

NONCOMMUTATIVE FURSTENBERG BOUNDARY

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ABSTRACT. We introduce and study the notions of boundary actions and of the Furstenberg boundary of a discrete quantum group. As for classical groups, properties of boundary actions turn out to encode significant properties of the operator algebras associated with the discrete quantum group in question; for example we prove that if the action on the Furstenberg boundary is faithful, the quantum group C^* -algebra admits at most one KMS-state for the scaling automorphism group. To obtain these results we develop a version of Hamana's theory of injective envelopes for quantum group actions, and establish several facts on relative amenability for quantum subgroups. We then show that the Gromov boundary actions of free orthogonal quantum groups, as studied by Vaes and Vergnioux, are also boundary actions in our sense; we obtain this by proving that these actions admit unique stationary states. Moreover, we prove these actions are faithful, hence conclude a new unique KMS-state property in the general case, and a new proof of unique trace property when restricted to the unimodular case. We prove equivalence of simplicity of the crossed products of all boundary actions of a given discrete quantum group, and use it to obtain a new simplicity result for the crossed product of the Gromov boundary actions of free orthogonal quantum groups.

1. INTRODUCTION

The concept of boundary actions in topological dynamics was introduced by Furstenberg in 1950s. It was also Furstenberg who noted the existence of a universal boundary action for any locally compact group G ; nowadays this action, as well as the relevant space, is called the Furstenberg boundary of G . In the last few years, beginning from the article [KK17] this notion found unexpected and groundbreaking applications in the study of operator algebras associated with discrete groups, in particular leading to a resolution of an old open problem regarding the relationship between the simplicity of the group C^* -algebra and the uniqueness of its trace (see [KK17], [BKKO17], [LB17] and also the survey [Rau19]). In particular it was shown in these works that for a discrete group Γ its reduced group C^* -algebra $C_r^*(\Gamma)$ is simple if and only if the action of Γ on its Furstenberg boundary is free, and that $C_r^*(\Gamma)$ admits unique trace if and only if this action is faithful, if and only if the amenable radical of Γ is trivial.

As has been known for over twenty years now, another source of operator algebras sharing many properties with these related to discrete groups is provided by the theory of compact (equivalently, discrete) quantum groups, as initiated by Woronowicz ([Wor98]), with the quantum theory encompassing its classical counterpart. Of particular interest is the class of universal quantum groups of Van Daele and Wang ([VDW96]), which leads to operator algebras behaving in many ways as these associated with the classical free groups ([VV07]),

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but recently shown by Brannan and the last named author of this paper to be non-isomorphic to these at the von Neumann algebra level ([BV18]).

In view of the statements in the beginning of this introduction it is natural to investigate a theory of boundary actions of discrete quantum groups and aim to apply it to operator algebraic questions. This is the subject of this paper, in which we introduce the notion of quantum boundary actions, establish the existence of the Furstenberg boundary for each discrete quantum group, record several consequences of properties of the action on the Furstenberg boundary for the features of the related operator algebras, and finally show that a geometric boundary action of the free orthogonal quantum group, as constructed in [VV07], is a faithful boundary action in our sense, which has several strong consequences.

Development of the theory in the quantum context offers several technical and conceptual difficulties. The notion of the Furstenberg boundary fits naturally with Hamana’s work on injective envelopes – which was a very important observation made in [KK17] – so that it is suitable for noncommutative generalizations. This allows us to establish the existence and the uniqueness of the noncommutative Furstenberg boundary, and use it for example to characterise the amenability of the quantum group in question. One should note that the relationship between actions of a discrete quantum group Γ on a C^* -algebra A and the $\ell^1(\Gamma)$ -structures on A is subtler in comparison to the classical case.

More importantly, quantum group actions in general do not admit ‘kernels’ viewed as quantum subgroups, but rather ‘cokernels’, identified with quantum subgroups of the dual quantum group. This is a key conceptual difference, and as its consequence in the main theorems regarding C^* -simplicity or the uniqueness of the trace (in the unimodular case) we were only able to obtain one-sided implications; however these are the ones that can be used to deduce something about the operator algebras from the knowledge of boundary actions, which is our main aim.

The difference mentioned above forces us also to study various notions of relative amenability/coamenability in the quantum world, which itself is of independent interest. In particular we show that certain relations between amenability of a subgroup Λ of a discrete quantum group Γ and existence of ucp liftings with particular properties, classically rather straightforward, persist also in the quantum world, but require far more advanced proofs, for example exploiting a natural equivalence relation on irreducible representations, introduced in [Ver04] and later studied in [DCKSS18].

Another phenomenon, this time visible only in the quantum world, is that the faithfulness of the action on the Furstenberg boundary turns out to be related to the uniqueness of the KMS-state for the scaling automorphism group – which in the unimodular case is the same as the uniqueness of the trace, and hence can be viewed as a natural generalization of the unique trace property.

Finally, in comparison with the classical case, it is much more difficult to produce non-trivial examples of boundary actions, or rather to show that certain geometric boundaries, as constructed in several examples, starting from [VV07] (see also [VVV10]), are boundary actions in our sense. To address this question, we produce a criterion based on unique stationarity (the concept which was recently very successfully applied in the classical framework by Hartman and the first-named author in [HK]), and then prove that the unique stationarity indeed holds for the ‘Gromov boundary’ action of the free orthogonal quantum group, studied in [VV07]. With this result in hand we can exploit several general theorems obtained earlier

for example to provide the ‘Ozawa-type’ embeddings of the exact group C^* -algebra $C(O_N^+)$ inside a nuclear C^* -algebra contained in the injective envelope of $C(O_N^+)$.

The main results of the paper are Theorem 3.7, concerning relative amenability for discrete quantum subgroups; Theorem 4.16, establishing the existence of the noncommutative Furstenberg boundary for arbitrary discrete quantum group; Theorem 5.3 on faithfulness of the boundary action implying the existence of at most one KMS-state for the scaling automorphism group; Theorems 6.3 and 6.4 on the Ozawa type embedding of the crossed product by boundary actions and connections between simplicity of the group C^* -algebra of a discrete quantum group and boundary crossed products; and Theorems 7.2 and 7.13 on the unique stationarity and faithfulness of the Gromov boundary actions of free orthogonal quantum groups, together with their corollaries.

A detailed plan of the paper is as follows: In Section 2 we recall preliminary facts on locally compact quantum groups, establishing notation and basic conventions concerning quantum group actions, associated crossed products and quantum subgroups. We introduce there also the notions of Poisson transforms and the co-kernel for a given discrete quantum group action. Section 3 is devoted to the study of relative amenability and coamenability for the pairs quantum group-subgroup; among other things we characterise amenability of a discrete quantum subgroup via the existence of certain ucp lifts and establish the existence of the amenable radical in the quantum context. In Section 4 we define boundary actions of discrete quantum groups, prove the existence of the unique universal boundary, called the non-commutative Furstenberg boundary, and deduce its key properties. Then in Section 5 we characterise the co-kernel of the Furstenberg boundary action and prove that faithfulness of any boundary action leads to the unique trace property for a unimodular discrete quantum group, and for a non-unimodular one implies that there is no KMS-invariant state for the scaling automorphism group. In Section 6 we discuss the connections between simplicity of the reduced C^* -algebra of a discrete quantum group and that of the associated boundary crossed product. In Section 7, the most concrete and technical part of the paper, we study in detail the action of a free orthogonal quantum group on its Gromov boundary, prove that it is uniquely stationary and faithful, and deduce several consequences for the relevant operator algebras. Finally in Section 8 we list several open questions naturally arising from our work.

2. PRELIMINARIES

In this section we recall some definitions and basic results from the theory of locally compact quantum groups, and gather some preliminary results that we will use in the rest of the paper.

All scalar products will be linear on the right. The symbol \otimes will denote the tensor product of maps and minimal spatial tensor product of C^* -algebras, $\bar{\otimes}$ will denote the ultraweak tensor product of von Neumann algebras. Given two C^* -algebras A and B , a *morphism* from A to B is a $*$ -homomorphism Φ from A into the *multiplier algebra* $M(B)$ of B , which is *non-degenerate*, i.e. the set $\Phi(A)B$ of linear combinations of products of the form $\Phi(a)b$ ($a \in A$, $b \in B$) is dense in B . The set of all morphisms from A to B will be denoted by $\text{Mor}(A, B)$. The non-degeneracy of morphisms ensures that each $\Phi \in \text{Mor}(A, B)$ extends uniquely to a unital $*$ -homomorphism $M(A) \rightarrow M(B)$ which we will usually denote by the same symbol and use implicitly when composing the morphisms. On the multiplier C^* -algebras we will occasionally use apart from the norm topology also the *strict topology*. Unital completely positive will be often abbreviated to *ucp*.

For operators acting on tensor products of spaces we will use the familiar leg notation: so for example if V is a vector space and $T \in L(V^{\otimes 2})$ then, depending on which legs of the triple tensor product we want to act with T , we have the natural operators $T_{12}, T_{13}, T_{23} \in L(V^{\otimes 3})$ (the notation will be also applied in a formally more complicated case of completed tensor products). Tensor flip between spaces and algebras will be denoted by σ .

2.1. Locally compact quantum groups – basic facts. Throughout the paper symbols \mathbb{G} , \mathbb{H} will denote *locally compact quantum groups* in the sense of Kustermans and Vaes ([KV00]) – we refer the reader to the latter paper, as well as [Kus05] and [DKSS12] for detailed definitions of the objects to be introduced below. A locally compact quantum group (often simply called a quantum group in what follows) \mathbb{G} is defined in terms of a von Neumann algebra $L^\infty(\mathbb{G})$ equipped with a unital normal, coassociative $*$ -homomorphism $\Delta : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$, called the *coproduct* or *comultiplication*. The symbols φ and ψ will denote respectively *left* and *right invariant Haar weights* of \mathbb{G} , which are unique up to a positive scalar multiple, and $L^2(\mathbb{G})$ will denote the GNS Hilbert space of the left Haar weight φ (on which $L^\infty(\mathbb{G})$ acts). The Tomita-Takesaki anti-unitary conjugation associated with φ will be denoted by $J : L^2(\mathbb{G}) \rightarrow L^2(\mathbb{G})$. We will also frequently use the corresponding C^* -algebra of “continuous functions on \mathbb{G} vanishing at infinity”, $C_0(\mathbb{G}) \subset L^\infty(\mathbb{G})$. The comultiplication Δ restricts to a (still coassociative) morphism $\Delta \in \text{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$. Finally we have the universal object related to \mathbb{G} , i.e. a C^* -algebra which we will denote by $C_0^u(\mathbb{G})$, endowed with a comultiplication $\Delta^u \in \text{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{G}) \otimes C_0^u(\mathbb{G}))$, introduced and studied in [Kus01]. There is a canonical surjective *reducing morphism* $\Lambda \in \text{Mor}(C_0^u(\mathbb{G}), C_0(\mathbb{G}))$, intertwining the respective coproducts. If Λ is injective, we say that \mathbb{G} is *coamenable*; further we say that a locally compact quantum group \mathbb{G} is *amenable*, if there exists a state $m \in L^\infty(\mathbb{G})^*$ such that for every $\omega \in L^1(\mathbb{G}) := L^\infty(\mathbb{G})_*$ we have

$$m \circ ((\omega \otimes \text{id}) \circ \Delta) = m \circ ((\text{id} \otimes \omega) \circ \Delta) = \omega(\cdot) \mathbf{1}.$$

A fundamental object in the study of \mathbb{G} turns out to be the *left multiplicative unitary* $W \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$, which satisfies the pentagonal equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$ (see [KV00, Proposition 3.18]). In fact W determines \mathbb{G} completely, as we have on one hand the equality: $L^\infty(\mathbb{G}) = \{(\text{id} \otimes \omega)(W) : \omega \in B(L^2(\mathbb{G}))_*\}''$, and on the other W implements the coproduct:

$$\Delta(x) = W^*(\mathbf{1} \otimes x)W, \quad x \in L^\infty(\mathbb{G}).$$

We also have the equality $C_0(\mathbb{G}) = \overline{\{(\text{id} \otimes \omega)(W) : \omega \in B(L^2(\mathbb{G}))_*\}}^{\|\cdot\|}$.

As already mentioned above, the predual of $L^\infty(\mathbb{G})$ is denoted, by the analogy with the classical case, $L^1(\mathbb{G})$. It is a Banach algebra with respect to the convolution product given by the pre-adjoint of the comultiplication.

The multiplicative unitary W allows for a straightforward description of the *dual locally compact quantum group* of \mathbb{G} , which we will denote by $\widehat{\mathbb{G}}$. We have $L^\infty(\widehat{\mathbb{G}}) \subset B(L^2(\mathbb{G}))$ and $L^\infty(\widehat{\mathbb{G}}) = \{(\omega \otimes \text{id})(W) : \omega \in B(L^2(\mathbb{G}))_*\}''$. It turns out that the representation of $L^\infty(\widehat{\mathbb{G}})$ on $L^2(\mathbb{G})$ can be promoted to the standard form of the von Neumann algebra $L^\infty(\widehat{\mathbb{G}})$, so that we have a canonical identification $L^2(\mathbb{G}) = L^2(\widehat{\mathbb{G}})$. In what follows, when we consider more than one quantum group, we will adorn the respective symbols with the upper index describing which group we refer to: so for example another (equivalent) way of defining $\widehat{\mathbb{G}}$ would be via the equality $W^{\widehat{\mathbb{G}}} = \sigma((W^{\mathbb{G}})^*)$. Since $L^\infty(\widehat{\mathbb{G}})$ is naturally represented on $L^2(\mathbb{G})$, the same

holds for $C_0(\widehat{\mathbb{G}})$, and it can be proved that $W^{\mathbb{G}} \in M(C_0(\mathbb{G}) \otimes C_0(\widehat{\mathbb{G}})) \subset L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$. In fact W admits a universal version, $\mathbb{W} \in M(C_0^u(\mathbb{G}) \otimes C_0^u(\widehat{\mathbb{G}}))$, such that $W = (\Lambda_{\mathbb{G}} \otimes \Lambda_{\widehat{\mathbb{G}}})(\mathbb{W})$.

Occasionally we will also need the *right multiplicative unitary* $V \in L^\infty(\widehat{\mathbb{G}})' \overline{\otimes} L^\infty(\mathbb{G})$, given by the formula

$$V = (\widehat{J} \otimes \widehat{J}) W^{\widehat{\mathbb{G}}} (\widehat{J} \otimes \widehat{J})$$

where $\widehat{J} : L^2(\mathbb{G}) \rightarrow L^2(\mathbb{G})$ is the antiunitary conjugation associated with the Haar weight $\widehat{\varphi}$ of the dual quantum group $\widehat{\mathbb{G}}$. The right multiplicative unitary implements the coproduct of \mathbb{G} via the formula

$$\Delta(x) = V(x \otimes \mathbf{1})V^*, \quad x \in L^\infty(\mathbb{G}).$$

We will also sometimes denote the objects related to $\widehat{\mathbb{G}}$ simply by using hats, so for example $\widehat{\varphi}$ and $\widehat{\psi}$ denote the left and right Haar weights of $\widehat{\mathbb{G}}$, respectively. Occasionally we will use the notation $C_r^*(\mathbb{G})$ for $C_0(\widehat{\mathbb{G}})$.

Recall that a locally compact quantum group is said to be *compact* if the C^* -algebra $C_0(\mathbb{G})$ or, equivalently, $C_0^u(\mathbb{G})$, is unital, and then these C^* -algebras are denoted simply by $C(\mathbb{G})$ and $C^u(\mathbb{G})$. It is said to be *discrete* if $\widehat{\mathbb{G}}$ is compact. A compact quantum group \mathbb{G} can be fully described by its associated Hopf*-algebra $\text{Pol}(\mathbb{G})$, which is densely contained in both $C(\mathbb{G})$ and $C^u(\mathbb{G})$. A discrete quantum group \mathbb{G} is said to be unimodular if its left and right Haar weights coincide, which is not automatic in the quantum case. This is equivalent to the fact that the dual compact quantum group $\widehat{\mathbb{G}}$ is *of Kac type* – the Haar state of $\widehat{\mathbb{G}}$ is tracial.

A (*unitary*) *representation* of \mathbb{G} on a Hilbert space \mathbb{H} is a unitary $\mathcal{U} \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(\mathbb{H}))$ such that $(\Delta \otimes \text{id})(\mathcal{U}) = \mathcal{U}_{13} \mathcal{U}_{23}$. It turns out that $W \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(L^2(\mathbb{G})))$ is a unitary representation of \mathbb{G} on $L^2(\mathbb{G})$ (the quantum analog of the left regular representation). For convenience we shall also call a unitary representation its ‘right version’, that is a unitary operator $\mathcal{V} \in M(\mathcal{K}(\mathbb{H}) \otimes C_0(\mathbb{G}))$ satisfying $(\text{id} \otimes \Delta)(\mathcal{V}) = \mathcal{V}_{12} \mathcal{V}_{13}$. It turns out that the right regular representation $V \in M(\mathcal{K}(L^2(\mathbb{G})) \otimes C_0(\mathbb{G}))$ is an example of a unitary representation in the latter sense.

2.2. Actions of quantum groups on von Neumann algebras and the crossed product construction. We will now recall the rudiments of the theory of actions of locally compact quantum groups on von Neumann algebras.

Definition 2.1. *Let \mathbb{G} be a locally compact quantum group and \mathbb{N} be a von Neumann algebra. We say that an injective normal $*$ -homomorphism $\alpha : \mathbb{N} \rightarrow L^\infty(\mathbb{G}) \overline{\otimes} \mathbb{N}$ is a left action of \mathbb{G} on \mathbb{N} (or \mathbb{N} is a left \mathbb{G} -von Neumann algebra) if the action equation holds: $(\Delta \otimes \text{id}) \circ \alpha = (\text{id} \otimes \alpha) \circ \alpha$.*

Similarly we define right \mathbb{G} -von Neumann algebras. In what follows we shall sometimes write simply a \mathbb{G} -von Neumann algebra having either a right or left case in mind, depending on the context.

