V\otimes W, V'\otimes W')$ is injective. Show that the natural inclusion $V^*\otimes W^* \subset (V\otimes W)^*$ is not surjective when V and W are infinite-dimensional. Show that the natural inclusion $W\otimes V^* \subset L(V,W)$ is not surjective when V and W are infinite-dimensional. What can be said when V or W is finite-dimensional? Give a general description of the subspace $W\otimes V^* \subset L(V,W)$.

Exercise 3. Let V be a f.-d. vector space, and choose $f \in GL(V)$. Consider the linear map $g = f\otimes {}^t f^{-1} \in L(V\otimes V^*)$. Give the expression of g as an element of $L(L(V))$ via the isomorphism $V\otimes V^* \simeq L(V)$.

Given three vector spaces V_1, V_2, V_3 one can form the tensor products $(V_1\otimes V_2)\otimes V_3$ and $V_1\otimes(V_2\otimes V_3)$. They are not equal but canonically isomorphic via $(\eta\otimes\zeta)\otimes\xi \mapsto \eta\otimes(\zeta\otimes\xi)$, and we denote one of them by $V_1\otimes V_2\otimes V_3$. One proceeds similarly with more than three vector spaces, and we denote $V^{\otimes k} = V\otimes\cdots\otimes V$.

One can build linear maps between such multiple tensor products using the so-called *leg notation*. Take e.g. $V_1 = V_2 = V_3 = V$. For any $\Phi \in L(V)$, we denote e.g. $\Phi_3 = \mathrm{id}\otimes\mathrm{id}\otimes\Phi \in L(V\otimes V\otimes V)$ the map Φ “acting on the third leg of $V\otimes V\otimes V$”. For $\Phi \in L(V\otimes V)$, we denote similarly $\Phi_{12} = \Phi\otimes\mathrm{id}$ the map Φ “acting on the two first legs of $V\otimes V\otimes V$”. More interestingly, Φ_{13} is the map Φ “acting on legs 1 and 3 of $V\otimes V\otimes V$”: using the flip map, it is given by $\Phi_{13} = (\mathrm{id}\otimes\Sigma)(\Phi\otimes\mathrm{id})(\mathrm{id}\otimes\Sigma)$.

Exercise 4. Compute the following maps on elementary tensors of $V\otimes V\otimes V$: $\Sigma_{13}\Sigma_{23}$, $\Sigma_{13}\Sigma_{23}\Sigma_{13}$. Show that the map $(k\ k+1) \mapsto \Sigma_{k,k+1}$ extends to a well-defined representation of the symetric group $\mathfrak{S}_n$ on $V^{\otimes n}$, for any vector space V.

Exercise 5. Given a group G and representations $\pi : G \to GL(V)$, $\rho : G \to GL(W)$, show that the formula $\pi\otimes\rho : g \mapsto \pi(g)\otimes\rho(g)$ defines a representation of G on $V\otimes W$. When $\pi = \rho$ we denote it by $\pi^{\otimes 2}$. Generalize to $V^{\otimes n}$ and check that any of the representations obtained in this way commute to the representation of $\mathfrak{S}_n$ constructed in the previous exercise.

1.1.3 Tensor products are important for us because they allow to describe some algebraic structures only in terms of linear maps, thus offering a convenient framework to dualize them.

Example 1.3 Let A be an algebra. The multiplication of A is bilinear and hence it defines a linear map $m : A\otimes A \to A$. The associativity property of the multiplication can be written as the following equality between maps from $A\otimes A\otimes A$ to A:

$$m(m\otimes\mathrm{id}) = m(\mathrm{id}\otimes m).$$

Any element $a \in A$ defines a linear map $\eta : k \to A$, $\lambda \mapsto \lambda a$. It is easily checked that $a = 1_A$ is a unit element for A **iff** the following equality between maps from $A \simeq A\otimes k \simeq k\otimes A$ to A holds:

$$m(\eta\otimes\mathrm{id}) = \mathrm{id} = m(\mathrm{id}\otimes\eta).$$

□

Exercise 6. Let A be an algebra and denote by $m : A\otimes A \to A$ the associated multiplication map. Give a characterization of the commutativity of A in terms of m. Give a characterization of bilateral ideals $I \subset A$ in terms of m and describe the multiplication map of the quotient algebra.

Exercise 7. Rephrase the definition of Lie algebras in the language of linear maps between tensor product spaces.

1.2 From groups to algebras and coalgebras

1.2.1 Let G be a group. To G one can associate two unital algebras:

1. The function algebra $\mathscr{F}(G) = \{\zeta : G \to k\}$, equipped with the pointwise algebra operations coming from the algebra structure of k, ie

$$\zeta + \lambda\xi = (g \mapsto \zeta(g) + \lambda\xi(g)), \qquad \zeta\xi = (g \mapsto \zeta(g)\xi(g)).$$

   The algebra $\mathscr{F}(G)$ posesses many interesting subalgebras, especially when G is a topological group.

2. The group algebra $k[G]$. As a vector space it is freely generated by G, ie $k[G]$ comes with a distinguished basis which we identify with G. The multiplication is bilinearly induced from the product in G, ie

$$(\textstyle\sum_{g\in G} \lambda_g g)(\sum_{h\in G} \mu_h h) = \sum_{g,h\in G} \lambda_g \mu_h\, (gh).$$

   Note that all the sums above are finite by definition.

These algebras can be used to study some properties of G in the language of algebras rather than groups. For instance, any representation $\pi : G \to GL(V)$ clearly extends by linearity to a unital representation of the algebra $k[G]$, and conversely unital representations $\pi : k[G] \to L(V)$ yield representations of G by restriction. Similarly, unital representations of $\mathscr{F}(G)$ correspond to vector space gradings by G.

However, some features of group theory are lost when passing to algebras. For instance, there is no general notion of tensor product between representations of a given algebra. Moreover, one cannot recover the group G from one of the algebras $\mathscr{F}(G)$, $k[G]$ alone: $\mathscr{F}(G)$ clearly recalls only the cardinality of G, whereas $k[G] \simeq k^n$ for all abelian groups of order n.

To recover G, one needs in fact both algebras as well as a duality between them. More precisely we define $\langle\cdot,\cdot\rangle : \mathscr{F}(G) \times k[G] \to k$ by the formula $\langle f, g\rangle = f(g)$ for $f \in \mathscr{F}(G)$, $g \in G$ and extending by linearity to $k[G]$. This bracket is non-degenerate: if $\langle f, x\rangle = 0$ for all f (resp. x), then $x = 0$ (resp. $f = 0$). Now we have:

Exercise 8. Let G, G' be two groups with unital algebra isomorphisms $\phi : \mathscr{F}(G') \to \mathscr{F}(G)$, $\psi : k[G] \to k[G']$ such that $\langle\phi(f), a\rangle = \langle f, \psi(a)\rangle$ for all $f \in \mathscr{F}(G')$, $a \in k[G]$. Show that ϕ maps characteristic functions to characteristic functions. Then, show that ψ maps elements of G to elements of G' — and hence restricts to an isomorphism $G \simeq G'$.

In the case of finite abelian groups and $k = \mathbb{C}$, the duality bracket above encodes Pontrjagin duality. Denote by $\hat{G}$ the set of group morphisms $f : G \to \mathbb{C}^*$, also called the characters of G. It becomes a group with respect to pointwise multiplication, called the dual group of G. When G is a finite abelian group, it is known (from elementary representation theory) that $\hat{G}$ forms a basis of $\mathscr{F}(G)$. Restricting the duality bracket to $\hat{G} \times G \subset \mathscr{F}(G) \otimes \mathbb{C}[G]$ we obtain therefore a bicharacter on $\hat{G} \times G$ which induces two group isomorphisms

$$\hat{G} \to \mathrm{Hom}(G, \mathbb{C}^*),\ \ f \mapsto \langle f, \cdot\rangle \quad \text{and} \quad G \mapsto \mathrm{Hom}(\hat{G}, \mathbb{C}^*), \ \ g \mapsto \langle\cdot, g\rangle.$$

In particular the bidual of G is canonically isomorphic to G. When G is not abelian, the set $\hat{G}$ of characters is to small and this result does not hold anymore.

1.2.2 The following definition gives the right notion to answer the following questions: how does the group structure of G reflect on its function algebra? how can we make tensor products of algebra representations? how can a finite group be reconstructed from a single algebra?

Definition 1.4 *Let A be a vector space. A coproduct, or comultiplication, on A is a linear map $\Delta : A \to A\otimes A$ such that $(\Delta\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta$ (coassociativity). A counit for (A, Δ) is a linear form $\epsilon : A \to k$ such that $(\epsilon\otimes\mathrm{id})\Delta = \mathrm{id} = (\mathrm{id}\otimes\epsilon)\Delta$. The triple (A, Δ, ϵ) is then called a (counital) coalgebra.*

The first example, as anounced above, is for $A = \mathscr{F}(G)$ with G finite. We can indeed use the product of G to define a coproduct on A in a very natural way: we take $A\otimes A = \mathscr{F}(G \times G)$ and we put $\Delta_G(f)(g, h) = f(gh)$. An easy calculation shows that coassociativity reduces to associativity of the group product: for all g, h, $k \in G$,

$$\begin{aligned}((\Delta_G\otimes\mathrm{id})\Delta_G(f))(g, h, k) &= \Delta_G(f)(gh, k) \\ &= f((gh)k) = f(g(hk)) \\ &= \Delta_G(f)(g, hk) = ((\mathrm{id}\otimes\Delta_G)\Delta_G(f))(g, h, k).\end{aligned}$$

Exercise 9. With G and Δ_G as above, check that G is abelian **iff** $\Sigma \circ \Delta_G = \Delta_G$. We say that Δ_G is symetric, or cocommutative. For a given $a \in G$, define $\epsilon : \mathscr{F}(G) \to k$ by $\epsilon(f) = f(a)$. Show that a is the unit element of G **iff** ϵ is a counit for Δ_G. We put then $\epsilon = \epsilon_G$.

Note that the definition of Δ_G does not work when G is infinite: indeed there is no reason why for any function $f : G \to k$ one could find *finitely many* pairs of functions g_i, $h_i : G \to k$ such that $f(xy) = \sum g_i(x)h_i(y)$ for all x, $y \in G$ — even for the characteristic function of $e \in G$ this is not possible.

There is however a nice subalgebra of $\mathscr{F}(G)$ on which the coproduct Δ_G makes sense even in the infinite case: this is the so-called algebra of representative functions on G defined by

$$R(G) = \{f : G \to k \mid \dim \mathrm{span}\, G \cdot f < \infty\},$$

where $G \cdot f$ is the orbit of f under the action of G given by $(y \cdot f)(x) = f(xy)$. Take indeed a finite basis h_i of this orbit, and denote by $g_i(x)$ the coefficients of $x \cdot f$ in this basis, then we have $\Delta(f) = \sum g_i\otimes h_i$. However it can be that $R(G)$ is not "rich" enough to remember G, e.g. there are examples with G infinite but $R(G) = k$.

1.2.3 The coproduct Δ_G on $\mathscr{F}(G)$, and also a "dual" coproduct $\hat{\Delta}_G$ on $k[G]$, can also be introduced as a consequence of the "generalized Pontrjagin" duality between $\mathscr{F}(G)$ and $k[G]$. For a finite group G, the braket $\langle\cdot,\cdot\rangle$ induces indeed linear isomorphisms $k[G] \to \mathscr{F}(G)^*$, $a \mapsto \langle\cdot, a\rangle$ and $\mathscr{F}(G) \to k[G]^*$, $f \mapsto \langle f, \cdot\rangle$, and also at the level of double tensor products, e.g. $\mathscr{F}(G)\otimes\mathscr{F}(G) \to k[G]^*\otimes k[G]^* \simeq (k[G]\otimes k[G])^*$.

One can then transport the algebra structure maps from one side to another by dualizing: the multiplication map $m : A\otimes A \to A$ yields a coproduct $\Delta = {}^t m : A^* \to A^*\otimes A^*$, and the unit map $\eta : k \to A$ yields a counit $\epsilon = {}^t\eta : A^* \to k$. It is clear that the identities of Example 1.3 correspond to the ones of Definition 1.4. More generally, it is clear that the dual of a f.-d. algebra (resp. coalgebra) is naturally a coalgebra (resp. an algebra).

Let us compute the coproduct $\hat{\Delta}_G$ obtained in this way on $k[G]$. For e, $f \in \mathscr{F}(G)$ and $g \in G$ we have by definition

$$\langle e\otimes f, \hat{\Delta}_G(g)\rangle = \langle m(e\otimes f), g\rangle = (ef)(g) = e(g)f(g) = \langle e\otimes f, g\otimes g\rangle,$$

hence $\hat{\Delta}_G(g) = g\otimes g$. By linearity, this determines $\hat{\Delta}_G$ on $k[G]$. Note that $\hat{\Delta}_G$ is cocommutative. Note also that this formula clearly defines a coproduct on $k[G]$ even if G is infinite.

Exercise 10. Check that the coproduct and counit on $\mathscr{F}(G)$ induced by duality from the multiplication and unit of $k[G]$ coincide with Δ_G and ϵ_G as above. Show that the counit on $k[G]$ is given by $\hat{\epsilon}_G(g) = 1$ for all $g \in G \subset k[G]$.

Now, it is clear that the algebra $\mathscr{F}(G)$ together with Δ_G and ϵ_G completely remembers the pair of algebras $(\mathscr{F}(G), k[G])$ in duality, and hence the group G itself by Exercise 8. Concretely, any algebra isomorphism $\psi : \mathscr{F}(G') \to \mathscr{F}(G)$ such that $\Delta_G \circ \psi = (\psi \otimes \psi) \circ \Delta_{G'}$ and $\epsilon_G \circ \psi = \epsilon_{G'}$ is associated to a group isomorphism $\phi : G \to G'$ by the formula $\psi(f) = f \circ \phi$.

This means that we can use $(\mathscr{F}(G), \Delta_G, \epsilon_G)$ as a replacement for the finite group G. Moreover we have a candidate for the dual object to it, namely $(k[G], \hat{\Delta}_G, \hat{\epsilon}_G)$. Of course, in the non abelian case this is not a function algebra associated to any group, because it is not commutative: we need to enlarge the category we are working in, and this will be the purpose of the next section. Note that one can also adopt the following symetric point of view: think of $(k[G], \hat{\Delta}_G, \hat{\epsilon}_G)$ as a replacement of G, and of $(\mathscr{F}(G), \Delta_G, \epsilon_G)$ as its dual.

1.2.4 Finally, let us underline how a coproduct Δ allows to take tensors products of representations π, ρ of an algebra A. One cannot put anymore $(\pi \otimes \rho)(a) = \pi(a) \otimes \rho(a)$ since this formula is not linearly consistent. But we can use $\Delta(a)$ as a replacement for $a \otimes a$: when $A = k[G]$ and $\Delta = \hat{\Delta}_G$ this will be consistent with the usual notion for group representations since $\hat{\Delta}_G(g) = g \otimes g$. Hence we define, abusing notation, $\pi \otimes \rho = (\pi \otimes \rho) \circ \Delta$.

Since we want to stay in the category of representations of A, it remains to ensure that $(\pi \otimes \rho) \circ \Delta$ is a homomorphism. First note that the vector space $A \otimes B$ of two (unital) algebras is very naturally endowed with the following (unital) algebra structure:

$$(a \otimes b)(c \otimes d) = (ac) \otimes (bd), \quad 1_{A \otimes B} = 1_A \otimes 1_B.$$

so that the tensor product $f \otimes g$ of two homomorphisms remains a homomorphism. Hence the natural requirement is that the coproduct Δ be a unital homomorphism itself. This leads to the notion of (unital) bialgebra, where some compatibility conditions are required between an algebra and a coalgebra structure on the same vector space.

Exercise 11. Check that the coproducts Δ_G, $\hat{\Delta}_G$ associated to a finite group G — as well as ϵ_G and $\hat{\epsilon}_G$ — are unital homomorphisms.

2 Hopf algebras

2.1 On the definition

In this section we introduce and discuss the formal definition of a Hopf algebra. As explained in the previous section, we are interested in vector spaces equipped with compatible structures of an algebra and a coalgebra. To allow for a full reconstruction theorem in the classical case, we also need a structure map, called the antipode, playing the role of the inversion in groups.

Recall that if A is an algebra, then the tensor product $A \otimes A$ is also an algebra with multiplication $(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$. If A is unital, then so is $A \otimes A$ with unit $1_A \otimes 1_A$. This can be written

$$m_{A\otimes A} = (m \otimes m)(\mathrm{id} \otimes \Sigma \otimes \mathrm{id}), \quad \eta_{A\otimes A} = \eta_A \otimes \eta_A,$$

where $k \otimes k$ is identified with k via complex multiplication. Similarly, if A is a coalgebra, there is a canonical coalgebra structure on $A \otimes A$ given by

$$\Delta_{A\otimes A} = (\mathrm{id} \otimes \Sigma \otimes \mathrm{id})(\Delta_A \otimes \Delta_A), \quad \epsilon_{A\otimes A} = \epsilon_A \otimes \epsilon_A.$$

Definition 2.1 *A Hopf algebra is a collection $(A, m, \eta, \Delta, \epsilon, S)$ where:*

1. *(A, m, η) is a unital algebra and (A, Δ, ϵ) is a counital coalgebra,*
2. *Δ, ϵ are algebra homomorphisms,*
3. *$S : A \to A$ is a linear map such that $m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S)\Delta = \eta\epsilon$.*

Let us recall the definition of a (co)algebra: $m : A \otimes A \to A$, $\eta : k \to A$, $\Delta : A \otimes A \to A$ and $\epsilon : A \to k$ are linear maps such that

$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m), \quad m(\eta \otimes \mathrm{id}) = m(\mathrm{id} \otimes \eta) = \mathrm{id},$$
$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \quad (\epsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}.$$

As a first, trivial example, the algebra k is a Hopf algebra with structure maps given by $\Delta(1) = 1 \otimes 1$, $m(1 \otimes 1) = 1$, $\epsilon = \eta = S = \mathrm{id}$. To check this, do not forget to identify $k \otimes k$ with k where appropriate!

Note that condition 2 seems to break the symmetry of the Definition. However it turns out that it is equivalent for Δ, ϵ to be unital algebra morphisms, or for m, η to be counital coalgebra morphisms. Let us state first the evident definitions: $f : A \to B$ is said to be a coalgebra morphism if $(f \otimes f)\Delta_A = \Delta_B f$, and to be counital if $\epsilon_B f = \epsilon_A$.

Now it suffices to write down definitions to solve the following:

Exercise 12. Let A be equipped with the structure of an algebra (A, m, η) and of a coalgebra (A, Δ, ϵ). Show that Δ, ϵ are unital algebra morphisms **iff** m, η are counital coalgebra morphisms.

Let us now discuss condition 3 of the Definition: it turns out that antipodes of Hopf algebras are unique, so that only the existence of S is really required in this Definition. To see this, we first define a convolution product between linear endomorphisms of A by putting $f * g = m(f \otimes g)\Delta$. Then:

Exercise 13. Show that the convolution product above turns the space $L(A)$ of linear endomorphisms of A into an algebra with unit $\eta\epsilon$. Deduce that if A is both an algebra and a coalgebra, then there exists at most one linear map $S : A \to A$ satisfying the formulae of condition 3 above.

One could also wonder why no compatibility condition is required between the antipode and the algebra or coalgebra structure. In fact such conditions are automatic:

Proposition 2.2 *Let A be a Hopf algebra with antipode S. Then S is a unital anti-algebra morphism and a counital anti-coalgebra morphism, i.e. $Sm = m(S\otimes S)\Sigma$ and $\Delta S = \Sigma(S\otimes S)\Delta$.*

PROOF. We endow the space $L(A, A\otimes A)$ with an algebra structure as above, given by the product $f * g = m_2(f\otimes g)\Delta$ and the unit element $\eta_2\epsilon$, where $\eta_2 = \eta\otimes\eta$ and $m_2 = (m\otimes m)(\mathrm{id}\otimes\Sigma\otimes\mathrm{id})$ are the algebra structure maps of $A\otimes A$. The following computation, where we use the fact that Δ is multiplicative, shows that ΔS is the inverse of Δ in this algebra:

$$m_2(\Delta\otimes\Delta S)\Delta = \Delta m(\mathrm{id}\otimes S)\Delta = \Delta\eta\epsilon = \eta_2\epsilon.$$

On the other hand, $\Sigma(S\otimes S)\Delta$ is also the inverse of Δ in this algebra. Indeed, let us use the notation $\Delta^3 = (\Delta\otimes\Delta)\Delta = (\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)\Delta$ and $\tilde{m} = m(\mathrm{id}\otimes S)$. We have

$$\begin{aligned} m_2(\Delta\otimes\Sigma(S\otimes S)\Delta)\Delta &= (m\otimes m)\Sigma_{(234)}(\mathrm{id}\otimes\mathrm{id}\otimes S\otimes S)\Delta^3 \\ &= (\tilde{m}\otimes\tilde{m})\Sigma_{(234)}\Delta^3 \\ &= (\tilde{m}\otimes\tilde{m})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\mathrm{id}\otimes\Sigma)(\mathrm{id}\otimes\Delta)\Delta \\ &= (\tilde{m}\otimes\eta)(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)\Delta \\ &= (\tilde{m}\Delta)\otimes\eta = (\eta\otimes\eta)\epsilon = \eta_2\epsilon. \end{aligned}$$

Hence $\Delta S = \Sigma(S\otimes S)\Delta$: the antipode is an anti-coalgebra morphism.

Finally, composing the axioms for the antipode by the counit on the left, and using the multiplicativity of the counit, we have

$$\epsilon = \epsilon\eta\epsilon = \epsilon m(S\otimes\mathrm{id})\Delta = (\epsilon\otimes\epsilon)(S\otimes\mathrm{id})\Delta = \epsilon S,$$

hence S is counital.

If A is finite-dimensional, applying the beginning of the proof to A^* yields the second part. Otherwise one has to check that the computations above still make sense when dualized formally. ■

Finally, one could wonder why there is no dual object to S in the definition, since all other structure maps obviously come by pairs. In fact it is easily seen that the axiom of the antipode is "self-dual", so that we have the following result, which is a major motivation for the definition.

Proposition 2.3 *Let $(A, m, \eta, \Delta, \epsilon, S)$ be a finite-dimensional Hopf algebra. Then $(A^*, {}^t\Delta, {}^t\epsilon, {}^tm, {}^t\eta, {}^tS)$ is also a Hopf algebra.*

PROOF. Straightforward. ■

In the infinite-dimensional case the result above does not hold, mainly because the canonical injection $A^*\otimes A^* \to (A\otimes A)^*$ is not an isomorphism anymore, so that ${}^tm$ is not a well-defined coproduct anymore.

However there is a natural subalgebra of A^*, equal to A^* in the finite-dimensional case, which becomes a Hopf algebra in general: it is the so-called finite dual A° of A, which consists of linear forms ϕ on A that factor through a finite-dimensional *algebra*, i.e. for which there exists a finite-codimensional ideal $I \subset A$ such that $\phi(I) = 0$.

Note however that A° may be trivial even if A is not; in fact A° separates the points of A **iff** A is residually finite-dimensional (or proper), i.e. it has "enough" finite-dimensional representations as an algebra. For a more precise discussion of the duality for infinite-dimensional Hopf algebra we refer to [5, Chapter 9].

We discuss finally the construction of opposite and coopposite bialgebras. If A is an algebra, we denote by A^{op} the opposite algebra, i.e. the same vector space with multiplication given by $m = m_A \circ \Sigma$, so that $x \cdot_{\mathrm{op}} y = yx$ for x, $y \in A$. It is clear that multiplicativity of Δ, ϵ is the same on A or A^{op}. However if A is a Hopf algebra then its antipode is not necessarily an antipode for A^{op} — in fact A^{op} need not even be a Hopf algebra.

Proposition 2.4 *Let A be a Hopf algebra with antipode S. Then $A^{\rm op}$ is a Hopf algebra **iff** $S : A \to A$ is bijective, and its antipode is then S^{-1}. It is then called the opposite Hopf algebra to A.*

PROOF. We compose the axioms for the antipode T of $A^{\rm op}$ by S on the left and use Proposition 2.2:

$$\begin{aligned} Sm\Sigma(T\otimes \mathrm{id})\Delta &= m(ST\otimes S)\Delta \\ &= S\eta\epsilon = \eta\epsilon. \end{aligned}$$

If $A^{\rm op}$ is a Hopf algebra, this shows that ST is the convolution inverse of S, hence the identity. Composing similarly by S on the right we see that S is bijective with composition inverse T. Conversely if S is composition invertible, we put $T = S^{-1}$ and the computation above shows that T is an antipode for $A^{\rm op}$. ■

Denote by $\Delta^{\rm op} = \Sigma\Delta$ the opposite coproduct on A, and by $A^{\rm cop}$ the algebra A equipped with $\Delta^{\rm op}$. Again, the only axiom that is not automatic for $A^{\rm cop}$ to be a Hopf algebra is the existence of the antipode. Moreover it is easy to check that $A^{\rm op,cop}$ is a Hopf algebra with the same antipode as A. Hence the result above holds also for $A^{\rm cop}$.

Notice also the following corollary of the previous Proposition: when A is commutative, i.e. $A = A^{\rm op}$, by unicity of the antipode we must have $S^2 = \mathrm{id}$. Similarly $S^2 = \mathrm{id}$ if A is cocommutative. Note that this is a very natural property of S, which is supposed to play the role of the inverse in groups — however it does not always hold.

We will see later that in the finite-dimensional case, S is automatically bijective. Let us also quote the following deeper results in the finite-dimensional case over k: S has automatically finite order in $GL(A)$, and if moreover A is semisimple and $k = \mathbb{C}$ we have automatically $S^2 = \mathrm{id}$. The question whether $S^2 = \mathrm{id}$ for f.-d. semisimple Hopf algebra over an arbitrary field is known as Kaplansky's conjecture.

Exercise 14. Take $k = \mathbb{C}$ and recall that an involutive algebra A is an algebra A endowed with an antilinear, antimultiplicative involution $(a \mapsto a^*)$. If A, B are involutive algebras, so is $A\otimes B$ with respect to $(a\otimes b)^* = a^*\otimes b^*$. Let A be an involutive algebra and a Hopf algebra such that $\Delta(a^*) = \Delta(a)^*$ for all $a \in A$. Using the uniqueness of the antipode, show that $S \circ * \circ S \circ * = \mathrm{id}$. Similarly, show that $\epsilon(a^*) = \overline{\epsilon(a)}$ for all $a \in A$.