Given a \mathbb{G} -von Neumann algebra \mathbb{N} one defines the corresponding fixed point subalgebra $\mathbb{N}^\alpha = \{x \in \mathbb{N} \mid \alpha(x) = \mathbf{1} \otimes x\}$ and the crossed product von Neumann algebra

$$\mathbb{G} \ltimes \mathbb{N} = ((L^\infty(\widehat{\mathbb{G}}) \otimes \mathbf{1}) \alpha(\mathbb{N}))'' \subset B(L^2(\mathbb{G})) \overline{\otimes} \mathbb{N}.$$

The crossed product von Neumann algebra admits the right (dual) action of $\widehat{\mathbb{G}}$ denoted by $\widehat{\alpha} : \mathbb{G} \ltimes \mathbb{N} \rightarrow \mathbb{G} \ltimes \mathbb{N} \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$ and defined as follows:

$$\widehat{\alpha}(y) = \widehat{V}_{13} y_{12} \widehat{V}_{13}^*, \quad y \in \mathbb{G} \ltimes \mathbb{N} \subset B(L^2(\mathbb{G})) \overline{\otimes} \mathbb{N},$$

where $\widehat{V} \in L^\infty(\mathbb{G})' \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$ is the right multiplicative unitary for $\widehat{\mathbb{G}}$: $\widehat{V} = (J \otimes J)W^{\mathbb{G}}(J \otimes J)$. It is clear that

- if $y = x \otimes \mathbf{1}$ where $x \in L^\infty(\widehat{\mathbb{G}})$, then $\widehat{\alpha}(y) = \widehat{\Delta}(x)_{13}$;
- if $y = \alpha(z)$ where $z \in \mathbb{N}$ then $\widehat{\alpha}(y) = y_{12}$.

Given $\widehat{\alpha}$ one defines a normal faithful operator valued weight $T : \mathbb{G} \times \mathbb{N} \rightarrow \alpha(\mathbb{N})$, which is roughly speaking given by the formula

$$T = (\text{id} \otimes \text{id} \otimes \widehat{\varphi}) \circ \widehat{\alpha}.$$

The crossed product von Neumann algebra admits also a left \mathbb{G} -von Neumann structure $\beta : \mathbb{G} \times \mathbb{N} \rightarrow L^\infty(\mathbb{G}) \overline{\otimes} \mathbb{G} \times \mathbb{N}$ given by

$$(2.1) \quad \beta(y) = W_{12}^* y_{23} W_{12}, \quad y \in \mathbb{G} \times \mathbb{N} \subset B(L^2(\mathbb{G})) \overline{\otimes} \mathbb{N}.$$

It is clear that

- if $y = x \otimes \mathbf{1}$ where $x \in L^\infty(\widehat{\mathbb{G}})$, then $\beta(y) = (W^*(\mathbf{1} \otimes x)W)_{12}$; the left action

$$L^\infty(\widehat{\mathbb{G}}) \ni x \mapsto W^*(\mathbf{1} \otimes x)W \in L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$$

will be referred to as the adjoint action of \mathbb{G} on $L^\infty(\widehat{\mathbb{G}})$ and occasionally will also be denoted by β .

- if $y = \alpha(z)$ where $z \in \mathbb{N}$ then $\beta(y) = (\Delta \otimes \text{id})(\alpha(z)) = (\text{id} \otimes \alpha)(\alpha(z))$; identifying \mathbb{N} with the image $\alpha(\mathbb{N}) \subset \mathbb{G} \times \mathbb{N}$ we see that $\beta|_{\mathbb{N}} = \alpha$.

2.3. Quantum subgroups. Given two locally compact quantum groups \mathbb{G} and \mathbb{H} , a morphism from \mathbb{H} to \mathbb{G} is represented via a C^* -morphism $\pi \in \text{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$ intertwining the respective coproducts:

$$(\pi \otimes \pi) \circ \Delta_{\mathbb{G}} = \Delta_{\mathbb{H}} \circ \pi.$$

It can be equivalently described via a *bicharacter* from \mathbb{H} to \mathbb{G} , i.e. a unitary $U \in M(C_0(\mathbb{H}) \otimes C_0(\widehat{\mathbb{G}}))$ such that

$$\begin{aligned} (\Delta_{\mathbb{H}} \otimes \text{id}_{C_0(\widehat{\mathbb{G}})})(U) &= U_{13} U_{23}, \\ (\text{id}_{C_0(\mathbb{H})} \otimes \Delta_{\widehat{\mathbb{G}}})(U) &= U_{13} U_{12}. \end{aligned}$$

In fact $U = ((\Lambda^{\mathbb{H}} \circ \pi) \otimes \Lambda^{\widehat{\mathbb{G}}})(W^{\mathbb{G}})$. Each morphism π from \mathbb{H} to \mathbb{G} , determines uniquely a *dual morphism* $\widehat{\pi}$ from $\widehat{\mathbb{G}}$ to $\widehat{\mathbb{H}}$ such that $(\pi \otimes \text{id})(W^{\mathbb{G}}) = (\text{id} \otimes \widehat{\pi})(W^{\mathbb{H}})$. Finally note that although $\pi \in \text{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$ describing a morphism from \mathbb{H} to \mathbb{G} need not have a reduced version $\pi_r \in \text{Mor}(C_0(\mathbb{G}), C_0(\mathbb{H}))$ (such that $\pi_r \circ \Lambda^{\mathbb{G}} = \Lambda^{\mathbb{H}} \circ \pi$), if we are given $\pi_r \in \text{Mor}(C_0(\mathbb{G}), C_0(\mathbb{H}))$ intertwining the coproducts, then it always admits the universal version $\pi \in \text{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$. For more information on this equivalence and other pictures of morphisms we refer to [MSW12], [DKSS12] (the cautious reader will note that our choice of the "left" convention yields a corresponding difference in the definition of bicharacter).

Definition 2.2. *We say that a morphism $\pi \in \text{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$ from \mathbb{H} to \mathbb{G} identifies \mathbb{H} with a closed quantum subgroup of \mathbb{G} (in the sense of Vaes) if there exists an injective normal unital $*$ -homomorphism $\gamma : L^\infty(\widehat{\mathbb{H}}) \rightarrow L^\infty(\widehat{\mathbb{G}})$ such that*

$$\gamma|_{C_0(\widehat{\mathbb{H}})} \circ \Lambda_{\widehat{\mathbb{H}}} = \Lambda_{\widehat{\mathbb{G}}} \circ \widehat{\pi}.$$

Often in this case we simply say that \mathbb{H} is a closed quantum subgroup of \mathbb{G} .

The above definition is equivalent to the existence of an injective normal unital $*$ -homomorphism $\gamma : L^\infty(\widehat{\mathbb{H}}) \rightarrow L^\infty(\widehat{\mathbb{G}})$ intertwining the respective coproducts. It then follows $\pi(C_0^u(\mathbb{G})) = C_0^u(\mathbb{H})$ – if the latter condition holds, we say that the underlying quantum group morphism identifies \mathbb{H} with a closed quantum subgroup of \mathbb{G} *in the sense of Woronowicz*. These two notions are studied in detail in [DKSS12], where in particular one can find the proofs of the facts stated above.

Note that it follows from [BV05, Proposition 10.5] that there is a bijective correspondence between closed quantum subgroups of \mathbb{G} and the so-called *Baaj-Vaes* subalgebras of $L^\infty(\widehat{\mathbb{G}})$ (i.e. those von Neumann subalgebras $\mathbb{N} \subset L^\infty(\widehat{\mathbb{G}})$ for which $\Delta_{\widehat{\mathbb{G}}}(\mathbb{N}) \subset \mathbb{N} \overline{\otimes} \mathbb{N}$, $\widehat{R}(\mathbb{N}) = \mathbb{N}$ and $\widehat{\tau}_t(\mathbb{N}) = \mathbb{N}$ for all $t \in \mathbb{R}$). More precisely, if \mathbb{N} is a Baaj-Vaes subalgebra of $L^\infty(\widehat{\mathbb{G}})$ then there exists a locally compact quantum group \mathbb{H} such that $\mathbb{N} = L^\infty(\mathbb{H})$ (and the coproducts match), and more or less by definition $\widehat{\mathbb{H}}$ is a closed quantum subgroup of $\widehat{\mathbb{G}}$.

Assume that \mathbb{H} is a closed quantum subgroup of \mathbb{G} , determined by a morphism $\pi \in \text{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$. Then \mathbb{H} acts on $L^\infty(\mathbb{G})$ on the left (we will modify the language slightly and say simply that \mathbb{H} acts on \mathbb{G}) by the following formula

$$(2.2) \quad \alpha_{\mathbb{H}}(x) = U^*(\mathbf{1} \otimes x)U, \quad x \in L^\infty(\mathbb{G}),$$

where $U \in M(C_0(\mathbb{H}) \otimes C_0(\widehat{\mathbb{G}}))$ denotes the bicharacter associated to the morphism π . We then call the fixed point space of $\alpha_{\mathbb{H}}$ the *algebra of bounded functions on the quantum homogeneous space* $\mathbb{H} \backslash \mathbb{G}$ and denote it by $L^\infty(\mathbb{H} \backslash \mathbb{G})$. One can similarly define the right quotient \mathbb{G} / \mathbb{H} and actually we have $R(L^\infty(\mathbb{H} \backslash \mathbb{G})) = L^\infty(\mathbb{G} / \mathbb{H})$.

Finally, we say that a von Neumann subalgebra $\mathbb{M} \subset L^\infty(\mathbb{G})$ is \mathbb{G} -invariant if $\Delta(\mathbb{M}) \subset L^\infty(\mathbb{G}) \overline{\otimes} \mathbb{M}$. If \mathbb{H} is a closed quantum subgroup of \mathbb{G} , then $L^\infty(\mathbb{G} / \mathbb{H})$ is a \mathbb{G} -invariant von Neumann subalgebra of $L^\infty(\mathbb{G})$. On the other hand if \mathbb{H} a closed quantum subgroup of $\widehat{\mathbb{G}}$ then $L^\infty(\widehat{\mathbb{H}})$ is a Baaj-Vaes subalgebra, hence a \mathbb{G} -invariant von Neumann subalgebra of $L^\infty(\widehat{\mathbb{G}})$.

2.4. Normal quantum subgroups and quotient quantum groups. The definition of a closed normal quantum subgroup was introduced in [VV06].

Definition 2.3. *Let \mathbb{G} be a locally compact quantum group and \mathbb{K} its closed quantum subgroup identified by an injective morphism $\gamma : L^\infty(\widehat{\mathbb{K}}) \rightarrow L^\infty(\widehat{\mathbb{G}})$. We say that \mathbb{K} is a normal quantum subgroup of \mathbb{G} if $\gamma(L^\infty(\widehat{\mathbb{K}}))$ is a normal coideal in $L^\infty(\widehat{\mathbb{G}})$, that is*

$$W^*(\mathbf{1} \otimes \gamma(L^\infty(\widehat{\mathbb{K}})))W \subset L^\infty(\mathbb{G}) \overline{\otimes} \gamma(L^\infty(\widehat{\mathbb{K}})).$$

Using terminology introduced in Subsection 2.2, $\mathbb{K} \subset \mathbb{G}$ is a normal quantum subgroup of \mathbb{G} if $\gamma(L^\infty(\widehat{\mathbb{K}}))$ is preserved by the adjoint action β of \mathbb{G} on $L^\infty(\widehat{\mathbb{G}})$. The key consequence (and actually a characterization) of the fact that \mathbb{K} is a normal quantum subgroup of \mathbb{G} is that $L^\infty(\mathbb{G} / \mathbb{K})$ is a Baaj-Vaes subalgebra of $L^\infty(\mathbb{G})$, so that \mathbb{G} / \mathbb{K} becomes a locally compact quantum group, naturally called a *quotient quantum group of \mathbb{G}* . As $L^\infty(\mathbb{G} / \mathbb{K})$ inherits the full quantum group structure from $L^\infty(\mathbb{G})$, quotient quantum groups of quantum groups of Kac type are again of Kac type (we defined the Kac property only for compact quantum groups, the general definition can be found for example in [ES92]).

The above facts lead naturally to the concept of short exact sequences, studied in detail in [VV06] (see also [KS]). Denoting $\mathbb{H} = \mathbb{G} / \mathbb{K}$ we have a short exact sequence

$$(2.3) \quad \{e\} \rightarrow \mathbb{K} \rightarrow \mathbb{G} \rightarrow \mathbb{H} \rightarrow \{e\}$$

together with the dual exact sequence

$$(2.4) \quad \{e\} \rightarrow \widehat{\mathbb{H}} \rightarrow \widehat{\mathbb{G}} \rightarrow \widehat{\mathbb{K}} \rightarrow \{e\}$$

where the embedding $L^\infty(\mathbb{H}) \subset L^\infty(\mathbb{G})$, yields the identification of $\widehat{\mathbb{H}}$ with a (Vaes) closed normal subgroup of $\widehat{\mathbb{G}}$. Thus we have a bijective correspondence between normal subgroups of \mathbb{G} and $\widehat{\mathbb{G}}$ given by (2.3) and (2.4).

2.5. Discrete quantum groups and their actions. Through most of the paper we will be primarily interested in discrete quantum groups. If Γ is a discrete quantum group, we usually just write $\ell^\infty(\Gamma)$ for $L^\infty(\Gamma)$, $c_0(\Gamma)$ for $C_0(\Gamma)$ and $\ell^1(\Gamma)$ for the predual of $L^\infty(\Gamma)$. The set of equivalence classes of unitary representations of $\widehat{\Gamma}$ will be denoted by $\text{Irr}(\widehat{\Gamma})$; we will often use the fact that $\ell^\infty(\Gamma) \cong \prod_{\gamma \in \text{Irr}(\Gamma)} M_{n_\gamma}$, $c_0(\Gamma) \cong \bigoplus_{\gamma \in \text{Irr}(\Gamma)} M_{n_\gamma}$ where we use the symbols \prod , \bigoplus to denote ℓ^∞ -, resp. c_0 -direct sums of normed spaces. Discrete quantum groups are always coamenable: $c_0(\Gamma) \cong c_0^u(\Gamma)$. Moreover $\ell^\infty(\Gamma) \cong M(c_0(\Gamma))$ and the counit character extends continuously to $\ell^\infty(\Gamma)$.

Recall that a discrete quantum group Γ is amenable if and only if $\widehat{\Gamma}$ is co-amenable, i.e. the reduced and universal C^* -norms on the canonical Hopf*-algebra $\text{Pol}(\widehat{\Gamma})$ coincide.

We will now discuss the notion of C^* -algebraic actions of discrete quantum groups.

Definition 2.4. *Let Γ be a discrete quantum group, let A be a C^* -algebra, and let $\alpha \in \text{Mor}(A, c_0(\Gamma) \otimes A)$ be an injective morphism. We say that α is a (left) action of Γ on A if*

- ▶ $(\text{id} \otimes \alpha) \circ \alpha = (\Delta \otimes \text{id}) \circ \alpha$;
- ▶ $(c_0(\Gamma) \otimes \mathbb{1})\alpha(A)$ spans a dense subspace of $c_0(\Gamma) \otimes A$ (the Podleś condition holds).

In this case we say A is a Γ - C^* -algebra.

Occasionally we will need the corresponding notion of a right action.

Given an action α of Γ on a C^* -algebra A and $a \in A$, $\mu \in c_0(\Gamma)^*$ and $\nu \in A^*$ we define

- ▶ $\mu * \nu = (\mu \otimes \nu) \circ \alpha \in A^*$;
- ▶ $a * \mu = (\mu \otimes \text{id})(\alpha(a)) \in A$.

Note that for $\tilde{\mu} \in c_0(\Gamma)^*$ we have $(a * \mu) * \tilde{\mu} = a * (\mu * \tilde{\mu})$.

A linear map $\Phi : A \rightarrow B$ between two Γ - C^* -algebras A and B is said to be Γ -equivariant if for all $a \in A$ and $\mu \in A^*$ we have

$$\Phi(a * \mu) = \Phi(a) * \mu;$$

if the map Φ is itself a morphism, the above condition is equivalent to the equality $(\text{id} \otimes \Phi) \circ \alpha_A = \alpha_B \circ \Phi$.

Remark 2.5. *Note that for any C^* -algebra A we have $c_0(\Gamma) \otimes A \cong \bigoplus_{\gamma \in \text{Irr}(\widehat{\Gamma})} M_{n_\gamma}(A)$. In particular if M is a von Neumann algebra then*

$$M(c_0(\Gamma) \otimes M) \cong \prod_{\gamma \in \text{Irr}(\Gamma)} M_{n_\gamma}(M) \cong \ell^\infty(\Gamma) \overline{\otimes} M.$$

This implies that if M is a von Neumann algebra equipped with an action $\beta : N \rightarrow \ell^\infty(\Gamma) \overline{\otimes} M$ then identifying $\ell^\infty(\Gamma) \overline{\otimes} M \cong M(c_0(\Gamma) \otimes M)$ we can view β as an action of Γ on the C^ -algebra M . Indeed β is unital; for every $\omega \in \ell^1(\Gamma)$ and $x \in M$ we have $(\omega \otimes \text{id})\beta(x) \in M$ and $(\varepsilon \otimes \text{id})\beta(x) = x$. Thus we can use [BSV03, Proposition 5.8], and the fact that any discrete quantum group is regular, to see that the action $\beta : M \rightarrow M(c_0(\Gamma) \otimes M)$ satisfies the Podleś condition.*

Let Γ be a discrete quantum group and let α be an action of Γ on A . The *Poisson transform* associated to $\nu \in A^*$ is the map $\mathcal{P}_\nu : A \rightarrow \ell^\infty(\Gamma)$ defined by

$$\mathcal{P}_\nu(a) = (\text{id} \otimes \nu)\alpha(a), \quad a \in A.$$

The Poisson transforms \mathcal{P}_ν are completely bounded and strict, that is continuous (on bounded subsets) with respect to the strict topology of A and the strict topology of $\ell^\infty(\Gamma) \cong M(c_0(\Gamma))$.