2.2 Classical and non-classical examples

2.2.1 In this section we present some important examples of finite-dimensional and infinite-dimensional Hopf algebras.

EXAMPLES 2.5 We recall now briefly the examples arising from a finite group G as in the previous sections. First we have the function algebra $A = \mathscr{F}(G)$. Identifying $A\otimes A$ with $\mathscr{F}(G \times G)$ as explained in Section 1.1, the structure maps are given as follows:

$$\begin{gathered} m(f\otimes g) = (x \mapsto f(x)g(x)), \quad \eta(\lambda) = (x \mapsto \lambda), \\ \Delta(f) = ((x, y) \mapsto f(xy)), \quad \epsilon(f) = f(e), \\ S(f) = (x \mapsto f(x^{-1})). \end{gathered}$$

Here e denotes the unit element of G. Then we have the group algebra $A = k[G]$, equipped with the following structure maps given on the relevant canonical bases:

$$\begin{gathered} m(g\otimes h) = gh, \quad \eta(\lambda) = \lambda e, \\ \Delta(g) = g\otimes g, \quad \epsilon(g) = 1, \\ S(g) = g^{-1}. \end{gathered}$$

These examples are dual to each other, i.e. we have $\mathscr{F}(G)^* =\simeq k[G]$ and $k[G]^* \simeq \mathscr{F}(G)$. Hence finite-dimensional Hopf algebra form a category, containing finite groups as a subcategory, and equipped with a duality induced by duality of vector spaces, which extends Pontrjagin duality for finite abelian groups.

As explained in Section 1.2, for infinite groups the construction of $k[G]$ works unchanged, but one has to replace $\mathscr{F}(G)$ with the space of representative functions $R(G)$, which is nothing but the finite dual $k[G]^\circ$ of $k[G]$. □

Exercise 15. Check that the axiom of the antipode is satisfied in the examples above.

An important question regarding the previous "classical example" is whether we can recognize them easily inside the general category of f.-d. Hopf algebras. Take $k = \mathbb{C}$ for simplicity, and recall then that function algebras $\mathscr{F}(E)$ on finite sets are characterized by their commutativity and semisimplicity — this is a consequence of Artin-Wedderburn's Theorem.

Hence one would like to recognize the Hopf algebras $\mathscr{F}(G)$, for G a finite group, exactly as the commutative semisimple Hopf algebras. This fact is proved at Proposition 2.7, and thus we can consider general Hopf algebras as "quantum groups" in the meaning of noncommutative geometry.

Note that the dual Hopf algebra $\mathbb{C}[G]$ to $\mathscr{F}(G)$ is also semisimple — this is Maschke's Theorem in the zero characteristic case — so finite groups are the commutative objects in the category of semisimple and "cosemisimple" complex Hopf algebras, whose self-duality induces Pontrjagin's duality of finite abelian groups.

Lemma 2.6 *Let A be a Hopf algebra, and $G(A) = \{x \in A \mid x \neq 0,\ \Delta(x) = x\otimes x\}$ the set of group-like elements in A. Then $G(A)$ is a linearly independent family in A, and the algebra structure of A restricts to a group structure on $G(A)$.*

PROOF. First we notice that $\epsilon(x) = 1$ for all $x \in G(A)$: indeed the identity $(\epsilon\otimes\mathrm{id})\Delta = \mathrm{id}$ yields $\epsilon(x)x = x$, and x is non-zero by definition.

Now let $H \subset G(A)$ be a maximal linearly independent subset. For any $x \in G(A)$ we can write $x = \sum_{y\in H} \lambda_y y$. Then we have

$$\sum_{y\in H} \lambda_y (y\otimes y) = \Delta(x) = x\otimes x = \sum_{y,z\in H} \lambda_y\lambda_z (y\otimes z).$$

Since H is linearly independent, so is $(y\otimes z)_{y,z\in H}$. Hence the identity above implies that at most one λ_y does not vanish: we have $x = \lambda y$ with $\lambda \in k^*$ and $y \in H$. Applying ϵ we see that $\lambda = 1$, hence $x = y \in H$. This shows that $G(A) = H$, hence $G(A)$ is linearly independent.

Since S is anti-comultiplicative, for any $x \in G(A)$ the element $S(x)$ is zero or group-like. Considering the identity $m(S\otimes\mathrm{id})\Delta = \eta\epsilon$ we see that $S(x)x = 1$, hence $S(x)$ is non-zero and x has an inverse in $G(A)$. In particular for x, $y \in G(A)$ the product xy does not vanish, hence the identity $\Delta(xy) = \Delta(x)\Delta(y) = xy\otimes xy$ shows that $xy \in G(A)$. ■

Proposition 2.7 *Let A be a finite-dimensional, semisimple commutative Hopf algebra over $\mathbb{C}$. Then there exists a finite group G such that $A \simeq \mathscr{F}(G)$ as a Hopf algebra.*

PROOF. The proof essentially relies on Artin-Wedderburn's Theorem: a semisimple complex algebra A is isomorphic to a finite sum of matrix algebras. Since A is moreover commutative, we must have a unital algebra isomorphism $A \simeq \mathscr{F}(G)$, with G a finite set.

Define $\mathbb{C}[G] = \bigoplus_{g\in G} \mathbb{C}g$ and a duality bracket $\langle f, g\rangle = f(g)$ on $\mathscr{F}(G) \times \mathbb{C}[G]$. We obtain a counital coalgebra isomorphism $A^* \simeq \mathbb{C}[G]$ with respect to $\Delta(g) = g\otimes g$, $\epsilon(g) = 1$: we have already done this computation at paragraph 1.2.3.

Now, we transport the algebra structure of A^* onto $\mathbb{C}[G]$. Since $G \subset \mathbb{C}[G]$ is a generating family of group-like elements of the Hopf algebra $\mathbb{C}[G]$, it is the set of group-like elements of $\mathbb{C}[G]$. Hence the multiplication of $\mathbb{C}[G]$ turns G into a group such that $\mathbb{C}[G]$ is the group algebra of G.

Finally, this shows that $A \simeq \mathscr{F}(G)$ where $\mathscr{F}(G)$ is equipped with its usual Hopf algebra structure, dual to the one of $\mathbb{C}[G]$. ■

In fact the result above still holds over on arbitrary field, up to an extension of scalars. Let us quote two more results that hold only over fields of zero characteristic: f.-d. semisimple Hopf algebras over $\mathbb{C}$ are automatically cosemisimple; f.-d. commutative Hopf algebras over $\mathbb{C}$ are of the form $\mathscr{F}(G)$ without semisimplicity assumption.

2.2.2 We present now some noncommutative, noncocommutative examples which, beside their independent interest, provide counter-examples to some properties of the antipode considered in Section 2.1:

EXAMPLES 2.8 If char $k \neq 2$, the smallest noncommutative, noncocommutative Hopf algebra has dimension 4 and is unique (hence self-dual). The underlying algebra is defined by generators and relations as follows:

$$H_4 = \langle 1, g, x \mid g^2 = 1, x^2 = 0, xg = -gx \rangle_{\text{alg}}.$$

The coalgebra structure is given by

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} = g,$$
$$\Delta(x) = x \otimes 1 + g \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -gx.$$

It is an easy exercise to check that these formulae indeed define a 4-dimensional Hopf algebra, and that its antipode has order 4.

Let us give now an infinite-dimensional example. Choose $q \neq 0$ in k and consider the "quantum plane" algebra $\mathscr{O}_q(k^2) = \langle 1, x, y \mid xy = qyx \rangle$. It becomes a (unital, counital) bialgebra if we put $\Delta(x) = x \otimes x$, $\Delta(y) = y \otimes 1 + x \otimes y$, $\epsilon(x) = 1$, $\epsilon(y) = 0$. This bialgebra is not a Hopf algebra since we should have $xS(x) = S(x)x = 1$ and x is not invertible in $\mathscr{O}_q(k^2)$, but we can consider

$$A = \langle 1, x, x^{-1}, y \mid xy = qyx, xx^{-1} = x^{-1}x = 1 \rangle,$$

which becomes a Hopf algebra with $S(x) = x^{-1}$, $S(y) = -x^{-1}y$. In this case the antipode is invertible with infinite order. □

Exercise 16. Check the facts stated in the Examples above. Give the expression of a nondegenerate bilinear form on $H_4 \otimes H_4$ implementing the self-duality of H_4. Is it unique?

Finally, we present the important example of quantum $sl(2)$, which can be generalized to "deform" any semisimple complex Lie algebra, yielding the well-known "quantum groups" à la Drinfel'd and Jimbo.

EXAMPLE 2.9 Let q be a complex number such that $q \neq 0$, $q^4 \neq 1$. We denote by $U_q(sl(2))$ the unital algebra generated by generators E, F, K, K^{-1} and relations $KK^{-1} = K^{-1}K = 1$ and

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K^2 - K^{-2}}{q^2 - q^{-2}}.$$

It admits a Hopf algebra structure given by

$$\Delta(E) = E \otimes K^{-1} + K \otimes E, \quad S(E) = -q^{-2} E, \quad \epsilon(E) = 0,$$
$$\Delta(F) = F \otimes K^{-1} + K \otimes F, \quad S(F) = -q^{-2} F, \quad \epsilon(F) = 0,$$
$$\Delta(K) = K \otimes K, \quad S(K) = K^{-1}, \quad \epsilon(K) = 1.$$

One can show that this defines indeed an infinite-dimensional, noncommutative and noncocommutative Hopf algebra.

The motivation for these definitions is as follows: heuristically, one should think that $K = e^{tH}$ and $q = e^t$, and consider the relations up to the first non-vanishing order as $t \to 0$. This yields

$$\begin{aligned} (1+tH)E &= (1+2t)E(1+tH) + O(t^2) &\implies& \quad [H,E] = 2E + O(t) \\ (1+tH)F &= (1-2t)F(1+tH) + O(t^2) &\implies& \quad [H,F] = -2F + O(t) \\ [E,F] &= \frac{(1+2tH)-(1-2tH)+O(t^2)}{(1+2t)-(1-2t)+O(t^2)} &\implies& \quad [E,F] = H + O(t), \end{aligned}$$

and one can recognize the relations satisfied by the following matrices of $sl(2) = \{M \in M_2(k) \mid \operatorname{Tr} M = 0\}$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These relations $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$ in fact generate $sl(2)$ as a Lie algebra, and its universal envelopping algebra $U(sl(2))$ as a unital algebra. So $U_q(sl(2))$ can be thought of as a deformation of $U(sl(2))$.

To check that the formulae above define algebra morphisms Δ, ϵ, it suffices to check that the images of E, F, K satisfy the defining relations of $U_q(sl(2))$. This is quite evident for ϵ, let us do the check for the coproduct Δ:

$$\begin{aligned} \Delta(K)\Delta(E) &= (K\otimes K)(E\otimes K^{-1} + K\otimes E) = KE\otimes 1 + K^2\otimes KE \\ &= q^2 EK\otimes 1 + q^2 K^2\otimes EK = q^2(E\otimes K^{-1} + K\otimes E)(K\otimes K) \\ &= q^2\Delta(E)\Delta(K), \end{aligned}$$

similarly $\Delta(K)\Delta(F) = q^2\Delta(F)\Delta(K)$, and, noting that $EK\otimes K^{-1}F = KE\otimes FK^{-1}$ by the defining relations:

$$\begin{aligned} [\Delta(E),\Delta(F)] &= EF\otimes K^{-2} + EK\otimes K^{-1}F + KF\otimes EK^{-1} + K^2\otimes EF \\ &\quad - FE\otimes K^{-2} + KE\otimes FK^{-1} + FK\otimes K^{-1}E + K^2\otimes FE \\ &= [E,F]\otimes K^{-2} + K^2\otimes [E,F] \\ &= (q-q^{-2})^{-1}((K^2-K^{-2})\otimes K^{-2} + K^2\otimes(K^2-K^{-2})) \\ &= (q-q^{-2})^{-1}(\Delta(K)^2 - \Delta(K)^{-2}). \end{aligned}$$

The same kind of computations show that $S : A \to A$ is well-defined, although some care must be taken since S is only antimultiplicative. Finally, it suffices to check that the axioms for Hopf algebras are satisfied on the generators E, F, K, whith again some extra care for the axiom of the antipode. For instance one compares

$$\begin{aligned} &(\Delta\otimes\mathrm{id})\Delta(E) = \Delta(E)\otimes K^{-1} + \Delta(K)\otimes E = E\otimes K^{-1}\otimes K^{-1} + K\otimes E\otimes K^{-1} + K\otimes K\otimes E \quad \text{and} \\ &(\mathrm{id}\otimes\Delta)\Delta(E) = E\otimes\Delta(K^{-1}) + K\otimes\Delta(E) = E\otimes K^{-1}\otimes K^{-1} + K\otimes E\otimes K^{-1} + K\otimes K\otimes E. \end{aligned}$$

□

2.3 Modules and comodules

2.3.1 In this section we introduce modules and comodules over a Hopf algebra and present some related constructions. In the following definition we recall the notion of module over an algebra in the language of tensor product spaces, and we dualize it to obtain the notion of comodule over a coalgebra:

Definition 2.10 *Let (A, m, η) be a (unital) algebra. A (left, non-degenerate) A-module is a k-vector space M equipped with a linear map $\alpha : A\otimes M \to M$ such that $\alpha(\mathrm{id}\otimes\alpha) = \alpha(m\otimes\mathrm{id})$ and $\alpha(\eta\otimes\mathrm{id}) = \mathrm{id}$.*

Let (A, Δ, ϵ) be a (counital) coalgebra. A (left, non-degenerate) A-comodule is a k-vector space equipped with a linear map $\delta : M \to A\otimes M$ such that $(\mathrm{id}\otimes\delta)\delta = (\Delta\otimes\mathrm{id})\delta$ and $(\epsilon\otimes\mathrm{id})\delta = \mathrm{id}$.