Observe that for $\mu \in \ell^1(\Gamma)$, $\nu \in A^*$ and $a \in A$ we have

$$(2.5) \quad \mathcal{P}_{\mu*\nu}(a) = (\text{id} \otimes \mu)(\Delta(\mathcal{P}_\nu(a))).$$

Proposition 2.6. *The mapping $(\nu, a) \mapsto \mathcal{P}_\nu(a)$ is norm continuous (from $A^* \times A$ to $\ell^\infty(\Gamma)$). For any $\nu \in A^*$ the Poisson transform \mathcal{P}_ν is Γ -equivariant. Moreover, if $\nu \in A^*$ is a state and A is unital, the map \mathcal{P}_ν is unital and completely positive.*

Any Γ -equivariant map from a unital Γ - C^ -algebra A to $\ell^\infty(\Gamma)$ is a Poisson transform, and any ucp Γ -equivariant map from A to $\ell^\infty(\Gamma)$ is a Poisson transform associated to a state on A .*

Proof. The statements in the first paragraph are very easy to check. For the second part it suffices to note that if $\Phi : A \rightarrow \ell^\infty(\Gamma)$ is Γ -equivariant, then putting $\nu := \epsilon \circ \Phi$ we get $\mu(\mathcal{P}_\nu(a)) = \mu(\Phi(a))$ for all $\mu \in A^*$ and $a \in A$, so that $\Phi = \mathcal{P}_\nu$. \square

The following definition will play a role in the later parts of the paper.

Definition 2.7. *Let Γ be a discrete quantum group acting on a C^* -algebra A , and let $\mu \in \ell^1(\Gamma)$ be a state. A state $\omega \in S(A)$ is called μ -stationary if $\mu * \omega = \omega$; in other words*

$$\mathcal{P}_\omega = (\text{id} \otimes \mu) \circ \Delta \circ \mathcal{P}_\omega.$$

Finally we will need the notion of the co-kernel of a given action.

Definition 2.8. *Let $\alpha \in \text{Mor}(A, c_0(\Gamma) \otimes A)$ be an action of Γ on A . We define the co-kernel of α to be the von Neumann algebra*

$$N_\alpha = \{\mathcal{P}_\nu(a) : \nu \in A^*, a \in A\}'' \subset \ell^\infty(\Gamma).$$

We say that the action α is faithful if $N_\alpha = \ell^\infty(\Gamma)$.

If $\alpha \in \text{Mor}(A, c_0(\Gamma) \otimes A)$ is an action of a classical group Γ on a C^* -algebra A , then the cokernel of this action is canonically isomorphic to the algebra of bounded functions on the quotient group $\Gamma / \ker \alpha$, where we identify α with a homomorphism from Γ to $\text{Aut}(A)$.

Proposition 2.9. *Let α be an action of Γ on A . Then the co-kernel of α is a BaaJ-Vaes subalgebra of $\ell^\infty(\Gamma)$.*

Proof. Indeed, since for every $\mu \in \ell^1(\Gamma)$ and $\nu \in A^*$ we have $\mu * (\nu * a) = (\mu * \nu) * a \in N_\alpha$ and $(\nu * a) * \mu = \nu * (a * \mu) \in N_\alpha$ we conclude that $\Delta(N_\alpha) \subset N_\alpha \bar{\otimes} N_\alpha$. By the results of [NeY14] we see that N_α is a BaaJ-Vaes subalgebra. \square

Verifying that a given action is faithful may be based on a simple observation, which we formulate as a proposition for the ease of reference.

Proposition 2.10. *Denote by p_0 the support projection of the counit in $\ell^\infty(\Gamma)$. If $N \subset \ell^\infty(\Gamma)$ is a BaaJ-Vaes subalgebra and $p_0 \in N$, then $N = \ell^\infty(\Gamma)$.*

Proof. The desired fact follows from the formula

$$\Delta(p_0) = \sum_{\gamma \in \text{Irr } \widehat{\Gamma}} t_\gamma t_\gamma^*,$$

where $t_\gamma \in H_\gamma \otimes H_\gamma^c$ is a (suitably normalised) invariant vector for the tensor product of the representation γ with its contragredient γ^c , and the fact that this vector is necessarily non-degenerate, in the sense that slicing $t_\gamma t_\gamma^*$ on the right by functionals on $B(H_{\gamma^c})$ we can obtain the whole $B(H_\gamma)$. \square

The (reduced) crossed product for an action α of a discrete quantum group Γ on a C^* -algebra A is defined similarly to the von Neumann construction, see Subsection 2.2: $\Gamma \rtimes_r A$ is the C^* -algebra that is obtained as the closed linear span of $(C(\widehat{\Gamma}) \otimes \mathbb{1})\alpha(A)$ inside $M(\mathcal{K}(\ell^2(\Gamma)) \otimes A)$. Moreover $\Gamma \rtimes_r A$ is equipped with the right dual action $\widehat{\alpha}$ of $\widehat{\Gamma}$ and the left action β of Γ (c.f. Subsection 2.2). The action β restricts to the (adjoint) action on $C(\widehat{\Gamma})$ (which we also denote by β). Let us summarize these observations in the form of the following lemma.

Lemma 2.11. *If $\alpha \in \text{Mor}(A, c_0(\Gamma) \otimes A)$ is an action of a discrete quantum group Γ on a unital C^* -algebra A , then the crossed product $\Gamma \rtimes_r A$ is a Γ -space (when equipped with the canonical adjoint action), and both standard embeddings of $(C(\widehat{\Gamma}), \beta)$ and of (A, α) into $\Gamma \rtimes_r A$ are Γ -equivariant ucp maps.*

3. AMENABILITY AND COAMENABILITY

All notions of boundary actions are intimately connected to the notion of amenability. Non-triviality of boundary actions is in a sense a measure of non-amenability of a group. In fact, the main applications of theories of boundary actions of locally compact groups, as developed by Furstenberg, have been in the problems that are related to rigidity properties that are extreme opposites of amenability.

One of the main motivations behind this work is to develop new tools to tackle similar problems in the context of discrete quantum groups. For this, we first need to generalize some of the main properties of amenable and co-amenable subgroups to the quantum setting. As it turns out, some of the basic well-known facts in the group setting become rather non-trivial in the quantum world.

3.1. Relative amenability.

Definition 3.1. *Let Γ be a discrete quantum group, and let Λ be a quantum subgroup of Γ . We say Λ is relatively amenable in Γ if there is a Λ -invariant mean on $\ell^\infty(\Gamma)$.*

In the classical setting, a subgroup Λ of a discrete group Γ is relatively amenable iff it is amenable. This follows from the fact that there is a unital positive Λ -equivariant map from $\ell^\infty(\Lambda)$ into $\ell^\infty(\Gamma)$, which is easily constructed using a set of representatives for the space of cosets, Γ/Λ . This reasoning is not immediately available in the non-commutative case and providing the answer requires significantly more work.

Another easily seen characterization of amenability of a subgroup in the classical setting, is the existence of an equivariant map with values in the algebra of functions on the quotient space. Motivated by this, we introduce the following definition.

Definition 3.2. *Let Γ be a discrete quantum group. A Γ -invariant von Neumann subalgebra M of $\ell^\infty(\Gamma)$ is said to be relatively amenable in $\ell^\infty(\Gamma)$ if there is a ucp Γ -equivariant map $\Psi : \ell^\infty(\Gamma) \rightarrow M$.*

Recall that if \mathbb{A} is a quantum subgroup of Γ , then $\ell^\infty(\Gamma/\mathbb{A})$ is an invariant von Neumann subalgebra of $\ell^\infty(\Gamma)$. The main result of this section is the equivalence of the notions introduced above for $\ell^\infty(\Gamma/\mathbb{A})$ to amenability of the quantum subgroup \mathbb{A} , which will be shown to hold in Theorem 3.7.

We need some preparation before proving the above theorem. Let Γ be a discrete quantum group. Given $\gamma \in \text{Irr}(\widehat{\Gamma})$ we denote by \mathbf{H}_γ the corresponding Hilbert space, by $U_\gamma \in B(\mathbf{H}_\gamma) \otimes C(\widehat{\Gamma})$ the unitary representation and by $\pi_\gamma : \ell^\infty(\Gamma) \rightarrow B(\mathbf{H}_\gamma)$ the representation of $\ell^\infty(\Gamma)$ on \mathbf{H}_γ . There is a unique state $\text{qTr}_\gamma : \ell^\infty(\Gamma) \rightarrow \mathbb{C}$ satisfying

$$\text{qTr}_\gamma(x)\mathbf{1}_{\mathbf{H}_\gamma} = (\text{id} \otimes h^{\widehat{\Gamma}})(U_\gamma(\pi_\gamma(x) \otimes \mathbf{1})U_\gamma^*)$$

for all $x \in \ell^\infty(\Gamma)$. We will need the following result, which is essentially contained in [Izu02, Lemma 2.2] (see also [Tom10, Subsection 4.2]).

Theorem 3.3 (Izumi). *If $y \in Z(\ell^\infty(\Gamma))$ then $(\text{id} \otimes \text{qTr}_\gamma)\Delta(y) \in Z(\ell^\infty(\Gamma))$.*

Suppose that \mathbb{A} is a quantum subgroup of Γ and $\pi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\mathbb{A})$ is the corresponding surjective $*$ -homomorphism satisfying $(\pi \otimes \pi) \circ \Delta^\Gamma = \Delta^\mathbb{A} \circ \pi$. Let $\mathbf{1}_\mathbb{A} \in Z(\ell^\infty(\Gamma))$ be the central carrier of π . By the theorem above we have $(\text{id} \otimes \text{qTr}_{\gamma^c})\Delta^\Gamma(\mathbf{1}_\mathbb{A}) \in Z(\ell^\infty(\Gamma))$ for all $\gamma \in \text{Irr}(\widehat{\Gamma})$. Moreover since $\Delta^\Gamma(\mathbf{1}_\mathbb{A}) \in \ell^\infty(\mathbb{A} \setminus \Gamma) \bar{\otimes} \ell^\infty(\Gamma/\mathbb{A})$ we conclude that

$$(3.1) \quad (\text{id} \otimes \text{qTr}_{\gamma^c})\Delta^\Gamma(\mathbf{1}_\mathbb{A}) \in Z(\ell^\infty(\mathbb{A} \setminus \Gamma))$$

In what follows we shall denote by $\sim_\mathbb{A}$ the equivalence relation on $\text{Irr}(\widehat{\Gamma})$ induced by \mathbb{A} . Recalling that $\text{Irr}(\widehat{\mathbb{A}}) \subset \text{Irr}(\widehat{\Gamma})$, we write $\tau \sim_\mathbb{A} \sigma$ if there exists $\rho \in \text{Irr}(\widehat{\mathbb{A}})$ such that $\tau \subset \sigma \otimes \rho$. Equivalently (cf [Ver04] and [DCKSS18, Theorem 5.6])

$$(3.2) \quad \tau \sim_\mathbb{A} \sigma \text{ if } (\pi_\sigma \otimes \pi_{\tau^c})\Delta^\Gamma(\mathbf{1}_\mathbb{A}) \neq 0,$$

where τ^c denotes the contragredient representation of τ .

Let $(q_i)_{i \in \mathcal{I}}$ be the maximal family of non-zero central minimal projections in $\ell^\infty(\mathbb{A} \setminus \Gamma)$. The equivalence relation in \mathcal{I} as defined in [DCKSS18] will be denoted by \sim . We shall use following property of \sim (c.f. [DCKSS18, Theorem 5.6] and [DCKSS18, Theorem 5.2]): given $i \in \mathcal{I}$ there exists $\gamma \in \text{Irr}(\widehat{\Gamma})$ such that

$$(3.3) \quad \sum_{\sigma \sim_\mathbb{A} \gamma} p_\sigma = \sum_{j \sim i} q_j.$$

Moreover $\sigma \in \text{supp}(q_j)$, i.e. $\pi_\sigma(q_j) \neq 0$.

Lemma 3.4. *Let $i \in \mathcal{I}$ and $\gamma \in \text{Irr}(\widehat{\Gamma})$ be related as in (3.3). Suppose that*

$$(3.4) \quad \sum_{\sigma \sim_\mathbb{A} \gamma} t_\sigma p_\sigma = \sum_{j \sim i} s_j q_j$$

for some $t_\sigma, s_j \in \mathbb{C}$. Then $t_\sigma = s_j$ for all σ and j .

Proof. Applying π_γ to both sides of (3.4) we conclude that

$$t_\gamma \mathbf{1}_{\mathbf{H}_\gamma} = \sum_{j \sim i} s_j \pi_\gamma(q_j).$$

Since $(\pi_\gamma(q_j))_{j \sim i}$ is a family of non-zero mutually orthogonal projections (c.f. the line following (3.3)) we conclude that $s_j = t_\gamma$ for all $j \sim i$. Replacing γ by $\sigma \sim_\mathbb{A} \gamma$ in the above argument we can see that $t_\sigma = s_j$ for all σ and j as above. \square

Corollary 3.5. *For every $\gamma \in \text{Irr}(\widehat{\Gamma})$ there exists $t_\gamma > 0$ such that*

$$(\text{id} \otimes \text{qTr}_{\gamma^c}) \Delta^\Gamma(\mathbf{1}_\Lambda) = t_\gamma \sum_{\sigma \sim_\Lambda \gamma} p_\sigma$$

Proof. Using (3.2) and Theorem 3.3 we conclude that there exist $t_\sigma \in \mathbb{R}$ such that $(\text{id} \otimes \text{qTr}_{\gamma^c}) \Delta^\Gamma(\mathbf{1}_\Lambda) = \sum_{\sigma \sim_\Lambda \gamma} t_\sigma p_\sigma$. On the other hand using (3.1) and (3.3) we see that there exists $i \in \mathcal{I}$ and $s_j \in \mathbb{R}$ such that $\sum_{\sigma \sim_\Lambda \gamma} t_\sigma p_\sigma = \sum_{j \sim i} s_j q_j$. We conclude using Lemma 3.4. \square

This leads us to the following result.

Proposition 3.6. *Let Γ be a discrete quantum group and Λ a quantum subgroup of Γ . Then there exists a normal ucp Λ -equivariant map $\Phi : \ell^\infty(\Lambda) \rightarrow \ell^\infty(\Gamma)$.*

Proof. Identify $\ell^\infty(\Lambda)$ with $\ell^\infty(\Gamma) \mathbf{1}_\Lambda$. Given $\gamma \in \text{Irr}(\widehat{\Gamma})$ we define a ucp map $\Phi_\gamma : \ell^\infty(\Lambda) \rightarrow \bigoplus_{\sigma \sim_\Lambda \gamma} p_\sigma \ell^\infty(\Gamma)$ by the formula

$$\Phi_\gamma(x) = t_\gamma^{-1} (\text{id} \otimes \text{qTr}_{\gamma^c}) \Delta^\Gamma(x).$$

Let $X : \text{Irr}(\widehat{\Gamma}) / \sim_\Lambda \rightarrow \text{Irr}(\widehat{\Gamma})$ be a section of the canonical surjection $\text{Irr}(\widehat{\Gamma}) \rightarrow \text{Irr}(\widehat{\Gamma}) / \sim_\Lambda$. Now it suffices to note that $\Phi := \bigoplus_{x \in \text{Irr}(\widehat{\Gamma}) / \sim_\Lambda} \Phi_{X(x)} : \ell^\infty(\Lambda) \rightarrow \ell^\infty(\Gamma)$ is a ucp Λ -equivariant map, that is

$$\alpha \circ \Phi = (\text{id} \otimes \Phi) \circ \Delta^\Lambda,$$

where $\alpha : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Lambda) \bar{\otimes} \ell^\infty(\Gamma)$ is the left action of Λ on $\ell^\infty(\Gamma)$. \square

We are now ready to formulate and prove the main result of this section.

Theorem 3.7. *Let Γ be a discrete quantum group and Λ a quantum subgroup of Γ . The following are equivalent:*

- (1) Λ is relatively amenable in Γ ;
- (2) $\ell^\infty(\Gamma/\Lambda)$ is relatively amenable in $\ell^\infty(\Gamma)$;
- (3) Λ is amenable.

Proof. (1) \implies (3): suppose Λ is relatively amenable in Γ , i.e. there is a Λ -invariant mean on $\ell^\infty(\Gamma)$. Composing this mean with the Λ -equivariant map $\Phi : \ell^\infty(\Lambda) \rightarrow \ell^\infty(\Gamma)$ from Theorem 3.6, we obtain a Λ -invariant mean on $\ell^\infty(\Lambda)$.

(3) \implies (2): suppose Λ is amenable, and let m be an invariant state on $\ell^\infty(\Lambda)$. Denote by $\rho^r : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma) \bar{\otimes} \ell^\infty(\Lambda)$ the right action of Λ on $\ell^\infty(\Gamma)$. Consider the ucp map $\Psi := (\text{id} \otimes m) \circ \rho^r : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma)$. For any $a \in \ell^\infty(\Gamma)$ we have

$$\begin{aligned} \rho^r(\Psi(a)) &= \rho^r((\text{id} \otimes m)\rho^r(a)) = (\text{id} \otimes \text{id} \otimes m)((\rho^r \otimes \text{id})\rho^r(a)) \\ &= (\text{id} \otimes \text{id} \otimes m)((\text{id} \otimes \Delta_\Lambda)\rho^r(a)) = \mathbf{1} \otimes [(\text{id} \otimes m)\rho^r(a)] \\ &= \mathbf{1} \otimes \Psi(a), \end{aligned}$$

which shows $\Psi(a) \in \ell^\infty(\Gamma/\Lambda)$. Moreover, for any $a \in \ell^\infty(\Gamma)$ and $\omega \in \ell^1(\Gamma)$ we have

$$\begin{aligned} (\omega \otimes \text{id})(\Delta_\Gamma(\Psi(a))) &= (\omega \otimes \text{id})(\Delta_\Gamma((\text{id} \otimes m)\rho^r(a))) = (\omega \otimes \text{id} \otimes m)((\Delta_\Gamma \otimes \text{id})\rho^r(a)) \\ &= (\omega \otimes \text{id} \otimes m)((\text{id} \otimes \rho^r)\Delta_\Gamma(a)) = (\omega \otimes [(\text{id} \otimes m) \circ \rho^r])\Delta_\Gamma(a) \\ &= \Psi((\omega \otimes \text{id})\Delta_\Gamma(a)), \end{aligned}$$

which shows that Ψ is Γ -equivariant. Hence $\ell^\infty(\Gamma/\Lambda)$ is relatively amenable in $\ell^\infty(\Gamma)$.

(2) \implies (1): let $\Psi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma/\Lambda)$ be a ucp Γ -equivariant map. The restriction of the co-unit ε of $\ell^\infty(\Gamma)$ to $\ell^\infty(\Gamma/\Lambda)$ is Λ -invariant. Hence $\tilde{m} = \varepsilon \circ \Psi$ is a Λ -invariant state on $\ell^\infty(\Gamma)$. \square

Next we investigate the connection of the above notion of relative amenability for invariant subalgebras with respect to the structure of the dual compact quantum group $\mathbb{G} = \widehat{\Gamma}$. Recall indeed that invariant subalgebras of $\ell^\infty(\Gamma)$ can also arise as subalgebras $\ell^\infty(\widehat{\mathbb{H}})$ associated with closed quantum subgroups \mathbb{H} of \mathbb{G} .

Definition 3.8. *Let \mathbb{G} be a compact quantum group and let \mathbb{H} be a closed quantum subgroup of \mathbb{G} realized by $\pi \in \text{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}))$. Denote*

- ▶ $C^u(\mathbb{H} \setminus \mathbb{G}) = \{x \in C^u(\mathbb{G}) : (\pi \otimes \text{id})\Delta^u(x) = \mathbf{1} \otimes x\};$
- ▶ $C(\mathbb{H} \setminus \mathbb{G}) = \Lambda_{\mathbb{G}}(C^u(\mathbb{H} \setminus \mathbb{G})).$

We say that $\mathbb{H} \setminus \mathbb{G}$ (which is in general just a symbol, not a compact quantum group) is a co-amenable quantum quotient of \mathbb{G} if the restriction $\varepsilon|_{C^u(\mathbb{H} \setminus \mathbb{G})} : C^u(\mathbb{H} \setminus \mathbb{G}) \rightarrow \mathbb{C}$ admits a reduced version, i.e. there exists $\varepsilon^r : C(\mathbb{H} \setminus \mathbb{G}) \rightarrow \mathbb{C}$ satisfying

$$\varepsilon^r \circ \Lambda_{\mathbb{G}}|_{C^u(\mathbb{H} \setminus \mathbb{G})} = \varepsilon|_{C^u(\mathbb{H} \setminus \mathbb{G})}$$

where $\Lambda_{\mathbb{G}} : C^u(\mathbb{G}) \rightarrow C(\mathbb{G})$ is the reducing surjection.

Remark 3.9. *Let \mathbb{G} be a compact quantum group and let $\mathbb{H} \subset \mathbb{G}$ be a normal quantum subgroup. The quotient $\mathbb{H} \setminus \mathbb{G}$ viewed as a compact quantum group will be denoted by \mathbb{L} . Let us note that the quotient $\mathbb{H} \setminus \mathbb{G}$ is co-amenable if and only if \mathbb{L} is a co-amenable quantum group. Indeed the exact sequence of quantum groups*

$$\{e\} \rightarrow \mathbb{H} \rightarrow \mathbb{G} \rightarrow \mathbb{L} \rightarrow \{e\}$$

has the dual version

$$\{e\} \rightarrow \widehat{\mathbb{L}} \rightarrow \widehat{\mathbb{G}} \rightarrow \widehat{\mathbb{H}} \rightarrow \{e\}.$$

Using [KKSS, Theorem 3.2] we see that $C^u(\mathbb{H} \setminus \mathbb{G})$ can be identified with $C^u(\mathbb{L})$.

Theorem 3.10. *Let \mathbb{G} be a compact quantum group and let $\mathbb{H} \subset \mathbb{G}$ be a normal quantum subgroup. Then the normal Baaj-Vaes subalgebra $\ell^\infty(\widehat{\mathbb{H}}) \subset \ell^\infty(\widehat{\mathbb{G}})$ is relatively amenable iff $\mathbb{H} \setminus \mathbb{G}$ is a co-amenable quotient.*

Proof. By the above remark we have $\ell^\infty(\widehat{\mathbb{H}}) = \ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{L}})$. Thus, by Theorem 3.7, the relative amenability of $\ell^\infty(\widehat{\mathbb{H}})$ is equivalent to amenability of $\widehat{\mathbb{L}}$, hence equivalent to co-amenable of the compact quantum group \mathbb{L} ([Tom06]). The latter is equivalent to co-amenable of the quotient $\mathbb{H} \setminus \mathbb{G}$ as mentioned in Remark 3.9. \square

We do not know the relation between relative amenability of $\ell^\infty(\widehat{\mathbb{H}})$ in $\ell^\infty(\widehat{\mathbb{G}})$ and the co-amenable of the quotient $\mathbb{H} \setminus \mathbb{G}$ in general for non-normal quantum subgroups $\mathbb{H} \subset \mathbb{G}$.

Theorem 3.11. *Let \mathbb{G} be a compact quantum group and $\mathbb{H} \subset \mathbb{G}$ be a compact quantum subgroup given by $\pi \in \text{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}))$. Then $\mathbb{H} \setminus \mathbb{G}$ is co-amenable if and only if $\pi \in \text{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}))$ admits a reduced version, that is there exists $\tilde{\pi} \in \text{Mor}(C(\mathbb{G}), C(\mathbb{H}))$ such that $\tilde{\pi} \circ \Lambda_{\mathbb{G}} = \Lambda_{\mathbb{H}} \circ \pi$.*

Proof. Suppose that $\mathbb{H}\backslash\mathbb{G}$ is a co-amenable quantum quotient. In order to prove that π admits the reduced version it is enough to show that $\Lambda_{\mathbb{H}} \circ \pi$ viewed as a map from $C^u(\mathbb{G})$ to $B(L^2(\mathbb{H}))$ admits a reduced version $C(\mathbb{G}) \rightarrow B(L^2(\mathbb{H}))$. Note that the map $C^u(\mathbb{G}) \rightarrow B(L^2(\mathbb{H}))$ is obtained by the GNS construction applied to the state $h_{\mathbb{H}} \circ \pi \in C^u(\mathbb{G})^*$ where $h_{\mathbb{H}}$ is the Haar state on $C^u(\mathbb{H})$.

Let $E : C(\mathbb{G}) \rightarrow C(\mathbb{H}\backslash\mathbb{G})$ be the conditional expectation and $E^u : C^u(\mathbb{G}) \rightarrow C^u(\mathbb{H}\backslash\mathbb{G})$ its universal version:

$$E^u(c) = (h_{\mathbb{H}} \circ \pi \otimes \text{id})\Delta^u(c)$$

for all $c \in C^u(\mathbb{G})$ and note that $E(\Lambda_{\mathbb{G}}(c)) = \Lambda_{\mathbb{G}}(E^u(c))$. Let us apply the Rieffel induction ([Rie74]) to $\varepsilon_r : C(\mathbb{H}\backslash\mathbb{G}) \rightarrow \mathbb{C}$ and denote the induced representation by $\pi_R : C(\mathbb{G}) \rightarrow B(\mathbb{H}_R)$, where $\mathbb{H}_R = C(\mathbb{G}) \otimes_{C(\mathbb{H}\backslash\mathbb{G})} \mathbb{C}$. Clearly $\Omega = \mathbf{1} \otimes 1$ is the cyclic vector for π_R . Let us fix $a, b \in C(\mathbb{G})$ and $a^u, b^u \in C^u(\mathbb{G})$ such that $\Lambda_{\mathbb{G}}(a^u) = a$ and $\Lambda_{\mathbb{G}}(b^u) = b$. The scalar product in \mathbb{H}_R is given by

$$\langle a \otimes 1, b \otimes 1 \rangle = \varepsilon^r(E(a^*b)) = \varepsilon^r(\Lambda_{\mathbb{G}}(h_{\mathbb{H}} \circ \pi \otimes \text{id})(\Delta^u(a_u^*b_u))) = h_{\mathbb{H}}(\pi(a_u^*b_u))$$

In particular the state $\omega : C(\mathbb{G}) \ni a \rightarrow \langle \Omega | \pi_R(a) \Omega \rangle \in \mathbb{C}$ satisfies $h_{\mathbb{H}}(\pi(a_u)) = \omega(a)$ and π_R is the GNS representation assigned to ω . This shows that π_R is the reduced version of π .

Conversely, if a given morphism $\pi \in \text{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}))$ identifying \mathbb{H} with a closed quantum subgroup of \mathbb{G} admits a reduced version $\tilde{\pi} \in \text{Mor}(C(\mathbb{G}), C(\mathbb{H}))$ then $\mathbb{H}\backslash\mathbb{G}$ is co-amenable. Indeed, for every $x \in C(\mathbb{H}\backslash\mathbb{G})$ we have $(\tilde{\pi} \otimes \text{id})(\Delta(x)) = \mathbf{1} \otimes x$. In particular $\Delta_{\mathbb{H}}(\tilde{\pi}(x)) = \mathbf{1} \otimes \tilde{\pi}(x)$ and $\tilde{\pi}(x) \in \mathbb{C}\mathbf{1}$. It is easy to check that $\tilde{\pi}(x) = \varepsilon(x^u)\mathbf{1}$ where $x^u \in C^u(\mathbb{H}\backslash\mathbb{G})$ is such that $\Lambda_{\mathbb{G}}(x^u) = x$. \square

Theorem 3.12. *Let \mathbb{G} be a compact quantum group and \mathbb{H} a closed quantum subgroup of \mathbb{G} . Then the following are equivalent:*

- \mathbb{G} is co-amenable;
- \mathbb{H} is co-amenable quantum group and $\mathbb{H}\backslash\mathbb{G}$ is co-amenable quantum quotient of \mathbb{G} .

Moreover if $\mathbb{H}_1 \subset \mathbb{H}_2 \subset \mathbb{G}$ and $\pi_{\mathbb{H}_1} \in \text{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}_1))$ admits a reduced version, then $\pi_{\mathbb{H}_2} \in \text{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}_2))$ also admits a reduced version.

Proof. If \mathbb{G} is co-amenable then using [Tom07] we see that \mathbb{H} is co-amenable. Clearly the quotient $\mathbb{H}\backslash\mathbb{G}$ is co-amenable in this case. Conversely suppose that \mathbb{H} and $\mathbb{H}\backslash\mathbb{G}$ are co-amenable. Then $\pi \in \text{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}))$ admits the reduced version $\tilde{\pi} \in \text{Mor}(C(\mathbb{G}), C(\mathbb{H}))$ and thus $\varepsilon_{\mathbb{H}} \circ \tilde{\pi}$ defines the reduced counit on $C(\mathbb{G})$.

The last statement of the theorem follows from Theorem 3.11 and the fact that $C^u(\mathbb{G}/\mathbb{H}_2) \subset C^u(\mathbb{G}/\mathbb{H}_1)$. \square

3.2. The amenable radical. We finish the section by proving a quantum version of Day's result from [Day57], showing that every discrete quantum group Γ has a unique maximal normal amenable closed quantum subgroup, which contains all normal amenable closed quantum subgroups of Γ .

We will need the following special case of a recent result of Crann.

Theorem 3.13 ([Cra17], Theorem 3.2). *Let \mathbb{G} be a locally compact quantum group with a closed normal quantum subgroup \mathbb{H} . Then \mathbb{G} is amenable if and only if both \mathbb{H} and \mathbb{G}/\mathbb{H} are amenable.*

Proof. It suffices to observe that the fact that in the context above the amenability of the \mathbb{G} action on \mathbb{G}/\mathbb{H} as defined in [Cra17] is the same as the amenability of the locally compact quantum group \mathbb{G}/\mathbb{H} and apply Theorem 3.2 of that paper. \square

Suppose then that \mathbb{G} is a locally compact quantum group and we have a collection $(\mathbb{H}_i)_{i \in \mathcal{I}}$ of closed quantum subgroups of \mathbb{G} , so that we have the respective normal embeddings $\gamma_i : L^\infty(\widehat{\mathbb{H}}_i) \rightarrow L^\infty(\widehat{\mathbb{G}})$. We say that \mathbb{G} is *generated* by the family $(\mathbb{H}_i)_{i \in \mathcal{I}}$ if $L^\infty(\widehat{\mathbb{G}})$ is the smallest von Neumann algebra containing all $\gamma_i(L^\infty(\widehat{\mathbb{H}}_i))$. Further if \mathcal{I} is an ordered set, then we say that the net $(\mathbb{H}_i)_{i \in \mathcal{I}}$ is *increasing* if for all $i, j \in \mathcal{I}$ such that $i \leq j$ we have $\gamma_i(L^\infty(\widehat{\mathbb{H}}_i)) \subset \gamma_j(L^\infty(\widehat{\mathbb{H}}_j))$.

Lemma 3.14. *Let \mathbb{F} be a discrete quantum group and let $(\mathbb{H}_i)_{i \in \mathcal{I}}$ be an increasing net of amenable quantum subgroups of \mathbb{F} . Then the quantum subgroup generated by $\{\mathbb{H}_i : i \in \mathcal{I}\}$ is amenable.*

Proof. Without loss of generality we shall assume that \mathbb{F} is generated by $\{\mathbb{H}_i : i \in \mathcal{I}\}$. By Theorem 6.2 of [DKSS12] each \mathbb{H}_i is also discrete. Thus we have the respective Hopf embeddings $\gamma_i : \text{Pol}(\widehat{\mathbb{H}}_i) \rightarrow \text{Pol}(\widehat{\mathbb{F}})$, which are isometries with respect to the respective reduced C^* -norms. Using [KKSS, Theorem 3.2] in the case of $\mathbb{H}_i \subset \mathbb{F}$ these embeddings are isometric for the universal norms too. It is easy to see that $\text{Pol}(\widehat{\mathbb{F}}) = \bigcup_{i \in \mathcal{I}} \gamma_i(\text{Pol}(\widehat{\mathbb{H}}_i))$ (this follows from the orthogonality relations). In particular, since each \mathbb{H}_i is amenable, the universal and the reduced norm coincide on $\text{Pol}(\widehat{\mathbb{F}})$. Since $\text{Pol}(\widehat{\mathbb{F}})$ is dense in $C^u(\widehat{\mathbb{F}})$ these norms must be equal and we see that $\widehat{\mathbb{F}}$ is co-amenable, so that \mathbb{F} is amenable. \square

Proposition 3.15. *Every discrete quantum group \mathbb{F} admits the largest normal amenable quantum subgroup, that we call the amenable radical of \mathbb{G} .*

Proof. The proof follows the original argument of Section 4 of [Day57].

Consider \mathcal{H} , the collection of all normal amenable quantum subgroups of \mathbb{F} . Kuratowski-Zorn's lemma together with Lemma 3.14 show that \mathcal{H} contains a maximal element, say \mathbb{H} . Suppose then that there is $\mathbb{H}' \in \mathcal{H}$ which is not contained in \mathbb{H} . Consider, as in [CHK17] (Definition 2.1 and Definition 1.9), the subgroup generated by \mathbb{H} and \mathbb{H}' , denoted $\mathbb{H} \vee \mathbb{H}'$. Again, by Definitions 2.1 and 1.9 in [CHK17] and the discussion before the latter it follows that $\mathbb{H} \vee \mathbb{H}'$ is a normal quantum subgroup of \mathbb{F} (strictly containing \mathbb{H}). Then Corollary 4.2 of [CHK17], applicable to discrete quantum groups, says that $(\mathbb{H} \vee \mathbb{H}')/\mathbb{H}' = \mathbb{H}/(\mathbb{H} \wedge \mathbb{H}')$. By Theorem 3.13 first the right hand side is amenable, and then so is $\mathbb{H} \vee \mathbb{H}'$. This leads to a contradiction and ends the proof. \square

4. NONCOMMUTATIVE TOPOLOGICAL BOUNDARIES AND INJECTIVE ENVELOPES

In this section we define topological \mathbb{F} -boundaries, and prove the existence of a (unique) universal \mathbb{F} -boundary, by identifying that object with the \mathbb{F} -injective envelope of the trivial action, in the sense of Hamana.

All the \mathbb{F} - C^* -algebras in this section are assumed to be unital.

4.1. Topological \mathbb{F} -boundaries. We begin by defining boundary actions.

Definition 4.1. *Let \mathbb{F} be a discrete quantum group. A \mathbb{F} - C^* -algebra \mathbb{A} is a \mathbb{F} -boundary if for every state ν on \mathbb{A} the Poisson transform \mathcal{P}_ν is completely isometric.*

By $P(\Gamma)$ we shall denote the set of all states $\mu \in \ell^1(\Gamma)$. The next lemma connects the above notion with the classical notion of boundary actions as developed by Furstenberg. Note indeed that when A and $\ell^\infty(\Gamma)$ are commutative, \mathcal{P}_ν is isometric on the self-adjoint part of A **iff** it is completely isometric, and on the other hand the set $\{\mu * \nu : \mu \in P(\Gamma)\}$ is weak* dense in the state space of A **iff** the action of $\Gamma = \Gamma$ on the spectrum of A is minimal and strongly proximal.