Of course, one can also consider right modules and comodules: the latter correspond e.g. to linear maps $\delta : M \to M \otimes A$ such that $(\delta\otimes\mathrm{id})\delta = (\mathrm{id}\otimes\Delta)\delta$ and $(\mathrm{id}\otimes\epsilon)\delta = \mathrm{id}$.

When all spaces are finite-dimensional, it is clear that if M is a left A-module (resp. comodule), then M^* is naturally endowed with the structure of a left A^*-comodule (resp. module). One can also dualize modules and comodules without changing the Hopf algebra: if M is a left A-module (resp. comodule), then M^* carries a natural right A-module (resp. comodule) structure given by $(\phi \cdot a)(x) = \phi(a \cdot x)$ for $a \in A$, $\phi \in M^*$, $x \in M$, resp. $\delta_{M^*}(\phi)(x) = (\mathrm{id}\otimes\phi)\delta_M(x)$ via $M^*\otimes A \simeq L(M, A)$. Finally, one can turn a right A-module (resp. comodule) into a left one by using the antipode, see Exercise 21.

Since f.-d. representations of a finite group G correspond to unital representations of $k[G]$, i.e. to $k[G]$-modules, we deduce that they also correspond to f.-d. comodules over the function algebra $\mathscr{F}(G)$. Here is how the correspondence works concretely. If $((g, x) \mapsto g\cdot x)$ is a representation of G on M, then the following formula turns M into a $\mathscr{F}(G)$-comodule: $\delta(x) = (g \mapsto g \cdot x)$. Here we identify $\mathscr{F}(G)\otimes M$ with $\mathscr{F}(G, M)$ thank to the bilinear map $\mathscr{F}(G)\times M \ni (f, x) \mapsto (g \mapsto f(g)x) \in \mathscr{F}(G, M)$.

Examples 2.11 Any vector space M carries the left trivial module structure given by $\alpha = \epsilon\otimes\mathrm{id}$, the left trivial comodule structure given by $\delta = \eta\otimes\mathrm{id}$, and similarly on the right. Besides, the vector space A carries the left and right "regular" module and comodule structures given by $\alpha = m$ and $\delta = \Delta$. □

Modules M over an algebra A can also be considered as representations of A: this equivalence is induced by the isomorphism $L(A\otimes M, M) \simeq L(A, L(M))$, which holds even in the infinite-dimensional case. In the finite-dimensional case, we have an analogue for comodules.

More precisely, we identify $L(M)\otimes A$ with $L(M, M\otimes A)$ via

$$L(M)\otimes A \to L(M, M\otimes A), \quad \phi\otimes a \mapsto (x \mapsto \phi(x)\otimes a),$$

and similarly $A\otimes L(M)$ with $L(M, A\otimes M)$. Now if $\delta \in L(M, M\otimes A)$ is a right comodule structure map, denote by $u \in L(M)\otimes A$ the corresponding element on the left-hand side of the isomorphism, so that we have $\delta(x) = u(x\otimes 1)$.

The axioms for δ have their counterparts for u. We have

$$\begin{aligned}(\delta\otimes\mathrm{id})\delta(x) &= (\delta\otimes\mathrm{id})(u(x\otimes 1)) = u_{12}u_{13}(x\otimes 1\otimes 1)\\ (\mathrm{id}\otimes\Delta)\delta(x) &= (\mathrm{id}\otimes\Delta)(u)(x\otimes 1\otimes 1)\end{aligned}$$

so that the first axiom corresponds to the identity $(\mathrm{id}\otimes\Delta)(u) = u_{12}u_{13}$. Similarly, the second one corresponds to $(\mathrm{id}\otimes\epsilon)(u) = \mathrm{id}$. Such an element $u \in L(M)\otimes A$ is called a non-degenerate corepresentation.

When M is infinite-dimensional, non-degenerate corepresentations in $L(M)\otimes A$ still induce comodule structures on M, but not all comodule structures on M arise in this way. For instance the regular comodule $M = A$ comes from a corepresentation **iff** A is finite-dimensional: if this is the case, there exists a f.-d. subspace $B \subset A$ such that $\Delta(A) = u(A\otimes 1) \subset A\otimes B$, and applying $\epsilon\otimes\mathrm{id}$ we obtain $A \subset B$.

Using the axiom of the antipode, we can see that a non-degenerate corepresentation is invertible in the algebra $L(M)\otimes A$, with inverse $\tilde{u} = (\mathrm{id}\otimes S)(u)$:

$$\begin{aligned}\tilde{u}u &= (\mathrm{id}\otimes m)(\tilde{u}_{12}u_{13}) = (\mathrm{id}\otimes m)(\mathrm{id}\otimes S\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)(u)\\ &= (\mathrm{id}\otimes\eta\epsilon)(u) = \mathrm{id}\otimes 1.\end{aligned}$$

Exercise 17. Reproduce the steps above for a left comodule. What is the difference? Hint: use the co-opposite Hopf algebra, if possible.

Exercise 18. Let G be a finite group, and $u \in L(M)\otimes \mathscr{F}(G)$ such that $(\mathrm{id}\otimes\Delta)(u) = u_{12}u_{13}$ and $(\mathrm{id}\otimes\epsilon)(u) = \mathrm{id}$. Identifying $L(M)\otimes \mathscr{F}(G)$ with $\mathscr{F}(G, L(M))$, show that u corresponds to a representation of G.

Many constructions from the theory of group representations can be generalized to modules and comodules over a Hopf algebras. Our goal is not to go through the whole theory, let us just discuss rapidly tensor products, since it was one of our motivations.

Let A be a Hopf algebra, and M, N two left A-modules. Then the tensor product vector space $M\otimes N$ carries a natural structure of A-module given by $\alpha_{M\otimes N} = (\alpha_M\otimes\alpha_N)(\mathrm{id}\otimes\Sigma\otimes\mathrm{id})(\Delta_A\otimes\mathrm{id})$. In other words, we put $a\cdot(x\otimes y) = \Delta(a)(x\otimes y)$. Dualizing, we obtain a formula for the comodule structure on a tensor product of two left A-comodules M, N, which can be expressed by the formula $\delta(x\otimes y) = \delta(x)_{12}\delta(y)_{13}$.

In the case when $A = k[G]$ (for modules), or $\mathscr{F}(G)$ (for comodules), we recover the usual tensor product of group representations. However, in general the flip map Σ is not an intertwiner between the modules $M\otimes N$ and $N\otimes M$, and there are even cases (as in exercise 20 if G is not abelian) when $M\otimes N$ and $N\otimes M$ are not isomorphic as A-modules.

Exercise 19. Check that the formulae above define indeed a module (resp. comodule) structure on $M\otimes N$. If a group G acts by algebra morphisms on two algebras M, N, check that the tensor product action on $M\otimes N$ is still by algebra morphisms. Assume now that M, N are A-comodules and algebras, such that δ_M, δ_N are algebra morphisms. Is $\delta_{M\otimes N}$ and algebra morphism?

Exercise 20. We consider the case $A = \mathscr{F}(G)$. Show that the formula $\delta(\lambda) = g\otimes\lambda$ turns k into an A-comodule denoted k_g. Show that $k_g\otimes k_h \simeq k_{gh}$. More generally, show that an A-comodule structure on a vector space M corresponds to a G-grading of M.

We will be particularly interested by fixed or cofixed vectors:

Definition 2.12 *Let A be a Hopf algebra and M be a left A-module (resp. comodule). The subspace of fixed (resp. cofixed) vectors of M is*

$$M^A = \{x \in M \mid \alpha(\mathrm{id}\otimes x) = x\epsilon\}, \quad \textit{resp.}$$
$$M^{\mathrm{co}A} = \{x \in M \mid \delta(x) = 1\otimes x\}.$$

In the first formula we identify x with the map $(k \to M, \lambda \mapsto \lambda x)$.

In particular, if G is finite and M is the $k[G]$-module, or the $\mathscr{F}(G)$-comodule, associated with a representation $G \to L(M)$, then the space of (co)fixed vectors of M equals $\{x \in M \mid \forall g \in G \;\; g\cdot x = x\}$. If on the other hand M is the $k[G]$-comodule (or the $\mathscr{F}(G)$-module) associated with a G-grading on M, then $M^{k[G]}$ is the neutral component M_e of M.

Exercise 21. Let M be a f.-d. module over a f.-d. Hopf algebra A, and denote $\tilde\alpha : A \to L(M)$ the usual structure map, given by $\alpha(a\otimes x) = \tilde\alpha(a)(x)$.

We endow the vector space dual M^* with the A-module structure given by $(a\cdot\phi)(x) = \phi(S(a)x)$ for $a \in A$, $\phi \in M^*$, $x \in M$. Using $\tilde\alpha$ and Sweedler's notation, write a formula for the resulting action of an element $a \in A$ on an element $f \in L(M) \simeq M\otimes M^*$.

Show that element $I \in M\otimes M^*$ corresponding to $\mathrm{id} \in L(M)$ is a fixed vector. What is the obstruction for the same result to hold for the element $\Sigma(I) \in M^*\otimes M$? How can one modify the module structure on M^* so that $\Sigma(I)$ is fixed?

2.3.2 Given a Hopf algebra A, it is natural to consider vector spaces M that are at the same time modules and comodules over A. There are two natural compatibility conditions that can be imposed to these structures on M, namely that the comodule structure map is a module morphism, or that the module structure map is a comodule morphism. As for the definition of Hopf algebras, both conditions are in fact the same and we put:

Definition 2.13 *Let A be a Hopf algebra with multiplication m and comultiplication Δ. A (left, left, nondegenerate) A-Hopf module is a vector space M equipped with an A-module structure $\alpha : A{\otimes}M \to M$ and an A-comodule structure $\delta : M \to A{\otimes}M$ such that*

$$\delta\alpha = (m{\otimes}\alpha)(\mathrm{id}{\otimes}\Sigma{\otimes}\mathrm{id})(\Delta{\otimes}\delta). \tag{1}$$

By the very definition of Hopf algebras, the vector space A equipped with the module and comodule structures of Examples 2.11 is a A-Hopf module. Moreover, if N is a vector space equipped with the trivial module and comodule strctures, it is clear that $A{\otimes}N$ equipped with the tensor product structures is also an A-Hopf module. It turns out that this example is universal:

Theorem 2.14 *Let A be a Hopf algebra, and M an A-Hopf module. Then M is isomorphic as a Hopf module to $A{\otimes}M^{\mathrm{co}A}$, where $M^{\mathrm{co}A}$ is equipped with the trivial module and comodule structures.*

Proof. We still denote α the restriction $\alpha : A{\otimes}M^{\mathrm{co}A} \to M$, and we introduce a map $\beta : M \to A{\otimes}M$ by the formula

$$\beta = (\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}S{\otimes}\mathrm{id})(\mathrm{id}{\otimes}\delta)\delta = (\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}S{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})\delta.$$

We start by showing that β ranges in $A{\otimes}M^{\mathrm{co}A}$: it suffices to check that $\beta_2 = \alpha(S{\otimes}\mathrm{id})\delta$ ranges in $M^{\mathrm{co}A}$, i.e. that $\delta \circ \beta_2 = \eta{\otimes}\beta_2$. We use (1) and computation rules in Hopf algebras:

$$\begin{aligned}
\delta \circ \alpha(S{\otimes}\mathrm{id})\delta &= (m{\otimes}\alpha)\Sigma_{23}(\Delta{\otimes}\delta)(S{\otimes}\mathrm{id})\delta \\
&= (m{\otimes}\alpha)\Sigma_{23}(S{\otimes}S{\otimes}\mathrm{id}{\otimes}\mathrm{id})\Sigma_{12}(\Delta{\otimes}\delta)\delta \\
&= (m{\otimes}\alpha)(S{\otimes}\mathrm{id}{\otimes}S{\otimes}\mathrm{id})\Sigma_{23}\Sigma_{12}(\Delta{\otimes}\mathrm{id}{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})\delta \\
&= (m{\otimes}\alpha)(S{\otimes}\mathrm{id}{\otimes}S{\otimes}\mathrm{id})\Sigma_{23}\Sigma_{12}(\mathrm{id}{\otimes}\Delta{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})\delta \\
&= (m{\otimes}\alpha)(S{\otimes}\mathrm{id}{\otimes}S{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id}{\otimes}\mathrm{id})\Sigma_{12}(\Delta{\otimes}\mathrm{id})\delta \\
&= (\eta\epsilon{\otimes}\alpha)(\mathrm{id}{\otimes}S{\otimes}\mathrm{id})\Sigma_{12}(\Delta{\otimes}\mathrm{id})\delta \\
&= (\eta{\otimes}\alpha)(S{\otimes}\mathrm{id})(\mathrm{id}{\otimes}\epsilon{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})\delta = \eta{\otimes}(\alpha(S{\otimes}\mathrm{id})\delta).
\end{aligned}$$