Lemma 4.2. *Let Γ be a discrete quantum group, let A be a Γ - C^* -algebra, and let $\nu \in A^*$ be a state. The Poisson transform \mathcal{P}_ν is isometric on the self-adjoint part of A if and only if the set $\{\mu * \nu : \mu \in P(\Gamma)\}$ is weak* dense in the state space of A .*

Proof. Suppose there is a state $\rho \in A^*$ that is not contained in the weak* closure of $\{\mu * \nu : \mu \in P(\Gamma)\}$. Then by the Hahn-Banach separation theorem, there is a self-adjoint element $x \in A$ and $\varepsilon > 0$ such that

$$\mu(\mathcal{P}_\nu(x)) = \mu * \nu(x) \leq \rho(x) - \varepsilon \leq \|x\| - \varepsilon, \quad \forall \mu \in P(\Gamma).$$

Since the Poisson transform \mathcal{P}_ν is completely positive, $\mathcal{P}_\nu(x)$ is also self-adjoint, and therefore it follows $\|\mathcal{P}_\nu(x)\| = \sup\{|\mu(\mathcal{P}_\nu(x))| : \mu \in P(\Gamma)\} \leq \|x\| - \varepsilon$.

Conversely, suppose the weak* closure of the set $\{\mu * \nu : \mu \in P(\Gamma)\}$ contains every state on A . Let $x \in A$ be self-adjoint. Let $\varepsilon > 0$ be given. There is a state ρ on A such that $|\rho(x)| > \|x\| - \varepsilon$. By weak* density of the set above, there is $\mu \in P(\Gamma)$ such that $|\mu(\mathcal{P}_\nu(x))| = |\mu * \nu(x)| > \|x\| - 2\varepsilon$. This implies $\|\mathcal{P}_\nu(x)\| \geq \|x\| - 2\varepsilon$. Since ε was arbitrary, and \mathcal{P}_ν is a contraction, it follows $\|\mathcal{P}_\nu(x)\| = \|x\|$. \square

We record the following simple fact as we use it in several several places later.

Proposition 4.3. *Let A be a Γ - C^* -algebra. The following are equivalent:*

- (1) *A is a Γ -boundary;*
- (2) *every ucp Γ -equivariant from A into $\ell^\infty(\Gamma)$ is completely isometric;*
- (3) *every ucp Γ -equivariant map from A into any Γ - C^* -algebra B is completely isometric.*

Proof. By Proposition 2.6 every ucp Γ -equivariant map from A into $\ell^\infty(\Gamma)$ is the Poisson transform associated to some state on A , hence the implication (1) \implies (2). To show (2) \implies (3), let ψ be a ucp Γ -equivariant map from A into a Γ - C^* -algebra B . Let ν be a state on B , then $\mathcal{P}_\nu \circ \psi$ is a ucp Γ -equivariant map from A into $\ell^\infty(\Gamma)$, hence completely isometric assuming (2). Since \mathcal{P}_ν is a complete contraction, this implies ψ is completely isometric. The implication (3) \implies (1) is immediate from the definition. \square

Proposition 4.4. *Let Γ be a discrete quantum group and let B be a Γ -boundary. Suppose A is a Γ - C^* -algebra such that there is a completely isometric ucp Γ -equivariant map $\iota : A \rightarrow B$. Then A is a Γ -boundary.*

Proof. Denote by α_A and α_B the actions of Γ on A and B . Let ν be a state on A . Extend the state $\nu \circ \iota^{-1}$ on $\iota(A)$ to a state $\tilde{\nu}$ on B . Then we have $\mathcal{P}_\nu = \mathcal{P}_{\tilde{\nu}} \circ \iota$. Since B is a Γ -boundary, $\mathcal{P}_{\tilde{\nu}}$ is completely isometric, hence \mathcal{P}_ν is completely isometric and the proof is complete. \square

Proposition 4.5. *Let Γ be a discrete quantum group and let A be a Γ - C^* -algebra. If A is a Λ -boundary for some quantum subgroup $\Lambda \subset \Gamma$ with the canonical restriction action, then A is a Γ -boundary.*

Proof. Let $\pi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Lambda)$ be the surjective normal $*$ -homomorphism which identifies Λ as a quantum subgroup of Γ . Let ν be a state on \mathbf{A} . Denoting $\mathcal{P}_\nu^\Gamma : \mathbf{A} \rightarrow \ell^\infty(\Gamma)$ and $\mathcal{P}_\nu^\Lambda : \mathbf{A} \rightarrow \ell^\infty(\Lambda)$ the corresponding Poisson maps we have $\mathcal{P}_\nu^\Lambda = \pi \circ \mathcal{P}_\nu^\Gamma$. Since \mathcal{P}_ν^Λ is completely isometric the same must hold for \mathcal{P}_ν^Γ . \square

Proposition 4.6. *Let Γ be a discrete quantum group and let $\Lambda \subset \Gamma$ be a normal quantum subgroup. Then every Γ/Λ -boundary endowed with the canonical action of Γ is a Γ -boundary.*

Proof. Suppose that \mathbf{A} is a Γ/Λ boundary via $\alpha \in \text{Mor}(\mathbf{A}, c_0(\Gamma/\Lambda) \otimes \mathbf{A})$. Recall that $\ell^\infty(\Gamma/\Lambda)$ is a von Neumann subalgebra of $\ell^\infty(\Gamma)$ and that we have the corresponding injective morphism $\iota \in \text{Mor}(c_0(\Gamma/\Lambda), c_0(\Gamma))$. Then the canonical action of $\beta \in \text{Mor}(\mathbf{A}, c_0(\Gamma) \otimes \mathbf{A})$ of Γ on \mathbf{A} is $\beta = (\iota \otimes \text{id}) \circ \alpha$, and for a state ν on \mathbf{A} the corresponding Poisson maps $\mathcal{P}_\nu^\Gamma : \mathbf{A} \rightarrow \ell^\infty(\Gamma)$ and $\mathcal{P}_\nu^{\Gamma/\Lambda} : \mathbf{A} \rightarrow \ell^\infty(\Gamma/\Lambda)$ satisfy $\mathcal{P}_\nu^\Gamma = \iota \circ \mathcal{P}_\nu^{\Gamma/\Lambda}$. In particular $\mathcal{P}_\nu^{\Gamma/\Lambda}$ is completely isometric if and only if \mathcal{P}_ν^Γ is completely isometric. \square

4.2. Γ -Injective envelopes. In this section we show existence of a (unique) minimal injective object in the category of Γ - C^* -algebras with ucp Γ -equivariant maps as morphisms (and completely isometric ucp Γ -equivariant maps as injections). The construction and most arguments below follow the work of Hamana [Ham85] and some simplifications made in [Pau11], but we include all details for the convenience of the reader.

However this general categorical construction only yields an injective $\ell^1(\Gamma)$ -module object. In the classical case, for discrete group Γ , every $\ell^1(\Gamma)$ -module has an obvious natural Γ -action. But this fact is by no means obvious in the quantum setting. The main new result of this section (Theorem 4.9) is that in the case of discrete quantum groups Γ , certain $\ell^1(\Gamma)$ -module actions, including the one on the minimal Γ -injective object, indeed come from a Γ -action.

Let \mathcal{G} be the set of all ucp Γ -equivariant maps $\phi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma)$. On the set \mathcal{G} define the partial pre-order

$$\phi \leq \psi \text{ if } \|\phi(x)\| \leq \|\psi(x)\| \text{ for all } x \in \ell^\infty(\Gamma).$$

Proposition 4.7. *The set \mathcal{G} contains a minimal element.*

Proof. We show that every decreasing net $(\phi_i)_{i \in \mathcal{I}}$ in \mathcal{G} has a lower bound. Then the result follows from the Kuratowski-Zorn Lemma.

To show this, first note that \mathcal{G} is point-weak* compact. Therefore, there is a subnet $(\phi_{i_j})_{j \in \mathcal{J}}$ and $\phi_0 \in \mathcal{G}$ such that $\phi_{i_j}(x) \rightarrow \phi_0(x)$ in the weak* topology for all $x \in \ell^\infty(\Gamma)$. We obviously have $\|\phi_0(x)\| \leq \limsup_{i_j} \|\phi_{i_j}(x)\| = \inf_{i \in \mathcal{I}} \|\phi_i(x)\|$ for all $x \in \ell^\infty(\Gamma)$. Thus, $\phi_0 \leq \phi_i$ for all $i \in \mathcal{I}$. \square

Lemma 4.8. *If ϕ_0 is a minimal element of \mathcal{G} , then ϕ_0 is an idempotent.*

Proof. Let $\phi^{(n)} := \frac{1}{n} \sum_{k=1}^n \phi_0^k$. We have $\|\phi^{(n)}(x)\| \leq \frac{1}{n} \sum_{k=1}^n \|\phi_0^k(x)\| \leq \frac{1}{n} \sum_{k=1}^n \|\phi_0(x)\| = \|\phi_0(x)\|$ for all $x \in \ell^\infty(\Gamma)$ and $n \in \mathbb{N}$. Let $\Phi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma)$ be a point-weak* limit point of $(\phi^{(n)})$. Then $\phi_0 \Phi = \Phi \phi_0 = \Phi$, and moreover $\|\Phi(x)\| \leq \limsup_n \|\phi^{(n)}(x)\| \leq \|\phi_0(x)\|$ for all $x \in \ell^\infty(\Gamma)$. By minimality of ϕ_0 we have $\|\Phi(x)\| = \|\phi_0(x)\|$ for all $x \in \ell^\infty(\Gamma)$. Therefore

$$\begin{aligned} \|\phi_0(x) - \phi_0^2(x)\| &= \|\phi_0(x - \phi_0(x))\| = \|\Phi(x - \phi_0(x))\| \\ &= \|\Phi(x) - \Phi\phi_0(x)\| = 0 \end{aligned}$$

for all $x \in \ell^\infty(\Gamma)$, which implies that ϕ_0 is idempotent. \square

The standard Choi-Effros product makes the image of a (minimal) ucp Γ -equivariant idempotent as above a unital C^* -algebra, which naturally has an $\ell^1(\Gamma)$ -module structure. We will show in the following theorem that this $\ell^1(\Gamma)$ -module structure in fact comes from an action of Γ .

Theorem 4.9. *Let $\phi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma)$ be an idempotent Γ -equivariant ucp map, let $\mathbf{X} = \phi(\ell^\infty(\Gamma))$ and let \mathbf{A} denote the C^* -algebra that is obtained from equipping \mathbf{X} with the Choi-Effros product $a \cdot b := \phi(ab)$, $a, b \in \mathbf{X}$. Then the restriction of the coproduct defines a Γ -action on \mathbf{A} satisfying the Podleś condition.*

Proof. In order to distinguish the abstract C^* -algebra \mathbf{A} from the image \mathbf{X} of $\phi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma)$, we shall be using the embedding $j : \mathbf{A} \rightarrow \ell^\infty(\Gamma)$ of \mathbf{A} into $\ell^\infty(\Gamma)$. The product of elements a, b viewed as elements of abstract C^* -algebra \mathbf{A} will be denoted by ab . The definition of the Choi-Effros product can now be written as

$$j(ab) = \phi(j(a)j(b)), \quad a, b \in \mathbf{A}.$$

Note that j is a unital completely isometric map.

Consider the Fubini product map (see e.g. [ER00, Chapter 7])

$$\Phi := \text{id} \otimes_{\mathcal{F}} \phi : \ell^\infty(\Gamma) \bar{\otimes} \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma) \bar{\otimes} \ell^\infty(\Gamma).$$

Recall that Φ is a complete contraction, characterised by the formula

$$(\omega \otimes \mu)(\Phi(x)) = \mu(\phi((\omega \otimes \text{id})(x))), \quad x \in \ell^\infty(\Gamma) \bar{\otimes} \ell^\infty(\Gamma), \omega, \mu \in \ell^1(\Gamma).$$

The above formula allows us to deduce that for x and ω as above we have $(\omega \otimes \text{id})\Phi(x) = \phi((\omega \otimes \text{id})(x))$, and further that Φ is idempotent. The fact that ϕ is a ucp Γ -equivariant map implies after another easy computation that

$$\Phi \circ \Delta = \Delta \circ \phi,$$

and that the image of Φ equals

$$Z := \{z \in \ell^\infty(\Gamma) \bar{\otimes} \ell^\infty(\Gamma) : (\omega \otimes \text{id})(z) \in \mathbf{X} \text{ for all } \omega \in \ell^1(\Gamma)\}.$$

We then need another identification: as $\ell^\infty(\Gamma) \approx \prod_{\gamma \in \text{Irr}(\hat{\Gamma})} M_{n_\gamma}$, we can identify canonically $\ell^\infty(\Gamma) \bar{\otimes} \ell^\infty(\Gamma)$ with $\prod_{\gamma \in \text{Irr}(\hat{\Gamma})} M_{n_\gamma}(\ell^\infty(\Gamma))$. In this identification we have $\Phi(\prod_{\gamma \in \text{Irr}(\hat{\Gamma})} m_\gamma) = \prod_{\gamma \in \text{Irr}(\hat{\Gamma})} \phi(m_\gamma)$ and $Z \approx \prod_{\gamma \in \text{Irr}(\hat{\Gamma})} M_{n_\gamma}(\mathbf{X})$. Finally use the ‘pointwise’ application of the map $j^{-1} : \mathbf{X} \rightarrow \mathbf{A}$ to define the ucp isometry $J^{-1} : Z \rightarrow \prod_{\gamma \in \text{Irr}(\hat{\Gamma})} M_{n_\gamma}(\mathbf{A}) \approx M(c_0(\Gamma) \otimes \mathbf{A})$.

We are finally ready to define the action $\alpha : \mathbf{A} \rightarrow M(c_0(\Gamma) \otimes \mathbf{A})$; it is informally just the suitably interpreted coproduct, and formally

$$\alpha = J^{-1} \circ \Delta \circ j.$$

It is now easy to check that for any $z, w \in Z$ we have $J^{-1}(z)J^{-1}(w) = J^{-1}(\Phi(zw))$. Thus finally for $a, b \in \mathbf{A}$ we have

$$\begin{aligned} \alpha(ab) &= J^{-1}(\Delta(j(ab))) = J^{-1}(\Delta(\phi(j(a)j(b)))) \\ &= J^{-1}(\Phi(\Delta(j(a)j(b)))) = (J^{-1} \circ \Phi)(\Delta(j(a))\Delta(j(b))) \\ &= J^{-1}(\Delta(j(a)))J^{-1}(\Delta(j(b))) = \alpha(a)\alpha(b), \end{aligned}$$

which is what we intended to prove.

To prove that α satisfies the Podleś condition $[\alpha(\mathbf{A})(c_0(\Gamma) \otimes \mathbf{1})] = c_0(\Gamma) \otimes \mathbf{A}$ we shall use [BSV03, Proposition 5.8]. Let us first note that α , being a unital $*$ -homomorphism, is non-degenerate, i.e. $[\alpha(\mathbf{A})(c_0(\Gamma) \otimes \mathbf{A})] = c_0(\Gamma) \otimes \mathbf{A}$. Moreover for all $\omega \in \ell^1(\Gamma)$ and $a \in \mathbf{A}$ we have $(\omega \otimes \text{id})(\alpha(a)) \in \mathbf{A}$. The counit $\varepsilon \in c_0(\Gamma)^*$ satisfies $(\varepsilon \otimes \text{id})\alpha(a) = a$ and since $\varepsilon \in \ell^1(\Gamma)$ we see that the action α is weakly continuous, that is

$$[\{(\omega \otimes \text{id})\alpha(a) : \omega \in \ell^1(\Gamma), a \in \mathbf{A}\}] = \mathbf{A}.$$

Since discrete quantum groups are regular, [BSV03, Proposition 5.8] indeed applies to α . \square

In the following, for any idempotent ucp Γ -equivariant ϕ on $\ell^\infty(\Gamma)$, we consider $\text{im}(\phi)$ as a Γ - C^* -algebra, with the Γ -action given in Theorem 4.9.

Proposition 4.10. *Suppose ϕ_0 is a minimal element of \mathcal{G} (which is idempotent by Lemma 4.8). Then*

- (1) (Γ -rigidity) *the identity map is the unique ucp Γ -equivariant map on $\text{im}(\phi_0)$;*
- (2) (Γ -essentiality) *any ucp Γ -equivariant map $\psi : \text{im}(\phi_0) \rightarrow \mathbf{A}$ from $\text{im}(\phi_0)$ into any Γ - C^* -algebra \mathbf{A} is completely isometric.*

Proof. (1) Suppose $\psi : \text{im}(\phi_0) \rightarrow \text{im}(\phi_0)$ is a ucp Γ -equivariant map. Similarly as in the proof of Lemma 4.8 above we may find a ucp Γ -equivariant Ψ acting on $\text{im}(\phi_0)$ such that $\Psi\psi = \psi\Psi = \Psi$. Since Ψ is a contraction we have $\|\Psi\phi_0(x)\| \leq \|\phi_0(x)\|$ for all $x \in \text{im}(\phi_0)$. Hence by minimality of ϕ_0 , it follows that $\|x\| = \|\phi_0(x)\| \leq \|\Psi(\phi_0(x))\|$ for all $x \in \text{im}(\phi_0)$, which shows that Ψ is isometric on $\text{im}(\phi_0)$. Thus, $\|x - \psi(x)\| = \|\Psi(x - \psi(x))\| = 0$ for all $x \in \text{im}(\phi_0)$, which implies that $\psi = \text{id}_{\text{im}(\phi_0)}$.

(2) Let \mathbf{A} be a Γ - C^* -algebra and $\psi : \text{im}(\phi_0) \rightarrow \mathbf{A}$ be a ucp Γ -equivariant map. Let $\omega \in S(\mathbf{A})$ be a state on \mathbf{A} , and $\mathcal{P}_\omega : \mathbf{A} \rightarrow \ell^\infty(\Gamma)$ the corresponding Poisson transform. Then $\phi_0 \circ \mathcal{P}_\omega \circ \psi$ is a ucp Γ -equivariant map on $\text{im}(\phi_0)$, hence the identity map by part (1). Since both ϕ_0 and \mathcal{P}_ω are completely contractive, it follows that ψ is completely isometric. \square

Corollary 4.11. *If ϕ_0 is a minimal element of \mathcal{G} , then $\text{im}(\phi_0)$ is minimal among subspaces of $\ell^\infty(\Gamma)$ that are images of idempotent ucp Γ -equivariant maps.*

Proof. Let $\phi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma)$ be a ucp Γ -equivariant map such that $\text{im}(\phi)$ is contained in $\text{im}(\phi_0)$. Then the restriction of ϕ to $\text{im}(\phi_0)$ is a ucp Γ -equivariant map on $\text{im}(\phi_0)$, hence the identity map by Γ -rigidity (Proposition 4.10). This implies $\text{im}(\phi_0) \subseteq \text{im}(\phi)$. \square

Proposition 4.12. *The minimal image of a ucp Γ -equivariant idempotent on $\ell^\infty(\Gamma)$ is unique up to isomorphism (as a Γ - C^* -algebra).*

Proof. Suppose $\phi : \ell^\infty(\Gamma) \rightarrow \mathbf{X}$ and $\psi : \ell^\infty(\Gamma) \rightarrow \mathbf{Y}$ are ucp Γ -equivariant idempotents, and \mathbf{X} and \mathbf{Y} are minimal among subspaces of $\ell^\infty(\Gamma)$ that are images of ucp Γ -equivariant idempotents.

Then by Γ -rigidity (Proposition 4.10) the composition $\phi\psi$, when restricted to \mathbf{X} , is the identity map. Similarly, the restriction of $\psi\phi$ to \mathbf{Y} is the identity map. Hence $\psi : \mathbf{X} \rightarrow \mathbf{Y}$ is a Γ -isomorphism (of C^* -algebras). \square

Note that the construction above remains valid for any sub-class of Γ -invariant ucp maps, as long as it is closed under composition and pointwise weak*-limits.

The next fact is standard, but extremely useful; in fact injectivity is what makes the Γ -envelopes interesting.

Proposition 4.13. *Suppose ϕ_0 is an idempotent in \mathcal{G} . Then $\text{im}(\phi_0)$ is injective in the category of \mathbb{T} - C^* -algebras, i.e. for any \mathbb{T} - C^* -algebras A and B equipped with ucp \mathbb{T} -equivariant maps $\psi : A \rightarrow \text{im}(\phi_0)$ and $\iota : A \rightarrow B$, with ι completely isometric, there exists a ucp \mathbb{T} -equivariant map $\Psi : B \rightarrow \text{im}(\phi_0)$ such that $\psi = \Psi \circ \iota$.*

Proof. As $\text{im}(\phi_0) \subset \ell^\infty(\mathbb{T})$ and ϕ_0 is a ucp \mathbb{T} -equivariant projection, it is enough to show that $\ell^\infty(\mathbb{T})$ itself is injective as a \mathbb{T} - C^* -algebras. This fact is a special case of the next proposition. \square

An injective object in the category of \mathbb{T} - C^* -algebras is called \mathbb{T} -injective. Recall Remark 2.5.

Proposition 4.14. *Let \mathbb{T} be a discrete quantum group, and let H be a Hilbert space. The C^* -algebra $\ell^\infty(\mathbb{T}) \overline{\otimes} B(H)$ with the \mathbb{T} -action $\Delta \otimes \text{id}$ is \mathbb{T} -injective.*

Proof. Assume that we have \mathbb{T} - C^* -algebras A and B equipped with ucp \mathbb{T} -equivariant maps $\psi : A \rightarrow \ell^\infty(\mathbb{T}) \overline{\otimes} B(H)$ and $\iota : A \rightarrow B$, with ι completely isometric. Put $\phi := (\epsilon \otimes \text{id}) \circ \psi$, obtaining thus a ucp map $A \rightarrow B(H)$, which then extends to a ucp map $\tilde{\phi} : B \rightarrow B(H)$, by injectivity of $B(H)$. Then for any $b \in B$ define the map $\tilde{\psi}(b) : \ell^1(\mathbb{T}) \rightarrow B(H)$ by

$$\left(\tilde{\psi}(b) \right) (\omega) := \tilde{\phi}(b * \omega); \quad \omega \in \ell^1(\mathbb{T}).$$

The map $\tilde{\psi}(b)$ is obviously completely bounded, hence identifies with an element in $\ell^\infty(\mathbb{T}) \overline{\otimes} B(H)$ (see e.g. [ER00, Corollary 7.1.5]). Thus, we obtain the map $\tilde{\psi} : B \rightarrow \ell^\infty(\mathbb{T}) \overline{\otimes} B(H)$; it is straightforward to see $\tilde{\psi}$ is a ucp \mathbb{T} -equivariant map that extends ψ . \square

Corollary 4.15. *Every \mathbb{T} - C^* -algebra A embeds equivariantly into a \mathbb{T} -injective C^* -algebra.*

Proof. Let $\alpha : \mathbb{T} \curvearrowright A$ be an action of \mathbb{T} on A , and let A be faithfully represented on a Hilbert space H . Then $\alpha : A \rightarrow M(c_0(\mathbb{T}) \otimes A) \subset \ell^\infty(\mathbb{T}) \overline{\otimes} B(H)$ is a ucp \mathbb{T} -equivariant map, when we equip $\ell^\infty(\mathbb{T}) \overline{\otimes} B(H)$ with the \mathbb{T} -action $\Delta \otimes \text{id}$. \square

We are ready to define the central object of this paper.

Theorem 4.16. *Every discrete quantum group \mathbb{T} admits a unique (up to \mathbb{T} -isomorphism) universal \mathbb{T} -boundary $C(\partial_F(\mathbb{T}))$, in the sense that for any \mathbb{T} -boundary A there is a completely isometric ucp \mathbb{T} -equivariant map $A \rightarrow C(\partial_F(\mathbb{T}))$.*

We call $C(\partial_F(\mathbb{T}))$ the (algebra of continuous functions on the) Furstenberg boundary of \mathbb{T} .

Proof. Let $C(\partial_F(\mathbb{T}))$ be the the minimal image of a ucp \mathbb{T} -equivariant idempotent on $\ell^\infty(\mathbb{T})$, which is unique up to \mathbb{T} -isomorphism by Proposition 4.12. It follows from Proposition 4.10 that $C(\partial_F(\mathbb{T}))$ is indeed a \mathbb{T} -boundary.

Now, to show the universality property, suppose A is a \mathbb{T} -boundary, and let $\mathcal{P}_\nu : A \rightarrow \ell^\infty(\mathbb{T})$ be the Poisson transform associated to a state ν on A . Composing \mathcal{P}_ν with a ucp \mathbb{T} -equivariant idempotent $\ell^\infty(\mathbb{T}) \rightarrow C(\partial_F(\mathbb{T}))$ yields a ucp \mathbb{T} -equivariant map from A into $C(\partial_F(\mathbb{T})) \subset \ell^\infty(\mathbb{T})$, which is completely isometric by Proposition 4.3. \square

Corollary 4.17. *Any \mathbb{T} -equivariant ucp map between two \mathbb{T} -boundaries A and B is an injective $*$ -homomorphism, and any such map is unique.*

Proof. Suppose A and B are \mathbb{T} -boundaries and ϕ and ψ are \mathbb{T} -equivariant ucp maps from A to B . By Theorem 4.16 there are completely isometric \mathbb{T} -equivariant ucp maps $\iota_A : A \rightarrow C(\partial_F(\mathbb{T}))$ and $\iota_B : B \rightarrow C(\partial_F(\mathbb{T}))$. By \mathbb{T} -injectivity of $C(\partial_F(\mathbb{T}))$ we may extend $\iota_B \circ \phi \circ \iota_A^{-1} :$

$\iota_A(\mathbf{A}) \rightarrow \iota_B(\mathbf{B})$ and $\iota_B \circ \psi \circ \iota_A^{-1} : \iota_A(\mathbf{A}) \rightarrow \iota_B(\mathbf{B})$ to Γ -equivariant ucp maps on $C(\partial_F(\Gamma))$. By Γ -rigidity both extensions are the identity map, hence $\phi = \psi$ is completely isometric. Thus, it only remains to show ϕ is $*$ -homomorphism, and for this it is enough to show that $\iota_A(\mathbf{A})$ is a C^* -subalgebra of $C(\partial_F(\Gamma))$. By Proposition 4.14 we can extend $\alpha \circ \iota_A^{-1} : \iota_A(\mathbf{A}) \rightarrow \alpha(\mathbf{A})$ to a Γ -equivariant ucp map from $\Phi : C(\partial_F(\Gamma)) \rightarrow \ell^\infty(\Gamma) \overline{\otimes} B(\mathbf{H})$, where \mathbf{H} is a Hilbert space on which \mathbf{A} acts faithfully. The map Φ is completely isometric by Γ -essentiality, and hence a C^* -isomorphism between $C(\partial_F(\Gamma))$ and $\Phi(C(\partial_F(\Gamma)))$, where the latter gets its C^* -algebra structure from $\ell^\infty(\Gamma) \overline{\otimes} B(\mathbf{H})$ via the Choi-Effros product. Since $\alpha(\mathbf{A})$ is a subalgebra of $\ell^\infty(\Gamma) \overline{\otimes} B(\mathbf{H})$, it is closed under the Choi-Effros product, and hence a subalgebra of $\Phi(C(\partial_F(\Gamma)))$. This implies $\iota_A(\mathbf{A})$ is a C^* -subalgebra of $C(\partial_F(\Gamma))$. \square

Note that in particular by the above any Γ -boundary \mathbf{A} is identified with a Γ - C^* -subalgebra of Furstenberg boundary $C(\partial_F(\Gamma))$. We will use this fact later.

Proposition 4.18. *A discrete quantum group Γ is amenable if and only if its Furstenberg boundary $C(\partial_F(\Gamma))$ is trivial.*

Proof. Let Γ be an amenable discrete quantum group. Let $m \in \ell^\infty(\Gamma)$ be an invariant mean and ν a state on $C(\partial_F(\Gamma))$. Then $\eta = m \circ \mathcal{P}_\nu$ is easily seen to be an invariant state on $C(\partial_F(\Gamma))$. Then invariance of η implies that the range of the Poisson transform \mathcal{P}_η is the scalars. Hence $C(\partial_F(\Gamma)) = \mathbb{C}$.

Conversely, if $C(\partial_F(\Gamma)) = \mathbb{C}$, then the ucp Γ -equivariant idempotent $\phi : \ell^\infty(\Gamma) \rightarrow C(\partial_F(\Gamma))$ is obviously an invariant mean, and hence Γ is amenable. \square

We finish this section with providing a relatively explicit criterion allowing us to check that a given action of Γ is indeed a Γ -boundary action. It will be used in Section 7.

Theorem 4.19. *Let Γ be a discrete quantum group and let $\mu \in \ell^1(\Gamma)$ be a state. Suppose \mathbf{A} is a unital Γ - C^* -algebra that admits a unique μ -stationary state ν , and that the Poisson transform \mathcal{P}_ν is completely isometric. Then \mathbf{A} is a Γ -boundary.*

Proof. Recall that the Poisson boundary $\mathcal{H}_\mu := \{x \in \ell^\infty(\Gamma) : (\text{id} \otimes \mu) \circ \Delta(x) = x\}$ is the image of a ucp Γ -equivariant projection Φ acting on $\ell^\infty(\Gamma)$. The fact that ν is μ -stationary implies that \mathcal{P}_ν takes values in \mathcal{H}_μ . Suppose that we have any ucp Γ -equivariant map Ψ from \mathbf{A} to $\mathcal{H}_\mu \subset \ell^\infty(\Gamma)$. By Proposition 2.6 it must be a Poisson transform associated to a state $\omega \in S(\mathbf{A})$; and then Definition 2.7 implies that ω is μ -stationary. Hence $\Psi = \mathcal{P}_\omega$ is completely isometric.

Consider then any state $\omega \in S(\mathbf{A})$ and the map $\mathcal{P}_\omega : \mathbf{A} \rightarrow \ell^\infty(\Gamma)$. Then $\Phi \circ \mathcal{P}_\omega$ is a ucp Γ -equivariant map from \mathbf{A} to \mathcal{H}_μ ; hence by the above it is completely isometric. Thus \mathcal{P}_ω is also completely isometric, and \mathbf{A} is a Γ -boundary. \square

5. THE UNIQUE TRACE PROPERTY FOR UNIMODULAR DISCRETE QUANTUM GROUPS AND ITS GENERALIZATION

In this section we consider the problem of identifying discrete quantum groups Γ with the unique trace property, and more generally these whose reduced group C^* -algebras do not admit ad-invariant states. Similarly to the classical case, this problem is related to the faithfulness of the action $\Gamma \curvearrowright \partial_F \Gamma$.

By a result of Furman [Fur03], the kernel of the action $G \curvearrowright \partial_F G$ of a locally compact group G on its Furstenberg boundary is the amenable radical of G . The following theorem is a version of this result in the quantum setting.

Theorem 5.1. *The co-kernel \mathbf{N}_Γ of the action $\Gamma \curvearrowright C(\partial_F \Gamma)$ is the unique minimal relatively amenable Baa-j-Vaes subalgebra of $\ell^\infty(\Gamma)$. It is contained in every other relatively amenable Baa-j-Vaes subalgebra of $\ell^\infty(\Gamma)$.*

Proof. By Proposition 2.9 the co-kernel \mathbf{N}_Γ is a Baa-j-Vaes subalgebra of $\ell^\infty(\Gamma)$. Note that, in fact, \mathbf{N}_Γ is the von Neumann algebra generated by all different copies of $C(\partial_F \Gamma)$ in $\ell^\infty(\Gamma)$. By definition, there is an idempotent ucp Γ -equivariant map from $\ell^\infty(\Gamma)$ onto any copy of $C(\partial_F \Gamma)$. Thus, \mathbf{N}_Γ is relatively amenable. Now, let $\mathbf{M} \subset \ell^\infty(\Gamma)$ be a relatively amenable Baa-j-Vaes subalgebra, so that there is a Γ -equivariant ucp map $\Psi : \ell^\infty(\Gamma) \rightarrow \mathbf{M}$. Let ν be a state on $C(\partial_F \Gamma)$. Then $\Psi \circ \mathcal{P}_\nu : C(\partial_F \Gamma) \rightarrow \mathbf{M} \subset \ell^\infty(\Gamma)$ is a Γ -equivariant ucp map. Thus, by Proposition 2.6 there is a state ν' on $C(\partial_F \Gamma)$ such that $\Psi \circ \mathcal{P}_\nu = \mathcal{P}_{\nu'}$ and in particular $\mathcal{P}_{\nu'}(C(\partial_F \Gamma)) \subset \mathbf{M}$. Since $\Delta(\mathbf{M}) \subset \mathbf{M} \overline{\otimes} \mathbf{M}$, by Equation (2.5) we have

$$\mathcal{P}_{\mu * \nu'}(C(\partial_F \Gamma)) = (\text{id} \otimes \mu) \Delta(\mathcal{P}_{\nu'}(C(\partial_F \Gamma))) \subset \mathbf{M}$$

for all $\mu \in \ell^1(\Gamma)$. Since $\mathcal{P}_{\nu'}$ is isometric, Lemma 4.2 implies that the set

$$\{\mu * \nu' : \mu \in \mathbf{P}(\Gamma)\}$$

is weak* dense in the state space of $C(\partial_F \Gamma)$. Thus, it follows from Proposition 2.6 that $\{\nu'' * a : a \in C(\partial_F \Gamma)\} = \mathcal{P}_{\nu''}(C(\partial_F \Gamma)) \subset \mathbf{M}$ for all $\nu'' \in C(\partial_F \Gamma)^*$. Hence $\mathbf{N}_\Gamma \subset \mathbf{M}$. \square

Recall that the adjoint action of Γ on $C(\widehat{\Gamma})$ is given by the formula $\beta(x) = W^*(1 \otimes x)W$ for all $x \in C(\widehat{\Gamma})$. When we talk about $C(\widehat{\Gamma})$ as a Γ -space, this is the action we have in mind and we say for instance that a functional $\mu \in C(\widehat{\Gamma})^*$ is Γ -invariant if $(\text{id} \otimes \mu)(\beta(x)) = \mu(x)\mathbf{1}$ for all $x \in C(\widehat{\Gamma})$. Let us also recall that when W is viewed as an operator on $\ell^2(\Gamma) \otimes \ell^2(\Gamma)$ then $\Delta(y) = W^*(1 \otimes y)W$ for all $y \in \ell^\infty(\Gamma)$.

Lemma 5.2. *Let Γ be a discrete quantum group and $\mu \in C(\widehat{\Gamma})^*$ a state. Then μ is Γ -invariant iff it is a KMS-state for the scaling automorphism group $(\tau_t)_{t \in \mathbb{R}}$ (at the inverse temperature 1). In particular when Γ is unimodular, μ is Γ -invariant iff it is a trace. On the other hand if Γ is not unimodular, then the Haar state of Γ is not Γ -invariant.*

Proof. In this proof the symbols adorned with hats refer to Γ and these without hats to $\widehat{\Gamma}$. Thus we use the antipode \hat{S} of $\ell^\infty(\Gamma)$, which is defined in particular on *finitely supported* elements. More precisely, if p is a finite-rank central projection in $\ell^\infty(\Gamma)$ then $\hat{S}_p : x \rightarrow \hat{S}(px)$ is a well defined, bounded linear map on $\ell^\infty(\Gamma)$, and $\bar{p} := \hat{S}(p)$ is another finite-rank central projection in $\ell^\infty(\Gamma)$. We recall moreover that $(\hat{S}_p \otimes \text{id})(W) = (\bar{p} \otimes 1)W^*$ ([KV00, Proposition 8.3]). Since $S^2 = \tau_{-i}$ on finitely supported elements ([KV00, Proposition 5.22]) and $(\hat{\tau}_t \otimes \text{id})(W) = (\text{id} \otimes \tau_{-t})(W)$ for all $t \in \mathbb{R}$ (see before Proposition 8.25 in [KV00]) we have also $(\hat{S}_p \otimes \text{id})(W^*) = (\bar{p} \otimes \tau_i)(W^*)$.