Next we show that α and β are inverse to each other. First we have, using again (1) and the fact that $\delta = \eta{\otimes}\mathrm{id}$ when restricted to $M^{\mathrm{co}A}$:

$$\begin{aligned}
\beta \circ \alpha &= (\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}S{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})\delta\alpha \\
&= (\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}S{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})(m{\otimes}\alpha)\Sigma_{23}(\Delta{\otimes}\delta) \\
&= (\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}S{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})(m{\otimes}\alpha)\Sigma_{23}(\Delta{\otimes}\eta{\otimes}\mathrm{id}) \\
&= (\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}S{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})(\mathrm{id}{\otimes}\alpha)(\Delta{\otimes}\mathrm{id}) \\
&= (\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}S{\otimes}\mathrm{id}{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id}{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id}) \\
&= (\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}m{\otimes}\mathrm{id})(\mathrm{id}{\otimes}S{\otimes}\mathrm{id}{\otimes}\mathrm{id})(\mathrm{id}{\otimes}\Delta{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id}) \\
&= (\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}\eta\epsilon{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id}) = \mathrm{id}{\otimes}\mathrm{id}.
\end{aligned}$$

Secondly the computation rules in Hopf algebra and the axioms for modules and comodules give directly

$$\begin{aligned}
\alpha \circ \beta &= \alpha(\mathrm{id}{\otimes}\alpha)(\mathrm{id}{\otimes}S{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})\delta \\
&= \alpha(m{\otimes}\mathrm{id})(\mathrm{id}{\otimes}S{\otimes}\mathrm{id})(\Delta{\otimes}\mathrm{id})\delta \\
&= \alpha(\eta\epsilon{\otimes}\mathrm{id})\delta = \mathrm{id}.
\end{aligned}$$

Finally, α is a comodule morphism because M is a Hopf module, and a module morphism if we endow $M^{\mathrm{co}A}$ with the trivial module structure, by definition of module structure maps. ■

2.4 Invariant forms on finite-dimensional Hopf algebras

In this section we study invariant forms on f.-d. Hopf algebras, and discuss some consequences. We start with the relevant definition:

Definition 2.15 *Let A be a Hopf algebra. A linear form $\phi \in A^*$ is called left-invariant if we have* $(\mathrm{id}\otimes\phi)\Delta = \eta\phi$*, right-invariant if* $(\phi\otimes\mathrm{id})\Delta = \eta\phi$.

Since A^* separates the points of A, right invariance of ϕ means that $\phi\psi = \psi(1)\phi$ for all $\psi \in A^*$, where we use the product on A^* induced by the coproduct of A: in other words, ϕ is invariant for the natural right A^*-module structure on A^*.

When A is finite-dimensional, this right A^*-module structure is also associated to a left A-comodule structure on A^*: the corresponding structure map $\delta : A^* \to A\otimes A^*$ is given by the formula $\langle\delta(\phi), \psi\otimes a\rangle = \langle\phi\psi, a\rangle = (\phi\otimes\psi)\Delta(a)$. Note that this left A-comodule structure on A^* is also the one naturally associated with the right A-comodule structure on A given by the coproduct.

In the "classical case" of a function algebra $A = \mathscr{F}(G)$ with G a finite group, linear forms on A can be written as integrals, $\phi(f) = \sum \lambda_g f(g) = \int f\mathrm{d}\phi$, and the identity $(\mathrm{id}\otimes\phi)\Delta = \eta\phi$ corresponds to the usual left invariance for integrals: for all $g \in G$,

$$(\mathrm{id}\otimes\phi)\Delta(f)(g) = \int f(gh)\mathrm{d}\phi(h) \quad \text{and} \quad \eta\phi(f)(g) = \int f(h)\mathrm{d}\phi(h).$$

Taking for f the characteristic functions of points, one finds easily that left or right invariant forms correspond exactly to multiples of the counting measure in this case, i.e. $\phi(f) = \lambda\sum f(g)$.

Exercise 22. Let G be a finite group and consider the Hopf algebra $A = kG$. Identifying forms $\phi \in A^*$ with functions on G by restriction, show that left or right invariant forms on A exactly correspond to multiples of the characteristic function of the unit $e \in G$.

2.4.1 Let A be a finite-dimensional Hopf algebra. Our main objective in this section will be to prove that $(A^*)^{\mathrm{co}A}$ is one-dimensional, i.e. non-zero right invariant forms on A exist and are unique up to a scalar multiple. The idea is to apply Theorem 2.14 to A^* with the comodule structure above. Since A^* and A have the same dimension, the result will follow immediately.

Hence the point is to find a left A-module structure on A^*, which together with the comodule structure above turns it into a left A-Hopf module. Recall that this comodule structure arises from the natural right A-comodule structure on A, which can be completed to a right Hopf module structure by multiplication on the right. However, the natural left module and comodule structures on the dual M^* of a right Hopf module are not compatible in general.

In fact in our case the correct left A-module structure on A^* is the one induced from the natural left A-module structure on A using the antipode. In other words, we put $\alpha(a\otimes\phi) = a\cdot\phi = (b \mapsto \phi(S(a)b))$. Since we have not proved any general "dualization" result, we first need to do some sanity checks, and we then prove that we have indeed obtained a Hopf module.

Exercise 23. Let A be a Hopf algebra. Let M be a right A-comodule and a left A-module. Check that the formula

$$\alpha_{M^*}(a\otimes\phi) = a\cdot\phi = (x \mapsto \phi(S(a)\cdot x)) = (x \mapsto \phi(\alpha_M(S(a)\otimes x)))$$

turns M^* into a left A-module. Assuming that M and A are finite dimensional, check that the formula

$$\langle\delta_{M^*}(\phi), \psi\otimes x\rangle = (\phi\otimes\psi)\delta_M(x)$$

turns M^* into a left A-comodule. The formulae above are meant to be valid for any $a \in A$, $x \in M$, $\phi \in M^*$, $\psi \in A^*$.

Lemma 2.16 *Let A be a finite-dimensional algebra, and endow A^* with the left module and comodule structures as above. Then A^* is a Hopf module.*

PROOF. We have to prove that $\delta(a \cdot \phi) = \Delta(a) \cdot \delta(\phi)$ for all $a \in A$, $\phi \in A^*$, where we use the left $A\otimes A$-module structure on $A\otimes A^*$ arising from the multiplication of A and our A-module structure on A^*. We can evaluate this identity against an elementary tensor $\psi\otimes b \in A^*\otimes A$. First we have

$$\langle \delta(a \cdot \phi), \psi\otimes b\rangle = ((a \cdot \phi)\otimes\psi)\Delta(b) = (\phi\otimes\psi)((S(a)\otimes 1)\Delta(b)).$$

On the other hand we compute, using Sweedler's notation $\Delta(a) = \sum a_{(1)}\otimes a_{(2)}$:

$$\begin{aligned}\langle \Delta(a) \cdot \delta(\phi), \psi\otimes b\rangle &= \langle \delta(\phi), (\psi\otimes b) \cdot \Delta(a)\rangle \\ &= \textstyle\sum (\phi\otimes\psi)((1\otimes a_{(1)})\Delta(S(a_{(2)})b)).\end{aligned}$$

Here we used the right $A\otimes A$-module structure on $A^*\otimes A$ given by $\phi\cdot a = (b \mapsto \phi(ab))$ and $b\cdot a = S(a)b$. Now it is an easy computation in Hopf algebras to check that $\sum(1\otimes a_{(1)})\Delta(S(a_{(2)})b) = (S(a)\otimes 1)\Delta(b)$ — one can note that it suffices to take $b = 1$. ■

Theorem 2.17 *Let A be a finite-dimensional Hopf algebra. Then there exist non-zero right invariant forms on A, and they are unique up to a scalar.*

PROOF. By the Lemma, we can apply Theorem 2.14 and we obtain $A^* \simeq A\otimes(A^*)^{\mathrm{co}A}$, hence $\dim(A^*)^{\mathrm{co}A} = 1$. ■

One can use the same arguments "on the right" — defining in particular a right Hopf structure on A^* — to prove existence and uniqueness of non-zero left invariant forms on A. Or we can apply the previous Theorem to A^{cop}, if we know that it is a Hopf algebra, i.e. the antipode of A is invertible. In fact the antipode of a f.-d. Hopf algebra is always invertible, and this result is itself a consequence of the previous Theorem:

Corollary 2.18 *Let A be finite-dimensional Hopf algebra with antipode S. Then S is bijective. In particular A^{cop} is a Hopf algebra, with antipode S^{-1}.*

PROOF. We recall from the proof of Theorem 2.14 that the isomorphism $A\otimes(A^*)^{\mathrm{co}A} \simeq A^*$ used in the proof of Theorem 2.17 is given by $(a\otimes\phi \mapsto a \cdot \phi)$. Now we take a non-zero $\phi \in (A^*)^{\mathrm{co}A}$, and an element $a \in A$ such that $S(a) = 0$. Recall that our module structure on A^* is given by $(a \cdot \phi)(b) = \phi(S(a)b)$, hence $a \cdot \phi = 0$, so $a\otimes\phi = 0$ and finally $a = 0$. This proves that S is injective, hence bijective. ■

Let us discuss shortly the relation between left and right invariant forms, for a f.-d. Hopf algebra A. First we note that composition by the antipode turns left invariant forms into right invariant ones, and vice-versa: indeed if ϕ is left invariant, $\psi = \phi S$ satisfies $(\psi\otimes\mathrm{id})\Delta = (S^{-1}\otimes\phi)\Delta S = S^{-1}\eta\phi S = \eta\psi$.

Then we endow A^* with the multiplication induced from the comultiplication of A, and we notice that if $\phi \in A^*$ is left invariant, then so is $\phi\psi$ for any $\psi \in A^*$. From the "unicity" of left invariant forms we obtain a unique non-zero element $d \in A$ such that $\phi\psi = \psi(d)\phi$ for all left invariant $\phi \in A^*$ and all $\psi \in A^*$. This element is called the *modular element*, or the distinguished group-like element, of A, and A is called *unimodular* if $d = 1$.

Exercise 24. Let A be a f.-d. Hopf algebra, and let $d \in A$ be the modular element of A. Show that d is a group-like element of A, i.e. $\Delta(d) = d\otimes d$. Deduce that $\epsilon(d) = 1$ and d is invertible in A with inverse $d^{-1} = S(d)$. Show that $\psi\phi = \psi(d^{-1})\phi$ for all right invariant $\phi \in A^*$ and all $\psi \in A^*$. Show that A is unimodular **iff** the spaces of left and right invariant forms coincide.

Proposition 2.19 *If there exists a* unital *left or right invariant form on A, then A is unimodular.*

PROOF. Let ϕ be a left invariant unital form on A, and d the modular element of A. For any form $\psi \in A^*$ we have, by definition of d and of the product in A^*:

$$\psi(d) = \psi(d)\phi(1) = (\phi\psi)(1) = \phi(1)\psi(1) = \psi(1)$$

hence $d = 1$. ■

2.4.2 We discuss now the relation between invariant forms and semisimplicity. We say that a Hopf algebra A is cosemisimple if any left A-comodule is a direct sum of simple comodules, i.e. of comodules without non trivial subcomodules. Equivalently, for any left A-comodule M and any $N \subset M$ such that $\delta(N) \subset A \otimes N$, there exists $P \subset M$ such that $\delta(P) \subset A \otimes P$ and $M = N \oplus P$. We will show below that for a f.-d. Hopf algebra A, cosemisimplicity of A is charaterized by the existence of a *unital* left invariant form $\phi \in A^*$.

Note that in the finite-dimensional case it is easy to see that cosemisimplicity of A is equivalent to semisimplicity of A^* — to a subcomodule $N \subset M$ one can e.g. associate a submodule $N^\circ \subset M^*$. In particular when G is a finite group, $\mathscr{F}(G)$ is cosemisimple **iff** $k[G]$ is semisimple. On the other hand, the invariant integral $\phi(f) = \sum_{g\in G} f(g)$ satisfies $\phi(1) = \#G$, hence it can be made unital **iff** $\#G$ is not a multiple of char k. Hence in this case Theorem 2.20 below shows that $k[G]$ is semisimple **iff** char k does not divide $\#G$: this is known as Maschke's Theorem.

First we outline how invariant forms are used for the next Theorem: namely obtaining invariant elements in comodules by averaging. Let indeed A be a Hopf algebra with right invariant form $\phi \in A^*$, and M be a left A-comodule. Then for any $x \in M$ the following computations shows that $\tilde{x} = (\phi\otimes\mathrm{id})\delta(x)$ is invariant:

$$\begin{aligned}\delta(\tilde{x}) &= (\phi\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\delta)\delta(x) \\ &= (\phi\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id})\delta(x) = (\eta\phi\otimes\mathrm{id})\delta(x) = 1\otimes\tilde{x}.\end{aligned}$$

Of course $\tilde{x}$ can well be the zero element of M even if $x \neq 0$ — especially if M does not have non-zero invariant vectors! In some situations one can prove the non-vanishing of $\tilde{x}$ provided the right invariant form ϕ is unital.