Now let $\mu \in C(\widehat{\Gamma})^*$ be a $(\tau_t)_{t \in \mathbb{R}}$ -KMS state. By the KMS-property and antimultiplicativity of \hat{S} we can write, for any $x \in \ell^\infty(\Gamma)$ and p as above,

$$\begin{aligned} (5.1) \quad (\hat{S}_p \otimes \mu)(W^*(1 \otimes x)W) &= (\text{id} \otimes \mu)((\hat{S}_p \otimes \tau_i)(W)(\hat{S}_p \otimes \text{id})(W^*)(1 \otimes x)) \\ &= (\text{id} \otimes \mu)((\bar{p} \otimes \tau_i)(W^*)(\bar{p} \otimes \tau_i)(W)(1 \otimes x)) = \mu(x)\bar{p}. \end{aligned}$$

Applying \hat{S}^{-1} yields $(p \otimes \mu)\beta(x) = \mu(x)p$, hence the ad-invariance of μ .

On the other hand, assume that μ is ad-invariant. Then the computation above shows that (5.1) holds for all suitable x and p . Fix $\varphi \in \ell^1(\Gamma)$, a finite-rank central projection $p \in \ell^\infty(\Gamma)$

and consider $y = (\varphi \circ \hat{S}_p \otimes \text{id})(W) \in \text{Pol}(\widehat{\Gamma})$. Denote $(S \otimes \text{id})\Delta(y) = \sum y'_{(1)} \otimes y_{(2)}$ — the sum is naturally finite. Since $(\text{id} \otimes \Delta)(W) = W_{12}W_{13}$ and $(\text{id} \otimes S)(W) = W^*$ (we are now using the antipode of $C(\widehat{\Gamma})$) we have

$$\begin{aligned} \sum y'_{(1)} \otimes y_{(2)} &= (\varphi \circ \hat{S}_p \otimes \text{id} \otimes \text{id})(W_{12}^*W_{13}) \\ &= (\varphi \otimes \text{id} \otimes \text{id})((\hat{S}_p \otimes \text{id})(W)_{13}(\hat{S}_p \otimes \text{id})(W^*)_{12}). \end{aligned}$$

Hence, applying φ to the first equality in (5.1) yields that $\sum \mu(y'_{(1)}xy_{(2)}) = \sum \mu(\tau_i(y_{(2)})y'_{(1)}x)$.

Now take arbitrary elements $w, z \in \text{Pol}(\widehat{\Gamma})$. We know that $w \otimes z$ can be written as a finite sum of elements of the form $\sum y'_{(1)}x \otimes y_{(2)}$, where y is associated to some φ and p as above (cf e.g. [Wor87, Theorem 4.9]), and by bilinearity the above identity yields $\mu(wz) = \mu(\tau_i(z)w)$. Hence μ is $(\tau_t)_{t \in \mathbb{R}}$ -KMS. In the unimodular case $(\tau_t)_{t \in \mathbb{R}}$ is trivial, so that KMS states correspond to traces.

For the Haar state h we compute as above, using the corresponding modular group $(\sigma_t)_{t \in \mathbb{R}}$ (again p is central finite rank projection and $x \in \ell^\infty(\Gamma)$):

$$\begin{aligned} (\hat{S}_p \otimes h)(W^*(1 \otimes x)W) &= (\text{id} \otimes h)((\hat{S}_p \otimes \sigma_i)(W)(\hat{S}_p \otimes \text{id})(W^*)(1 \otimes x)) \\ &= (\text{id} \otimes h)((\bar{p} \otimes \sigma_i)(W^*)(\bar{p} \otimes \tau_i)(W)(1 \otimes x)). \end{aligned}$$

This is equal to $h(x)\bar{p}$ for any $x \in \ell^\infty(\Gamma)$ **iff** $(\bar{p} \otimes \sigma_i)(W^*)(\bar{p} \otimes \tau_i)(W) = \bar{p}$ **iff** $(\bar{p} \otimes \tau_i)(W) = (\bar{p} \otimes \sigma_i)(W)$, which happens for all central finite rank projections p **iff** $\tau_i = \sigma_i$, i.e. when we are in the Kac case. \square

Theorem 5.3. *Let Γ be a discrete quantum group and assume that the action $\Gamma \curvearrowright \mathbf{A}$ of Γ on some Γ -boundary \mathbf{A} is faithful. Then if Γ is unimodular, it has the unique trace property, i.e. $C(\widehat{\Gamma})$ admits a unique tracial state. If Γ is not unimodular, then it does not admit any Γ -invariant functional, nor any KMS-state for the scaling automorphism group at the inverse temperature 1.*

Proof. Note first that if \mathbf{A} is a Γ -boundary and the action of Γ on \mathbf{A} is faithful, so is the action of Γ on its Furstenberg boundary $C(\partial_F(\Gamma))$, as we have $\mathbf{N}_{\alpha_{\mathbf{A}}} \subset \mathbf{N}_{\alpha_F}$, where $\alpha_{\mathbf{A}}$ denotes the action of Γ on \mathbf{A} and α_F the action of Γ on $C(\partial_F(\Gamma))$. Indeed, it suffices to use Theorem 4.16 and note that if $\Psi : \mathbf{A} \rightarrow C(\partial_F(\Gamma))$ is an isometric ucp Γ -invariant map, then for any functional $\nu \in \mathbf{A}^*$ and $a \in \mathbf{A}$ we have

$$(\text{id} \otimes \nu)(\alpha_{\mathbf{A}}(a)) = (\text{id} \otimes \rho)(\alpha_F(\Psi(a))) \in \mathbf{N}_{\alpha_F},$$

where ρ is any functional on $C(\partial_F(\Gamma))$ extending the functional $\nu \circ \Psi^{-1} \in \Psi(\mathbf{A})^*$. Thus we may and do assume that $\mathbf{A} = C(\partial_F(\Gamma))$.

We want to show that if $\tau \in C(\widehat{\Gamma})^*$ is a Γ -invariant state on $C(\widehat{\Gamma})$, then $(\text{id} \otimes \tau)(W) \in \ell^\infty(\Gamma)$ is the support of the counit $\varepsilon : \ell^\infty(\Gamma) \rightarrow \mathbb{C}$.

In view of Lemma 2.11, we can extend τ to a ucp Γ -equivariant map $\tilde{\tau} : \Gamma \rtimes_r C(\partial_F(\Gamma)) \rightarrow C(\partial_F(\Gamma))$, using Γ -injectivity of $C(\partial_F(\Gamma))$. By Γ -rigidity of $C(\partial_F(\Gamma))$ (see Proposition 4.10) the restriction of $\tilde{\tau}$ to $\alpha(C(\partial_F(\Gamma)))$ equals α^{-1} . Thus, $\alpha(C(\partial_F(\Gamma)))$ lies in the multiplicative domain of $\tilde{\tau}$, and therefore we have

$$\tilde{\tau}(\alpha(a)(y \otimes 1)) = a\tau(y)$$

for all $a \in C(\partial_F(\Gamma))$ and $y \in C(\widehat{\Gamma})$.

Consider the action β from (2.1). Applying $(\text{id} \otimes \tilde{\tau})$ to the equation $(1 \otimes x)(W \otimes 1) = (W \otimes 1)\beta(x)$, with $x = \alpha(a)$, $a \in C(\partial_F(\Gamma))$ and remembering that β restricted to $\alpha(C(\partial_F(\Gamma)))$ coincides with $\text{id} \otimes \alpha$, we get

$$((\text{id} \otimes \tau)(W)) \otimes a = (((\text{id} \otimes \tau)(W)) \otimes 1)\alpha(a).$$

In particular, setting $x = (\text{id} \otimes \tau)(W)$ we have $\mu(a)x = x(\mu * a)$ for all $\mu \in C(\partial_F(\Gamma))^*$ and $a \in C(\partial_F(\Gamma))$. Since $\mu(a) = \varepsilon(\mu * a)$ this yields $\varepsilon(y)x = xy$ for all $y \in \mathbf{N}_\alpha$. By faithfulness of α we can take for y the support p_0 of ε and we see that $x = \varepsilon(x)p_0$. Since $(\varepsilon \otimes \text{id})(W) = 1$ we conclude that $x = p_0$; thus τ is the Haar state.

The rest of the statements follows then from Lemma 5.2; note that in the unimodular case the Haar state is tracial, and in the non-unimodular case the Haar state is not Γ -invariant. \square

Finally the next proposition gives a connection to the classical result of Furman mentioned in the beginning of the section. By combining Theorem 5.3 and the next proposition we see that faithfulness of the action on the Furstenberg boundary implies, in the unimodular case, triviality of the amenable radical. This also results from Theorem 5.1, but we do not know whether the converse holds since in principle there might be relatively amenable Baaĵ-Vaes subalgebras of $\ell^\infty(\Gamma)$ which do not arise from normal amenable subgroups.

Proposition 5.4. *Let Γ be a unimodular discrete quantum group with the unique trace property, and let $\mathbb{G} = \widehat{\Gamma}$. If $\mathbb{H} \subset \mathbb{G}$ is such that $\mathbb{H} \setminus \mathbb{G}$ is co-amenable then $\mathbb{H} = \mathbb{G}$; in particular the amenable radical of Γ is trivial.*

Proof. Suppose that \mathbb{H} is a subgroup of \mathbb{G} realized by $\pi \in \text{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}))$ such that $\mathbb{H} \setminus \mathbb{G}$ is co-amenable. Then by Theorem 3.11 the map π admit a reduced version $\tilde{\pi} \in \text{Mor}(C(\mathbb{G}), C(\mathbb{H}))$. If $\mathbb{H} \subset \mathbb{G}$ is a proper subgroup then $\tilde{\pi}$ is not injective and therefore $h_{\mathbb{H}} \circ \tilde{\pi}$ is a trace on $C(\mathbb{G})$ which is not faithful. This proves the first statement. To prove the second one, suppose $\mathbb{A} \subset \Gamma$ is a non-trivial normal amenable subgroup. Then $\mathbb{H} = \widehat{\mathbb{A}/\Gamma}$ is a subgroup of \mathbb{G} such that $\mathbb{H} \setminus \mathbb{G}$ is co-amenable. Thus, we have $\mathbb{H} = \mathbb{G}$, which implies $\mathbb{A} = \{e\}$. \square

6. FURTHER APPLICATIONS: C^* -SIMPLICITY AND CROSSED PRODUCTS

In this section we discuss some applications of the concepts discussed earlier to simplicity and existence of nuclear embeddings. These will find concrete interpretations in the last section.

Recall first that a discrete quantum group is said to be C^* -simple if the reduced C^* -algebra $C(\widehat{\Gamma})$ of the dual compact quantum group $\widehat{\Gamma}$ is simple.

Proposition 6.1. *Let Γ be a C^* -simple discrete quantum group, and let $\mathbb{G} = \widehat{\Gamma}$. If $\mathbb{H} \subset \mathbb{G}$ is a closed quantum subgroup such that $\mathbb{H} \setminus \mathbb{G}$ is co-amenable then $\mathbb{H} = \mathbb{G}$. In particular, Γ has trivial amenable radical.*

Proof. Let \mathbb{H} be a quantum subgroup of \mathbb{G} such that $\mathbb{H} \setminus \mathbb{G}$ is co-amenable. Then by Theorem 3.11 there is a canonical $*$ -homomorphism $\tilde{\pi} : C(\mathbb{G}) \rightarrow C(\mathbb{H})$. Since Γ is C^* -simple, i.e. $C(\mathbb{G})$ is simple, $\tilde{\pi}$ is injective, hence $\mathbb{H} = \mathbb{G}$. This proves the first statement. The second follows exactly as in Corollary 5.4. \square

Lemma 6.2. *Let \mathbb{H} be a Hilbert space. Consider the action $(\Delta \otimes \text{id})$ of Γ on $B(\ell^2(\Gamma) \otimes \mathbb{H})$. Suppose ψ is a ucp map on $B(\ell^2(\Gamma) \otimes \mathbb{H})$ whose restriction to $C(\widehat{\Gamma}) \otimes 1$ is the identity map. Then ψ is Γ -equivariant.*

Proof. The ucp map $\text{id} \otimes \psi$ on $B(\ell^2(\Gamma) \otimes \ell^2(\Gamma) \otimes \mathbf{H})$ restricts to the identity map on $c_0(\Gamma) \otimes C(\widehat{\Gamma}) \otimes \mathbb{1}$, and therefore contains the latter in its multiplicative domain. Thus, it follows that $\text{id} \otimes \psi$ also restricts to identity on $M(c_0(\Gamma) \otimes C(\widehat{\Gamma}) \otimes \mathbb{1})$. In particular, $W \otimes \mathbb{1}$ is in the multiplicative domain of $\text{id} \otimes \psi$. Hence, for every $x \in B(\ell^2(\Gamma) \otimes \mathbf{H})$ we get

$$\begin{aligned} (\text{id} \otimes \psi)(\Delta \otimes \text{id})(x) &= (\text{id} \otimes \psi)(W_{12}^* x_{23} W_{12}) = W_{12}^* ((\text{id} \otimes \psi)(x_{23})) W_{12} \\ &= W_{12}^* \psi(x)_{23} W_{12} = (\Delta \otimes \text{id})(\psi(x)). \end{aligned}$$

□

Before we formulate the next theorem, we recall that Γ acts canonically on $\ell^\infty(\Gamma)$ (by the coproduct, viewed as the right action), and also on $B(\ell^2(\Gamma))$; the latter right action, denoted by Δ^r , is given by the same implementation, namely

$$\Delta^r(x) = V(x \otimes \mathbb{1})V^*, \quad x \in B(\ell^2(\Gamma)),$$

with V denoting again the right multiplicative unitary. We note here that there exists a (necessarilly unique) Γ -equivariant faithful normal conditional expectation $\mathbb{E} : B(\ell^2(\Gamma)) \rightarrow \ell^\infty(\Gamma)$. It is given by the formula

$$\mathbb{E}(x) = (\omega_\Omega \otimes \text{id}) \circ \Delta^r(x), \quad x \in B(\ell^2(\Gamma)),$$

where Ω is the unit vector in $\ell^2(\Gamma)$ corresponding to the counit of $\ell^\infty(\Gamma)$. The fact that \mathbb{E} is a Γ -equivariant ucp idempotent with image $\ell^\infty(\Gamma)$ is easy to check. In order to justify the faithfulness of \mathbb{E} we invoke the crossed product construction $\Gamma \rtimes \ell^\infty(\Gamma) = [(L^\infty(\widehat{\Gamma}) \otimes \mathbb{1})\Delta(\ell^\infty(\Gamma))]''$ and the canonical faithful conditional expectation $\mathbb{E}_{\Gamma \rtimes \ell^\infty(\Gamma)} : \Gamma \rtimes \ell^\infty(\Gamma) \rightarrow \Delta(\ell^\infty(\Gamma))$ as described in Subsection 2.2. The latter is uniquely characterised by two conditions:

$$\begin{aligned} \mathbb{E}_{\Gamma \rtimes \ell^\infty(\Gamma)}(x \otimes \mathbb{1}) &= h(x) \\ \mathbb{E}_{\Gamma \rtimes \ell^\infty(\Gamma)}(\Delta(y)) &= \Delta(y) \end{aligned}$$

for all $x \in L^\infty(\widehat{\Gamma})$ and $y \in \ell^\infty(\Gamma)$, where $h : L^\infty(\widehat{\Gamma}) \rightarrow \mathbb{C}$ is the Haar state on the compact quantum group $\widehat{\mathbb{G}}$. Noting that

$$(x \otimes \mathbb{1})\Delta(y) = (x \otimes \mathbb{1})V(y \otimes \mathbb{1})V^* = V(xy \otimes \mathbb{1})V^*$$

we get the standard identification of $\Gamma \rtimes \ell^\infty(\Gamma)$ with $[L^\infty(\widehat{\Gamma})\ell^\infty(\Gamma)]'' = B(\ell^2(\Gamma))$. It is easy to check that $\mathbb{E}(x) = h(x)\mathbb{1}$ and $\mathbb{E}(y) = y$ for all $x \in L^\infty(\widehat{\Gamma})$ and $y \in \ell^\infty(\Gamma)$; thus under the identification $\Gamma \rtimes \ell^\infty(\Gamma) \cong B(\ell^2(\Gamma))$ both conditional expectations coincide. In particular \mathbb{E} is faithful.

We also recall that an inclusion of operator systems $\mathbf{V} \subset \mathbf{W}$ is said to be essential if a ucp map ϕ from \mathbf{W} into any operator system is completely isometric when its restriction to \mathbf{V} is completely isometric. We will use the facts that $\mathbf{V} \subset \mathbf{W}$ is essential iff $\mathbf{W} \subset I(\mathbf{V})$ and that any essential operator system inclusion of C^* -algebras is a C^* -inclusion (see [Ham79] for these facts and more details).

In the proof of the following theorem we will make use of the notion of Fubini tensor product $X \otimes_{\mathcal{F}} Y$ of two operator systems X and Y . For relevant definitions and properties we refer the reader to [Tom67] or to [ER00, Chapter 7].