In the proof of the next Theorem we will use this averaging argument for the left comodule structure on $L(M)$ induced from the structure of M. More precisely, recall that in the identification $A\otimes L(M) \simeq L(M, A\otimes M)$, the comodule map $\delta : M \to A\otimes M$ corresponds to a corepresentation $u \in A\otimes L(M)$ such that $\delta(x) = u(1\otimes x)$ for $x \in M$. Moreover if A is finite-dimensional we know that S is bijective, and hence we can consider $u^{-1} = (S^{-1}\otimes\mathrm{id})(u)$ which is indeed the inverse of u, see Exercise 17.

Then for any $f \in L(M)$ we define the element $\delta(f) \in A\otimes L(M)$ by putting $\delta(f) = u(1\otimes f)u^{-1}$. This is clearly a unital linear map — even an algebra morphism — and we have

$$\begin{aligned}(\mathrm{id}\otimes\delta)\delta(f) &= u_{23}u_{13}(1\otimes 1\otimes f)u_{13}^{-1}u_{23}^{-1} \\ &= (\Delta\otimes\mathrm{id})(u)(1\otimes 1\otimes f)(\Delta\otimes\mathrm{id})(u)^{-1} \\ &= (\Delta\otimes\mathrm{id})(u(1\otimes f)u^{-1}) = (\Delta\otimes\mathrm{id})\delta(f),\end{aligned}$$

hence we have defined a left A-comodule structure on $L(M)$, which clearly satisfies $\delta(f)\delta(x) = \delta(f(x))$ for all $f \in L(M)$, $x \in M$.

Theorem 2.20 *Let A be a finite-dimensional Hopf algebra. Then A is cosemisimple* ***iff*** *it admits a unital right invariant form.*

PROOF. First assume that A admits a unital right invariant form $\phi \in A^*$. Let M be a f.-d. left A-comodule, and $N \subset M$ a subcomodule. Let $p \in L(M)$ be any projection onto N. We consider the element $\tilde{p}$ averaged as above with respect to ϕ and the left comodule structure on $L(M)$.

For any $y \in N$ we have $\delta(y) \in A \otimes N$ hence $u(1 \otimes p) = (1 \otimes p)u(1 \otimes p)$. As a result we have

$$\tilde{p} = (\phi \otimes \mathrm{id})[(1 \otimes p)u(1 \otimes p)u^{-1}] = p\tilde{p}.$$

This proves that $\mathrm{Im}\,\tilde{p} \subset N$. Now we prove that $\tilde{p}$ is a projection onto N, using the fact that ϕ is unital. Applying $(S^{-1} \otimes \mathrm{id})$ to the identity above we obtain $u^{-1}(1 \otimes p) = (1 \otimes p)u^{-1}(1 \otimes p)$, hence we can write

$$\tilde{p}p = (\phi \otimes \mathrm{id})[u(1 \otimes p)u^{-1}(1 \otimes p)] = (\phi \otimes \mathrm{id})[uu^{-1}(1 \otimes p)] = \phi(1)p = p,$$

so that $\tilde{p}$ restricts to the identity on N.

Thus we have a found an invariant projection $\tilde{p} \in L(M)$ such that $\mathrm{Im}\,p = N$. Putting $P = \mathrm{Ker}\,p$ we have $M = N \oplus P$, and P is a subcomodule because p is invariant: for $x \in P$, we have $(\mathrm{id} \otimes p)\delta(x) = \delta(p)\delta(x) = \delta(px) = 0$ hence $\delta(P) \subset A \otimes P$.

For the reverse implication we use the left comodule A itself, with structure map given by the comultiplication Δ. Since Δ is unital, $k1_A$ is a subcomodule of A and hence there is a subcomodule $P \subset A$ such that $k1_A \oplus P = A$. Now we can define a unital linear form $\psi \in A^*$ by putting $\psi(1) = 1$ and $\psi(P) = \{0\}$. We claim that ψ is left invariant: indeed $(\mathrm{id} \otimes \psi)\Delta(1) = 1 \otimes \psi(1) = \eta\psi(1)$ on the one hand, and $(\mathrm{id} \otimes \psi)\Delta(P) \subset A \otimes \psi(P) = \{0\} = \eta\psi(P)$. Then $\phi = \psi S$ is right invariant and unital. ■

REMARK 2.21 Using the expressions $u^{-1} = (S^{-1} \otimes \mathrm{id})(u)$ and $\phi = \psi S$ where ψ is a left invariant form, one can compute the following formula for $\tilde{p}$:

$$\tilde{p}(x) = (\psi \otimes \mathrm{id})(m \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(\mathrm{id} \otimes \delta)(\mathrm{id} \otimes p)\delta(x).$$

It is possible to check by hand, using this formula, that $\tilde{p}$ is a projection onto N, and that it is an endomorphism of the comodule M. The advantage is that the proof of Theorem 2.20 becomes in this way independant from Therem 2.17 — we don't use u^{-1} and the inverse of S anymore — and works even for infinite-dimensional Hopf algebras (and comodules). □

EXAMPLE 2.22 Consider the case of H_4, with char $k \neq 2$. It is easy to check that the forms ϕ, ψ below are respectively left and right invariant:

$$\phi(1) = \phi(g) = \phi(x) = 0, \quad \phi(gx) = 1,$$
$$\psi(1) = \psi(g) = \psi(gx) = 0, \quad \psi(x) = 1.$$

As a result H_4 is not unimodular and does not admit unital invariant forms. Hence H_4 is not cosemisimple, and by selfduality, not semisimple — this can also be checked directly. □

3 Compact quantum groups

In this section we will switch from the algebraic setting of the previous section to a topological setting. The most easy topological groups are compact groups, whose representation theory e.g. is almost as simple as the one of finite groups. One can "map" compact groups G to the category of (infinite-dimensional) Hopf algebras via the algebras of representative functions $R(G)$. But there is no easy way to recognize such algebras inside the category, and the whole category of Hopf algebras is certainly too big to be a quantum analogue of the category of compact groups.

Moreover one of our motivation in the case of infinite groups is to consider questions of functional analytical nature. To this respect, the correct way to "replace" a compact space X with an algebra is to consider the algebra $C(X)$ of continuous functions on X endowed with a natural norm that allows to reconstruct X and its topology. This leads to the notion of C^*-algebra. It turns out that this framework allows to give an axiomatic definition of "compact quantum groups" where classical quantum groups correspond to commutative algebras, and which present many features analogous to the classical ones.

Of course, a theory of compact quantum groups cannot be self-dual, since duals of abelian compact groups are (possibly infinite) discrete groups. On the other hand, it turns out that "duals" of all discrete groups Γ will be included in the category of compact quantum groups, via some normed versions $C^*(\Gamma)$ of the group algebras $\mathbb{C}[\Gamma]$. Hence the theory of compact quantum groups can be dualy considered as a theory of discrete quantum groups. Discrete groups form a much wilder world than compact groups and the analytical properties of the algebras $C^*(\Gamma)$ are in particular very interesting.

3.1 About C^*-algebras

From now on the base field will be $k = \mathbb{C}$, the field of complex numbers. C^*-algebras are the topological algebras that will fit our purposes. Let us start with the definition, starting with some natural compatibilities between multiplications, involutions and norms.

Recall that a $\mathbb{C}$-algebra A is called involutive if it is equipped with an involutive, antilinear, antimultiplicative map $(a \mapsto a^*)$. In other words, we have $(a^*)^* = a$, $(\lambda a)^* = \bar{\lambda} a^*$, $(ab)^* = b^* a^*$ for all a, $b \in A$, $\lambda \in \mathbb{C}$.

An normed algebra is an algebra A equipped with a norm $\| \cdot \|$ such that $\|ab\| \leq \|a\| \|b\|$ for all a, $b \in A$. A normed involutive algebra is a normed algebra and an involutive algebra such that $\|a^*\| = \|a\|$ for all $a \in A$.

To obtain the notion of a C^*-algebra we add a simple but essential compatibility condition between the algebraic and the analytical structures.

Definition 3.1 *A C^*-algebra is a normed involutive algebra which is complete as a normed vector space, and such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.*

Examples 3.2 Let X be a locally compact space. Then the algebra $A = C_0(X)$ of continuous functions vanishing at infinity is a C^*-algebra with respect to pointwise involution and the supremum norm $\|f\| = \sup_{x \in X} |f(x)|$. It is commutative, and unital **iff** X is compact.

Let H be a Hilbert space. The space $B(H)$ of bounded linear maps from H to itself is equipped with the involution given by adjoint operators, and with the operator norm $\|T\| = \sup\{\|Tx\|_H \mid \|x\|_H \leq 1\}$. Then any norm closed $*$-subalgebra of $B(H)$ is a C^*-algebra with respect to the operator norm. □

We give now two of the culminating results of the general theory, which may give some indication on the kind of objects we are considering since they show that the examples above are universal.

Theorem 3.3 *(Gelfand-Naimark) Let A be a commutative C^*-algebra. Then there exists a locally compact space X and a $*$-isomorphism $A \simeq C_0(X)$.*

The proof of this theorem relies on the study of the notion of spectrum for C^*-algebras and their elements. Note also the following important feature of the theory, which shows that algebraic and analytical properties of C^*-algebras are tightly related: $*$-algebra morphisms between C^*-algebras are automatically contracting, hence continuous, and we call them C^*-morphisms. Moreover, injective $*$-algebra morphisms are automatically isometric, and in particular the $*$-isomorphism in the Theorem above is also an isomorphism of normed algebras.

Theorem 3.4 *Let A by any C^*-algebra. Then there exists a Hilbert space H, a norm closed $*$-subalgebra $B \subset B(H)$ and a C^*-isomorphism $A \simeq B$.*

The proof of this theorem relies notably on the study of the notion of positivity in C^*-algebras. An element a in a C^*-algebra A is called positive if it can be written $a = b^*b$. One can show that such elements form a salient convex cone $A_+ \subset A$, hence they define an ordering of A.

A linear form $\phi \in A^*$ is called positive if it maps positive elements to positive numbers — and then we have $\phi(a^*) = \overline{\phi(a)}$. One can show that positive linear forms are automatically continuous, with $\|\phi\| = \phi(1)$ if A is unital. A state on A is a positive linear form ϕ on A such that $\|\phi\| = 1$. One corollary of the previous theorem is that the states of A separate the points of A.

When $A = C_0(X)$ it is clear that positive elements of A are exactly functions in $C_0(X)$ which take only non-negative values, and positive linear forms correspond to positive Radon measures on X. When $A = B(H)$ for some Hilbert space H, one recovers the notion of positive operators.

As in the algebraic setting, we will need to make tensor products of C^*-algebras. This construction is not as easy as in the algebraic setting: the algebraic tensor product $A \otimes B$ of two C^*-algebras must be completed so as to yield a C^*-algebra, and there are many ways to do this in general.

Here we will only need the minimal, or spatial tensor product of C^*-algebra, which is easy to describe. First we note that there is an easy notion for the tensor product of two Hilbert spaces H, K: indeed the formula $(\zeta \otimes \xi | \zeta' \otimes \xi') = (\zeta|\zeta') \times (\xi|\xi')$ defines a hermitian product on the algebraic tensor product of H and K, and we denote by $H \otimes K$ its completion with respect to the associated norm.

Given two elements $a \in B(H)$, $b \in B(K)$ we have then a well-defined linear map $a \otimes b \in L(H \otimes K)$ which can easily be shown to be bounded. As a result if $A \subset B(H)$, $B \subset B(K)$ are sub-C^*-algebras, we obtain a natural $*$-subalgebra of $B(H \otimes K)$ spanned by elements $a \otimes b$, and which identifies with the algebraic tensor product of A and B. The minimal, or spatial tensor product of A and B, which we will simply denote $A \otimes B$, is defined to by the norm closure of this $*$-subalgebra. In the case $A = C_0(X)$, $B = C_0(Y)$ one can show that $A \otimes B = C_0(X \times Y)$ where elementary tensors are given by $(f \otimes g)(x, y) = f(x)g(y)$.

If $\phi : A \to A'$, $\psi : B \to B'$ are C^*-morphisms, one can also show that the map $\phi \otimes \psi$ defined on the algebraic tensor product extends to a C^*-morphism $\phi \otimes \psi : A \otimes B \to A' \otimes B'$. One can replace ϕ or ψ with a positive linear form, and the map obtained is again continuous.

3.2 Woronowicz C^*-algebras

Motivated by the algebraic Hopf structure on $\mathscr{F}(G)$ and by Gelfand-Naimark's theorem, we define the "space of continuous functions" on a compact quantum group as follows. Note that the coproduct takes its values in the completed tensor product $A \otimes A$: in general $\Delta(a)$ is not a finite sum of elementary tensors, only a limit of such sums.

Definition 3.5 *A Woronowicz C^*-algebra is a pair (A, Δ) where A is a unital C^*-algebra, and $\Delta : A \to A \otimes A$ is a C^*-morphism such that*

1. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
2. $\Delta(A)(1 \otimes A)$ *and* $\Delta(A)(A \otimes 1)$ *span dense subspaces of* $A \otimes A$.

This Definition might seem surprising since it does not include the existence of a counit, nor of an antipode, which are replaced by a somewhat obscure topological condition. Let us consider the commutative case:

Example 3.6 Let X be a compact space equipped with a continuous associative product $X \times X \to X$, and put $A = C(X)$. For each $f \in A$ we put $\Delta(f) = ((x, y) \mapsto f(xy))$, and this defines a C^*-morphism $\Delta : A \to A \otimes A$ satisfying the coassociativity condition of the definition.

On the other hand, since A is commutative $\Delta(A)(A \otimes 1)$ is clearly a $*$-subalgebra of $C(X \otimes X)$, hence by Stone-Weiestrass Theorem it is dense **iff** it separates the points. Let us check that this is equivalent to the left cancellativity of the product.

Assume first that the functions $((x, y) \mapsto f(xy)g(x))$ separate the points of $X \otimes X$. If x, y, $z \in X$ are such that $xy = xz$, then we have $f(xy)g(x) = f(xz)g(x)$ for all f, $g \in C(X)$ hence $(x, y) = (x, z)$ and in particular $y = z$. Conversely, assume that $f(xy)g(x) = f(x'y')g(x')$ for all f, g: then $x = x'$ by taking $f = 1$, and $xy = x'y' = xy'$. Hence $y = y'$ by left cancellativity, and $(x, y) = (x', y')$.

This shows that the Definition above includes bicancellative compact monoids as a special case. Now the following Lemma explains why we consider Woronowicz C^*-algebras as "algebras of functions" on compact quantum groups: □

Lemma 3.7 *Let X be a compact monoid. If X is bicancellative, then X is a group.*

Proof. Assume we have found two elements x, $e \in X$ such that $xe = x$. Multiplying on the right by any $y \in X$ and cancelling x we obtain $ey = y$. Proceeding similarly on the left we obtain $ze = z$ for all $z \in X$, hence e is a (necessarily unique) unit element in X. Similarly one proves that if x has a right inverse in X then it is invertible.

If X is finite, for any $x \in X$ there must exist $q > p \geq 1$ such that $x^q = x^p$, hence $e = x^{q-p}$ is a unit element in X, and x^{q-p-1} is an inverse of x if $x \neq e$.

In the general case, we fix $x \in X$ and we denote Y the closed set of limit points of the sequence (x^n). Let $y = \lim x^{m_\alpha}$, $z = \lim x^{n_\alpha}$ be two elements of Y, by taking a subnet of (n_α) if necessary one can assume that $p_\alpha = n_\alpha - m_\alpha \to \infty$, and by compactness and taking again subnets one can assume that $x^{p_\alpha} \to t$. We have then $yt = z$: this proves that $Y \subset yY$. The reverse inclusion is obvious, and in particular we can find $e \in Y$ such that $ye = y$, it is then a unit element for X. Finally, we have $xY = Y$, so we can find $x^{-1} \in Y$ such that $xx^{-1} = e$. ■

Theorem 3.8 *Let A be a commutative Woronowicz C^*-algebra. Then there exists a compact group G such that $A \simeq C(G)$ and Δ is the usual coproduct on $C(G)$.*

Proof. By Gelfand-Naimark's Theorem, we know that $A \simeq C(X)$ for some compact space X. More precisely X is the set of multiplicative and involutive forms $x : A \to \mathbb{C}$, endowed with the trace of the weak-$*$ topology, and the image of $a \in A$ in $C(X)$ is $(x \mapsto x(a))$.

Now for x, $y \in X$ we denote $xy = (x \otimes y)\Delta$. It is easy to check that this product on X is continuous and associative. Finally the considerations at Example 3.6 show that X is bicancellative, and Lemma 3.7 shows that X is a group. ■

Example 3.9 Let Γ be group. We endow the group algebra $\mathbb{C}[\Gamma]$ with the involution defined as the antilinear extension of the inversion map.

Consider the regular representation $\lambda : \mathbb{C}[\Gamma] \to B(\ell^2\Gamma)$ where the elements of Γ act by left translations, $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$. The reduced C^*-algebra of Γ is the completion $C^*_{\text{red}}(\Gamma)$ of $\mathbb{C}[\Gamma]$ with respect to the norm $\|a\|_{\text{red}} = \|\lambda(a)\|$. Since λ is faithful, this is indeed a norm and not a semi-norm, and it satisfies the C^*-condition because λ is a $*$-morphism and operator noms satisfy it.

The maximal C^*-algebra of Γ is the completion $C^*(\Gamma)$ of $\mathbb{C}[\Gamma]$ with respect to the norm $\|a\| = \sup \|\pi(a)\|$, where the supremum is taken over all $*$-representations $\pi : \mathbb{C}[\Gamma] \to B(H)$ with H a Hilbert space. This is again a C^*-norm wich dominates the reduced one, so that there is a canonical quotient map $C^*(\Gamma) \to C^*_{\text{red}}(\Gamma)$.

The coproduct $\Delta : g \to g\otimes g$ of $\mathbb{C}[\Gamma]$ extends by continuity to a C^*-morphism $\Delta : C^*(\Gamma) \to C^*(\Gamma)\otimes C^*(\Gamma)$. Indeed, for $a \in \mathbb{C}[\Gamma]$ the norm of $\Delta(a)$ in $C^*(\Gamma)\otimes C^*(\Gamma)$ is by definition $\|(\pi\otimes\pi)\Delta(a)\|$ where $\pi : C^*(\Gamma) \to B(H)$ is a faithful representation. Now $(\pi\otimes\pi)\Delta$ defines a $*$-representation of $\mathbb{C}[\Gamma]$, so this norm is not bigger than the one of a in $C^*(\Gamma)$.

The map Δ is clearly coassociative: it is enough to check it on the dense subspace $\mathbb{C}[\Gamma]$. Moreover we can write $g\otimes h = \Delta(g)(1\otimes g^{-1}h) = \Delta(h)(h^{-1}g\otimes 1)$ hence the dense subspace $\mathbb{C}[\Gamma]\otimes\mathbb{C}[\Gamma]$ is contained in $\Delta(C^*(\Gamma))$ $(1\otimes C^*(\Gamma))$ and $\Delta(C^*(\Gamma))(C^*(\Gamma)\otimes 1)$. As a result, $C^*(\Gamma)$ is a Woronowicz C^*-algebra.

The same arguments will show that $C^*_{\text{red}}(\Gamma)$ is a Woronowicz C^*-algebra, provided we can check that the coproduct Δ of $\mathbb{C}[\Gamma]$ is continuous with respect to the norm of $C^*_{\text{red}}(\Gamma)$. In other words, we have to show that $\|(\lambda\otimes\lambda)\Delta(a)\| \le \|\lambda(a)\|$ for all $a \in \mathbb{C}[\Gamma]$. This is true due to the following exercise: □

Exercise 25. Denote by $\lambda : \Gamma \to B(H) = B(\ell^2\Gamma)$ the regular representation of Γ, and let $\pi : \Gamma \to B(K)$ be any unitary representation. Find a unitary $W \in B(H\otimes K)$ such that $W(\lambda(g)\otimes\pi(g))W^* = (\lambda(g)\otimes\text{id})$ for all $g \in \Gamma$.

Note that the Woronowicz C^*-algebras of the preceeding example are cocommutative. This example is “almost” universal, as shown by the following Theorem — that can only be proved after the properties of corepresentations of Woronowicz C^*-algebras have been established.

Theorem 3.10 *Let A be a cocommutative Woronowicz C^*-algebra. Then there exists a group Γ and a faithful unitary representation $\pi : \Gamma \to B(H)$ weakly containing $\pi\otimes\pi$ such that A is isomorphic to the norm closure of $\pi(\mathbb{C}[\Gamma])$ in $B(H)$ and Δ restricts to the coproduct of $\mathbb{C}[\Gamma]$.*

3.3 Examples: the matrix case

3.3.1 Definition 3.5 is very compact and allows for a rapid proof of the existence of the Haar state, as we will see in Section 3.4. However the construction of examples, and in particular the verification of the density condition, is not straightforward. In this section we define “compact quantum groups embedded in a unitary group of matrices”, show that they are particular cases of compact quantum groups, and use them to build examples.

Definition 3.11 *Let A be a C^*-algebra and $u \in M_n(A) \simeq M_n(\mathbb{C})\otimes A$ a matrix with coefficients in A. Denote by $\mathscr{A}$ the unital $*$-subalgebra of A generated by the coefficients of u. We say that (A, u) is a matrix Woronowicz C^*-algebra if $\mathscr{A}$ is dense in A, u is invertible in $M_n(A)$ as well as $\bar{u} = (u^*_{ij})$, and there exist a unital C^*-morphism $\Delta : A \to A\otimes A$ such that $(\text{id}\otimes\Delta)(u) = u_{12}u_{13}$.*

We want to show that matrix Woronowicz C^*-algebras are C^*-Woronowicz algebras. For this we will use the language of corepresentations. In fact we can use the algebraic definition for finite-dimensional comodules: we consider f.-d. vector spaces M endowed with a linear map $\delta : M \to M\otimes A$ such that

$$(\delta\otimes\text{id})\delta = (\text{id}\otimes\Delta)\delta \tag{2}$$

Since M is finite-dimensional, it suffices to consider here the algebraic tensor product $M\otimes A$ — which is already complete for any “cross norm” on it. We do not impose counitality of δ anymore since we do not have a counit at hand.

Recall that the comodule structure can also be encoded in a corepresentation $v \in L(M)\otimes A$ — again this tensor product does not need to be completed. If $v \in B(N)\otimes A$ is another f.-d. corepresentation, we can build the direct sum $u \oplus v = u + v \in L(M \oplus N)\otimes A$, the tensor product $u\otimes v = u_{13}u_{23} \in L(M\otimes N)\otimes A$ and, given an antilinear automorphism j of $B(M)$, the conjugate $u^j = (j\otimes *)(u)$. Finally, we define the space of coefficients of v to be $A_v = \{(\phi\otimes\text{id})(v) \mid \phi \in L(M)^*\} \subset A$. It is straightforward to check that

$$A_{u\oplus v} = A_u + A_v, \quad A_{u\otimes v} = \text{Span } A_uA_v, \quad A_{u^j} = A_u^*.$$

Proposition 3.12 *If (A, u) is a matrix Woronwicz C^*-algebra with coproduct Δ, then (A, Δ) is Woronowicz C^*-algebra.*

Proof. Let us first note that if v is a corepresentation of (A, Δ) on M which is invertible in $L(M) \otimes A$, then $A_v \otimes 1 \subset \Delta(A)(1 \otimes A)$: indeed we have $v \otimes 1 = (\mathrm{id} \otimes \Delta)(v) v_{13}^{-1} \in L(M) \otimes (\Delta(A)(1 \otimes A))$. Similarly we have $1 \otimes A_v \subset (A \otimes 1)\Delta(A)$.

Now we remark that the fundamental corepresentation u is by definition invertible, as well as $\bar{u}$. Let us form multiple tensor products between u, $\bar{u}$ and take direct sums. Observing that these operations preserve invertibility, we see that all elements of the subalgebra generated by $A_u + A_u^*$ are coefficients of invertible corepresentations. This algebra is nothing but $\mathscr{A}$, hence we have $\mathscr{A} \otimes 1 \subset \Delta(A)(1 \otimes A)$ by our first remark. Multiplying on the right by $1 \otimes A$, this shows that $\Delta(A)(1 \otimes A)$ contains a dense subspace.

Similarly we see that $(A \otimes 1)\Delta(A)$ is dense in $A \otimes A$, and applying $* \otimes *$ we obtain the density of $\Delta(A)(A \otimes 1)$. ■

3.3.2 Now we are ready to present some examples.

Example 3.13 Let $F \in M_n(\mathbb{C})$ be an invertible matrix. Consider the unital C^*-algebra $A_u(F)$ generated by n^2 elements u_{ij} subject to the relations that make $u = (u_{ij})$ and $\tilde{u} = F\bar{u}F^{-1}$ unitary matrices. We claim that $(A_u(F), u)$ is a matrix Woronowicz C^*-algebra.

Indeed $u_{12}u_{13} \in M_n(A \otimes A)$ is unitary as a product of unitaries, as well as the following element:

$$(F \otimes 1 \otimes 1) u_{12} u_{13} (F^{-1} \otimes 1 \otimes 1) = \tilde{u}_{12} \tilde{u}_{13}.$$

Hence there exists $\Delta : A \to A \otimes A$ mapping u to $u_{12}u_{13}$, and the remaining conditions of Definition 3.11 are clear.

Similarly, one defines the unital C^*-algebra $A_o(F)$ generated by n^2 generators u_{ij} subject to the relations that make $u = (u_{ij})$ unitary and $F\bar{u}F^{-1}$ equal to u. The same arguments as above show that $(A_o(F), u)$ is a matrix Woronowicz C^*-algebra.

It turns out that these examples are "universal" in the following sense. If a Woronowicz C^*-algebra A admits a finite-dimensional corepresentation v whose coefficients generate A, then there exists an invertible matrix F and a surjective $*$-morphism $\phi : A_u(F) \to A$ such that $(\phi \otimes \phi)\Delta = \Delta\phi$. As a matter of fact, one can show, using the Haar state, that any corepresentation $v \in L(M) \otimes A$ can be made unitary by a suitable choice of hermitian structure on M. Note that there does not exist in general such a structure making v and v^j simultaneously unitary. Similarly, A is a quotient of some $A_o(F)$ if v is equivalent to v^j for some j. □

Exercise 26. Recall that a projection p in a C^*-algebra is an element p such that $p^2 = p = p^*$. A partition of the unit in a C^*-algebra is a finite family of projections p_i such that $\sum p_i = 1$. Show that we have then $p_i p_j = 0$ for $i \neq j$.

Consider the unital C^*-algebra $A_s(n)$ generated by n^2 projections u_{ij} forming a unitary matrix. Write the relations between u_{ij}'s corresponding to unitarity of $u = (u_{ij})$. Show that $(A_s(n), u)$ is a matrix Woronowicz C^*-algebra.

The corresponding compact quantum group is called the quantum permutation group. Show that $\delta : \mathbb{C}^n \to \mathbb{C}^n \otimes A_s(n)$, $e_i \mapsto \sum e_j \otimes u_{ji}$ is a coaction, i.e. a $*$-algebra morphism such that $(\delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \Delta)\delta$.

Example 3.14 Before introducing the compact quantum group $SU_q(2)$, we start with some motivation and we remark that for $\phi \in L(\mathbb{C}^2)$ we have $(\phi \otimes \phi)\xi_1 = (\det \phi)\xi_1$, where $\xi_1 = (e_1 \otimes e_2) - (e_2 \otimes e_1) \in \mathbb{C}^2$. Now we fix $q \in [-1, 1] \setminus \{0\}$ and put $\xi_q = (e_1 \otimes e_2) - q(e_2 \otimes e_1)$.

We consider the C^*-algebra $A = C_q(SU(2))$ generated by the entries of a matrix $u \in M_2(A)$ with the relations corresponding to unitarity of u and the identity $u_{13}u_{23}(\xi_q \otimes 1) = (\xi_q \otimes 1)$. For a unitary matrix, this identity is equivalent to $u_{23}(\xi_q \otimes 1) = u_{13}^*(\xi_q \otimes 1)$, which is again equivalent to having the matrix u of the form

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

The relations corresponding to unitarity of u can then be made explicit, and one sees that A can also be presented as the C^*-algebra generated by two elements α, γ subject to the relations

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1,$$
$$\gamma\gamma^* = \gamma^*\gamma, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.$$

Finally, one proves that (A, u) is a matrix Woronowicz C^*-algebra. Indeed $u_{12}u_{13}$ is clearly unitary, and it is easy to check that it satisfies the twisted unimodularity condition:

$$u_{13}u_{14}u_{23}u_{24}(\xi_q \otimes 1 \otimes 1) = u_{13}u_{23}(\xi_q \otimes 1 \otimes 1) = (\xi_q \otimes 1 \otimes 1).$$

Besides, u is invertible because it is unitary, and using the relations above one checks that the following matrix is the inverse of $\bar{u}$:

$$\bar{u}^{-1} = \begin{pmatrix} \alpha & q^2\gamma \\ -q^{-1}\gamma^* & \alpha^* \end{pmatrix}.$$

□

3.4 The Haar measure

3.4.1 Let A be a Woronowicz C^*-algebra. A state $h \in A^*$ is called left-invariant if $(\mathrm{id} \otimes h)\Delta = 1h$, right-invariant if $(h \otimes \mathrm{id})\Delta = 1h$. In this section we prove the existence of a unique left-invariant state and of a unique right-invariant state on A, which coincide. We call this invariant state the Haar state of A. The proof is of analytical nature, hence quite different from the one of Theorem 2.17. When $A = C(G)$ is commutative, we recover the Haar measure on G as a Radon measure.

Let us introduce some notation. As in the algebraic setting, the coproduct of A induces an associative product on the set A_+^* of positive linear forms on A, as follows: $\phi\psi = (\phi \otimes \psi)\Delta$. We will moreover denote, for $\phi \in A_+^*$, $a \in A$: $a * \phi = (\phi \otimes \mathrm{id})\Delta(a)$ and $\phi * a = (\mathrm{id} \otimes \phi)\delta(a)$, so that $\psi(\phi * a) = (\psi\phi)(a)$. In this way left-invariance (resp. right-invariance) of h means that $h * a = h(a)1$ (resp. $a * h = h(a)1$) for all $a \in A$, and also that $\phi h = \phi(1)h$ (resp. $h\phi = \phi(1)h$) for all $\phi \in A_+^*$.

In the proof of the following Lemma we will use the Cauchy-Schwartz inequality for positive forms on C^*-algebras. If ϕ is such a form, we can define a positive hermitian form on A by putting $(a|b)_\phi = \phi(a^*b)$. The associated semi-norm is given by $\|a\|_\phi^2 = \phi(a^*a)$, and Cauchy-Schwartz inequality reads $|\phi(a^*b)|^2 \le \phi(a^*a)\phi(b^*b)$. In particular if $\phi(a^*a) = 0$, then $\phi(a^*b) = 0$ for all $b \in A$.

Lemma 3.15 *Let A be a Woronowicz C^*-algebra. Let h be a state of A, and ϕ, $\psi \in A_+^*$.*

1. *We have $\phi h = \phi(1)h$ **iff** $(h \otimes \phi)((\Delta(a_h) - a_h \otimes 1)^*(\Delta(a_h) - a_h \otimes 1)) = 0$ for all $a \in A$, where we denote $a_h = h * a$.*
2. *If $\phi h = \phi(1)h$ and $\psi \le \phi$, then $\psi h = \psi(1)h$.*

PROOF. 1. For the implication $\Rightarrow$, we develop the quantity considered and compute. We have e.g.

$$(h \otimes \phi)(\Delta(a_h)^*(a_h \otimes 1)) = (h \otimes \phi \otimes h)(\mathrm{id} \otimes \Delta)(\Delta(a)^*(a_h \otimes 1))$$
$$= \phi(1)(h \otimes h)(\Delta(a)^*(a_h \otimes 1)) = \phi(1)h(a_h^* a_h).$$

We find the same value for the three other terms without their signs, and the sum vanishes.

For the implication $\Leftarrow$ we have to prove that $(\phi\otimes h)\Delta(x) = \phi(1)h(x)$ for all $x \in A$. Since $\Delta(A)(A\otimes 1)$ is dense in $A\otimes A$, $(h\otimes \mathrm{id})(\Delta(A)(A\otimes 1))$ is dense in A hence it suffices to prove that $(h\otimes\phi\otimes h)(\Delta^2(a)(b\otimes 1\otimes 1)) = \phi(1)(h\otimes h)(\Delta(a)(b\otimes 1))$ for all $a, b \in A$.

Now the hypothesis implies by Cauchy-Schwartz' inequality $(h\otimes\phi)\ ((\Delta(a_h) - a_h\otimes 1)^*(b\otimes 1)) = 0$ for all $a, b \in A$. Using the definition of a_h we get exactly the previous identity for a^* and b.

2. Put $b = \Delta(a_h) - a_h\otimes 1$. The element b^*b is positive, hence if $0 \le \psi \le \phi$ we have $0 \le (h\otimes\psi)(b^*b) \le (h\otimes\phi)(b^*b)$. Thus the implication follows from the first point. ■

Exercise 27. Rewrite the statement and proof of the Lemma in the case when $A = C(G)$ with G a compact group, using integrals on G.

The proof of the next Theorem relies on Banach-Alaoglu's Theorem that the unit ball in the dual space of a normed vector space is compact with respect to the weak-$*$ topology. Recall that a net ϕ_α in the space of continuous forms on A converges to ψ in the weak-$*$ topology **iff** $\phi_\alpha(a) \to \psi(a)$ for all $a \in A$. In particular it is clear that the subset of states is closed in this unit ball, hence also weak-$*$ compact.

Theorem 3.16 *Let A be a Woronowicz C^*-algebra. Then there exists a unique left (resp. right) invariant state on A, and it is also right invariant.*

Proof. Let ϕ be a non-zero positive state on A. For any n, the form $\phi_n = \frac{1}{n}\sum_{k=1}^n \phi^k$ is positive with norm $\phi_n(1) = 1$. Hence the sequence $\{\phi_n\}$ admits a limit point h_ϕ with respect to the weak-$*$ topology, which is a state on A. We have $\phi\phi_n - \phi_n = \frac{1}{n}(\phi^{n+1} - \phi)$ hence $\phi h_\phi = h_\phi$.

Now for any positive form ϕ on A, the preceeding paragraph shows that $K_\phi = \{h \text{ state on } A \mid \phi h = \phi(1)h\}$ is non-empty, and it is clearly weak-$*$ closed. For any positive ϕ, ψ we have $\phi, \psi \le \phi + \psi$ hence $K_{\phi+\psi} \subset K_\phi \cap K_\psi$ by Lemma 3.15. By weak-$*$ compactness of the state space, we can find a state h belonging to all K_ϕ, i.e. h is left invariant.

Now we can proceed exactly in the same way on the right and we obtain a right invariant state h'. For any such state h' we have $h = h'(1)h = h'h = h(1)h' = h'$. This shows the existence and unicity of h. ■

3.4.2 Complements. In general the Haar state h on a Woronowicz C^*-algebra A is not a trace, i.e. we do not have $h(ab) = h(ba)$ for $a, b \in A$. However, there is a way to describe this lack of commutation inside h in terms of a "modular group" σ_t. This one-parameter group will be defined only on a dense $*$-subalgebra $\mathscr{A} \subset A$, which is interesting on its own: in fact $\mathscr{A}$ turns out to be a "real" Hopf algebra inside A, thus yielding a strong connection between the topological and algebraic theories. Notice that the Hopf algebras $\mathscr{A}$ appearing in this way are always semisimple and cosemisimple.

We introduce $\mathscr{A} \subset A$ as the $*$-subalgebra formed by coefficients of all invertible finite-dimensional corepresentations. In the case of a matrix Woronowicz C^*-algebra, we have seen that $\mathscr{A}$ is dense in A. In the general case, this is still true, but to prove it one has to use the regular corepresentation of A, and we did not discuss infinite-dimensional corepresentations in the analytical setting.

It turns out that this dense $*$-subalgebra has a very rich structure: in fact it is a Hopf algebra as in Section 2. As a matter of fact it is clear that Δ restricts to an (algebraic) coproduct on $\mathscr{A}$ since we have $(\mathrm{id}\otimes\Delta)(v) = v_{12}v_{13}$ for any invertible f.-d. corepresentation v, and $v \in L(M)\otimes A$ is a finite sum of elementary tensors.

Then one wants to define a counit $\epsilon : \mathscr{A} \to \mathbb{C}$ and an antipode $S : \mathscr{A} \to \mathscr{A}$ by prescribing for all v's the following identities:

$$(\mathrm{id}\otimes\epsilon)(v) = \mathrm{id}_M \quad \text{and} \quad (\mathrm{id}\otimes S)(v) = v^{-1}.$$

If this can be done, then it is easy to see that ϵ, S are indeed a counit and antipode, by considering e.g. the following kind of computation already encoutered in Section 2.3:

$$\begin{aligned}(\mathrm{id}\otimes\eta\epsilon)(v) &= \mathrm{id}\otimes 1 = v(\mathrm{id}\otimes S)(v) = (\mathrm{id}\otimes m)(\mathrm{id}\otimes\mathrm{id}\otimes S)(v_{12}v_{13}) \\ &= (\mathrm{id}\otimes m)(\mathrm{id}\otimes\mathrm{id}\otimes S)(\mathrm{id}\otimes\Delta)(v).\end{aligned}$$

However to prove that the "definitions" of ϵ and S above make sense, one needs to develop the general theory of corepresentations of Woronowicz C^*-algebras.

Now this construction and the examples of Section 3.3 yield new examples of Hopf algebras. In the case when $A = C(G)$, the algebra of continuous functions on a compact group G, the dense sub-$*$-algebra $\mathscr{A}$ coincides with the algebra of representative functions $R(G)$. In the case when $A = C_q(SU(2))$, one can show that $\mathscr{A}$ is isomorphic as a Hopf algebra to (the type 1 part of) the finite dual $U_q(sl(2))^\circ$.

Then Woronowicz constructs a family of a complex one-parameter group $(z \mapsto f_z)$ of unital multiplicative linear forms $f_z \in \mathscr{A}^*$ such that $(z \mapsto f_z(a))$ is holomorphic for all $a \in \mathscr{A}$ and

1. $f_z(S(a)) = f_{-z}(a)$, $f_z(a^*) = \overline{f_{-\bar{z}}(a)}$
2. $S^2(a) = f_{-1} * a * f_1$
3. $h(ab) = h(b(f_1 * a * f_1))$

for all a, $b \in \mathscr{A}$, where we use the convolution notation introduced earlier in this section. The last relation concerning h can also be written $h(ab) = h(b\,\sigma_{-\mathrm{i}}(a))$ if we put $\sigma_z(a) = f_{\mathrm{i}z} * a * f_{\mathrm{i}z}$, and we notice that $(t \mapsto \sigma_{\mathrm{i}t})$ is a real one-parameter group of $*$-automorphisms of $\mathscr{A}$. This shows in particular that h is a so-called "KMS state" of A.

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