Theorem 6.3. *Let Γ be a discrete quantum group, and let \mathbf{A} be a Γ - C^* -algebra. Then \mathbf{A} is a Γ -boundary if and only if*

$$(6.1) \quad \Gamma \rtimes_r \mathbf{A} \subset I(C(\widehat{\Gamma})).$$

Proof. Suppose \mathbf{A} is a Γ - C^* -algebra such that $\Gamma \rtimes_r \mathbf{A} \subset I(C(\widehat{\Gamma}))$. Let $\phi : \mathbf{A} \rightarrow \ell^\infty(\Gamma)$ be a ucp Γ -equivariant map. Let \mathbf{K} be a Hilbert space on which \mathbf{A} acts. Extend ϕ to a ucp map from $B(\mathbf{K})$ to $\ell^\infty(\Gamma)$, still denoted by ϕ . The ucp map $\text{id} \otimes \phi : B(\ell^2(\Gamma)) \otimes B(\mathbf{K}) \rightarrow B(\ell^2(\Gamma)) \otimes \ell^\infty(\Gamma)$ restricts to identity on $C(\widehat{\Gamma}) \otimes \mathbf{1}$, hence restricts to a completely isometric map $\Gamma \rtimes_r \mathbf{A} \rightarrow B(\ell^2(\Gamma)) \otimes \ell^\infty(\Gamma)$ by essentiality. Denote by α the action of Γ on \mathbf{A} . Since ϕ is Γ -equivariant, for all $T \in M_n(\mathbb{C}) \otimes \mathbf{A}$ and $n \in \mathbb{N}$ we get

$$\|(\text{id}_{M_n(\mathbb{C})} \otimes \phi)(T)\| = \|(\text{id} \otimes \Delta)(\text{id}_{M_n(\mathbb{C})} \otimes \phi)(T)\| = \|(\text{id}_{M_n(\mathbb{C})} \otimes \text{id} \otimes \phi)(\text{id}_{M_n(\mathbb{C})} \otimes \alpha)(T)\| = \|T\|,$$

which shows that ϕ is completely isometric. Hence \mathbf{A} is a Γ -boundary.

Conversely, suppose \mathbf{A} is a Γ -boundary. Let $\iota : \mathbf{A} \rightarrow C(\partial_F(\Gamma))$ be a Γ -equivariant completely ucp map. Then $\text{id} \otimes \iota : \Gamma \rtimes_r \mathbf{A} \rightarrow \Gamma \rtimes_r C(\partial_F(\Gamma))$ is an operator system inclusion. Hence, it is enough to show that $\Gamma \rtimes_r C(\partial_F(\Gamma)) \subset I(C(\widehat{\Gamma}))$.

Since $C(\partial_F(\Gamma))$ is Γ -injective there is a ucp Γ -equivariant map $\psi : \ell^\infty(\Gamma) \otimes_{\mathcal{F}} C(\partial_F(\Gamma)) \rightarrow C(\partial_F(\Gamma))$ such that $\psi \circ \Delta$ is the identity map on $C(\partial_F(\Gamma))$. Consider the map $\Phi = (\text{id} \otimes \psi) \circ (\Delta^r \otimes \text{id})$ on $B(\ell^2(\Gamma)) \otimes_{\mathcal{F}} C(\partial_F(\Gamma))$, where as above Δ^r denotes the canonical right action of Γ on $B(\ell^2(\Gamma))$. We have

$$\begin{aligned} \Phi \circ \Phi &= (\text{id} \otimes \psi) \circ (\Delta^r \otimes \text{id}) \circ (\text{id} \otimes \psi) \circ (\Delta^r \otimes \text{id}) \\ &= (\text{id} \otimes \psi) \circ (\text{id} \otimes \text{id} \otimes \psi) \circ (\Delta^r \otimes \text{id} \otimes \text{id}) \circ (\Delta^r \otimes \text{id}) \\ &= (\text{id} \otimes \psi) \circ (\text{id} \otimes \text{id} \otimes \psi) \circ (\text{id} \otimes \Delta \otimes \text{id}) \circ (\Delta^r \otimes \text{id}) \\ &= (\text{id} \otimes \psi) \circ (\text{id} \otimes \Delta) \circ (\text{id} \otimes \psi) \circ (\Delta^r \otimes \text{id}) \\ &= (\text{id} \otimes \psi \circ \Delta) \circ (\text{id} \otimes \psi) \circ (\Delta^r \otimes \text{id}) \\ &= (\text{id} \otimes \psi) \circ (\Delta^r \otimes \text{id}) \\ &= \Phi, \end{aligned}$$

which shows that Φ is a projection. The C^* -algebra $C(\partial_F(\Gamma))$ is injective, as $\ell^\infty(\Gamma)$ is a direct product of matrix algebras, and $C(\partial_F(\Gamma))$ is the image of a ucp projection on $\ell^\infty(\Gamma)$. Thus $B(\ell^2(\Gamma)) \otimes_{\mathcal{F}} C(\partial_F(\Gamma))$ is also injective, for example by comments in [Pau11, Section 4]. Hence finally $\text{im}(\Phi)$ is injective. Furthermore, for $a \in C(\partial_F(\Gamma))$ and $\hat{x} \in C(\widehat{\Gamma})$ we have

$$\begin{aligned} \Phi((\hat{x} \otimes 1)\Delta(a)) &= (\text{id} \otimes \psi)((\Delta^r(\hat{x}) \otimes 1)(\Delta \otimes \text{id})\Delta(a)) \\ &= (\hat{x} \otimes 1)(\text{id} \otimes \psi \circ \Delta)(\Delta(a)) \\ &= (\hat{x} \otimes 1)\Delta(a), \end{aligned}$$

which shows that $\Gamma \rtimes_r C(\partial_F(\Gamma)) \subset \text{im}(\Phi)$.

So, in particular, $C(\widehat{\Gamma}) \otimes \mathbf{1} \subset \text{im}(\Phi)$ and by the afore-mentioned injectivity of $\text{im}(\Phi)$ we have an inclusion $I(C(\widehat{\Gamma}) \otimes \mathbf{1}) \subseteq \text{im}(\Phi)$. Let $\phi : \text{im}(\Phi) \rightarrow I(C(\widehat{\Gamma}) \otimes \mathbf{1})$ be a projection. Since ϕ restricted to $C(\widehat{\Gamma}) \otimes \mathbf{1}$ is the identity map, if we show that the restriction of ϕ to $\Delta(C(\partial_F(\Gamma)))$ is also the identity map then it finishes the proof (by multiplicative domain arguments, standard properties of the crossed product, and the obvious identification $I(C(\widehat{\Gamma}) \otimes \mathbf{1}) = I(C(\widehat{\Gamma}))$).

Extend ϕ to a ucp map $\phi : B(\ell^2(\Gamma) \otimes \mathbf{H}) \rightarrow I(C(\widehat{\Gamma}) \otimes \mathbf{1}) \subset B(\ell^2(\Gamma) \otimes \mathbf{H})$, where \mathbf{H} is a Hilbert space on which the C^* -algebra $C(\partial_F(\Gamma))$ acts faithfully. Consider the action $(\Delta \otimes \text{id})$ of Γ on $B(\ell^2(\Gamma) \otimes \mathbf{H})$. Then ϕ is Γ -equivariant by Lemma 6.2.

Denote by $\mathbb{E} : B(\ell^2(\Gamma)) \rightarrow \ell^\infty(\Gamma)$ the canonical conditional expectation, which is faithful and Γ -equivariant by the remarks before the theorem. Then for any $T \in B(\ell^2(\Gamma)) \otimes_{\mathcal{F}} C(\partial_F(\Gamma))$

we have

$$\begin{aligned}
\Delta((\varepsilon \otimes \text{id})(\mathbb{E} \otimes \text{id})\Phi(T)) &= \Delta((\varepsilon \otimes \text{id})(\mathbb{E} \otimes \text{id})(\text{id} \otimes \psi)(\Delta^r \otimes \text{id})(T)) \\
&= \Delta((\varepsilon \otimes \text{id})(\text{id} \otimes \psi)(\mathbb{E} \otimes \text{id} \otimes \text{id})(\Delta^r \otimes \text{id})(T)) \\
&= \Delta((\varepsilon \otimes \text{id})(\text{id} \otimes \psi)(\Delta \otimes \text{id})(\mathbb{E} \otimes \text{id})(T)) \\
&= \Delta((\varepsilon \otimes \text{id})(\Delta \circ \psi)(\mathbb{E} \otimes \text{id})(T)) \\
&= (\Delta \circ \psi)(\mathbb{E} \otimes \text{id})(T) \\
&= (\mathbb{E} \otimes \text{id})\Phi(T),
\end{aligned}$$

which shows that the conditional expectation $(\mathbb{E} \otimes \text{id})$ maps $\text{im}(\Phi)$ onto $\Delta(C(\partial_F(\Gamma)))$.

Thus, the composition of $(\mathbb{E} \otimes \text{id})$ with the restriction of ϕ to $\Delta(C(\partial_F(\Gamma)))$ yields the ucp Γ -equivariant map $(\mathbb{E} \otimes \text{id}) \circ \phi|_{\Delta(C(\partial_F(\Gamma)))} : \Delta(C(\partial_F(\Gamma))) \rightarrow \Delta(C(\partial_F(\Gamma)))$, which by Proposition 4.10 must be the identity map. Since the conditional expectation $(\mathbb{E} \otimes \text{id})$ is faithful, it follows that ϕ is identity on $\Delta(C(\partial_F(\Gamma)))$. \square

We now have all we need to generalize the main result of [KK17], which led to applications of the Furstenberg boundary action in C^* -simplicity problems.

The argument below follows that of [KK17, Theorem 6.2], with several necessary modifications.

Theorem 6.4. *Let Γ be a discrete quantum group. Then the following are equivalent:*

- (1) $C(\hat{\Gamma})$ is simple;
- (2) $\Gamma \rtimes_r A$ is simple for every Γ -boundary A ;
- (3) $\Gamma \rtimes_r A$ is simple for some Γ -boundary A ;
- (4) $\Gamma \rtimes_r C(\partial_F(\Gamma))$ is simple.

Proof. The implications (1) \Rightarrow (2) and (3) \Rightarrow (4) follow from Theorem 6.3, Corollary 4.17, and the general fact that if we are given a C^* -inclusion $B \subset C \subset I(B)$ and B is a simple C^* -algebra, then so is C . Indeed suppose that $J \subset C$ is an ideal. Since B is simple the map $B \rightarrow C/J$ is injective. Therefore there is a ucp map $\gamma : C/J \rightarrow I(B)$ that extends the canonical embedding $B \rightarrow I(B)$. Consider the induced map $\tilde{\gamma} : C \rightarrow I(B)$. Since C embeds into $I(B)$ there is an extension of $\tilde{\gamma}$ to $\tilde{\tilde{\gamma}} : I(B) \rightarrow I(B)$. Noting that the restriction of $\tilde{\tilde{\gamma}}$ to B is completely isometric we see that the map $\tilde{\tilde{\gamma}}$ must be isometric (we use here the enveloping property of $I(B)$). In particular $\tilde{\tilde{\gamma}}$ has trivial kernel and hence $J = \{0\}$.

The implication (2) \Rightarrow (3) is trivial, so it remains to prove (4) \Rightarrow (1). Suppose that $C(\hat{\Gamma})$ is not simple, let J be a non-trivial ideal of $C(\hat{\Gamma})$, and let $\pi : C(\hat{\Gamma}) \rightarrow C(\hat{\Gamma})/J$ denote the corresponding quotient map. Recall that the action β of Γ on $C(\hat{\Gamma})$ is implemented by (the left) multiplicative unitary $W \in M(c_0(\Gamma) \otimes C(\hat{\Gamma}))$, i.e.

$$\beta(x) = W^*(1 \otimes x)W, \quad x \in C(\hat{\Gamma}).$$

It is then easy to see that β induces an action $\tilde{\beta}$ on the quotient $C(\hat{\Gamma})/J$:

$$\tilde{\beta}(\pi(x)) := (\text{id} \otimes \pi)\beta(x) \quad x \in C(\hat{\Gamma}).$$

Observe that π is Γ -equivariant by definition.

By Corollary 4.15 the Γ - C^* -algebra $C(\hat{\Gamma})/J$ embeds Γ -equivariantly into a Γ -injective C^* -algebra, which we will denote by B .

Let $\tilde{\pi} : \Gamma \rtimes_r C(\partial_F(\Gamma)) \rightarrow B$ be a Γ -equivariant ucp extension of π . Note that the restriction of $\tilde{\pi}$ to $C(\partial_F(\Gamma))$ (or rather strictly speaking to its copy in the crossed product, which we

will identify with the original object), is isometric by the rigidity established in Proposition 4.3. Hence, when endowed with the corresponding Choi-Effros product, $\tilde{\pi}(C(\partial_F(\Gamma)))$ is a Γ -injective C^* -algebra isomorphic to $C(\partial_F(\Gamma))$.

The last fact implies that there is a Γ -equivariant projection $\phi : \mathbb{B} \rightarrow \tilde{\pi}(C(\partial_F(\Gamma)))$. Its restriction to the C^* -algebra $C^*(\tilde{\pi}(C(\partial_F(\Gamma))))$ is easily seen to be a Γ -equivariant surjective $*$ -homomorphism. Let I denote its kernel, so that I is a Γ -invariant ideal in $C^*(\tilde{\pi}(C(\partial_F(\Gamma))))$, and define $\tilde{I} = I \tilde{\pi}(C(\hat{\Gamma}))$. We will often use below the fact that $\Gamma \rtimes_r C(\partial_F(\Gamma))$ is spanned (as a normed space) by the products of elements in $C(\hat{\Gamma})$ and in $C(\partial_F(\Gamma))$, and by the multiplicative domain argument we have also that $\tilde{\pi}(\Gamma \rtimes_r C(\partial_F(\Gamma)))$ is spanned by the products of elements in $C(\hat{\Gamma})/J$ and in $\tilde{\pi}(C(\partial_F(\Gamma)))$.

Observing that I is contained in the multiplicative domain of ϕ we conclude that for $x \in \tilde{I}$ we have $\phi(x) = 0$. In particular since $\phi|_{C(\partial_F(\Gamma))} = \text{id}_{C(\partial_F(\Gamma))}$ we see that $\tilde{I} \cap \tilde{\pi}(C(\partial_F(\Gamma))) = \{0\}$, thus \tilde{I} is strictly contained in $C^*(\tilde{\pi}(\Gamma \rtimes_r C(\partial_F(\Gamma))))$.

Let us show that \tilde{I} is an ideal in $C^*(\tilde{\pi}(\Gamma \rtimes_r C(\partial_F(\Gamma))))$; for that it suffices to show that $I \tilde{\pi}(C(\hat{\Gamma})) = \tilde{\pi}(C(\hat{\Gamma}))I$. As $\tilde{\pi}$ is a ucp Γ -equivariant map, and Γ acts on the copy of $C(\partial_F(\Gamma))$ in the crossed product by the formula

$$\alpha(y) = W_{12}^*(1 \otimes y)W_{12}, \quad y \in C(\partial_F(\Gamma)) \subset \Gamma \rtimes C(\partial_F(\Gamma)) \subset M(\mathcal{K}(\ell^2(\Gamma)) \otimes C(\partial_F(\Gamma))),$$

the action of Γ on $\tilde{\pi}(C(\partial_F(\Gamma)))$ – hence also on $C^*(\tilde{\pi}(C(\partial_F(\Gamma))))$ – is implemented by the unitary operator $U = (\text{id} \otimes \tilde{\pi})(W) \in M(c_0(\Gamma) \otimes C(\hat{\Gamma})/J) \subset M(c_0(\Gamma) \otimes \mathbb{B})$. Further we have the equality $\tilde{\pi}(C(\hat{\Gamma})) = \overline{\{(\omega \otimes \text{id})(U) : \omega \in \ell^1(\Gamma)\}}$. For any $z \in I$, $\omega \in \ell^1(\Gamma)$ and $b \in c_0(\Gamma)$ we thus note that

$$\begin{aligned} z[(b\omega \otimes \text{id})(U)] &= (b\omega \otimes \text{id})(UU^*(1 \otimes z)U) = (\omega \otimes \text{id})(UU^*(1 \otimes z)U(b \otimes 1)) \\ &= (\omega \otimes \text{id})(U(\alpha(z)(b \otimes 1))) \in \overline{\tilde{\pi}(C(\hat{\Gamma}))}I, \end{aligned}$$

where in the last equality we use the Γ -invariance of I and the Podleś condition for the restricted action. This now implies that $I \tilde{\pi}(C(\hat{\Gamma})) = \tilde{\pi}(C(\hat{\Gamma}))I$.

Finally consider the quotient map $q : C^*(\tilde{\pi}(\Gamma \rtimes_r C(\partial_F(\Gamma)))) \rightarrow C^*(\tilde{\pi}(\Gamma \rtimes_r C(\partial_F(\Gamma)))/\tilde{I}$ and the composition $q \circ \tilde{\pi} : \Gamma \rtimes_r C(\partial_F(\Gamma)) \rightarrow C^*(\tilde{\pi}(\Gamma \rtimes_r C(\partial_F(\Gamma)))/\tilde{I}$. The latter is a priori just a ucp map, but we claim it is in fact multiplicative. Indeed, note that since $\tilde{\pi}$ is multiplicative on $C(\hat{\Gamma})$, so is $q \circ \tilde{\pi}$. Also, it follows from the construction that I (as the kernel of ϕ) contains all the elements of the form $\tilde{\pi}(xy) - \tilde{\pi}(x)\tilde{\pi}(y)$, $x, y \in C(\partial_F(\Gamma))$. Hence $q \circ \tilde{\pi}$ is also multiplicative on $C(\partial_F(\Gamma))$, so that another multiplicative domain argument yields the desired fact.

Now $q \circ \tilde{\pi}$ vanishes on J , which was assumed to be non-trivial and on the other hand is a non-zero map, as the target algebra is non-zero. This implies the crossed product $\Gamma \rtimes_r C(\partial_F(\Gamma))$ is not simple. \square

7. BOUNDARY ACTIONS FOR FREE ORTHOGONAL QUANTUM GROUPS

In this section we give concrete examples of boundary actions for free orthogonal quantum groups.

7.1. Free orthogonal quantum groups. We denote by $\mathbb{F}O_Q$ Van Daele and Wang's free orthogonal discrete quantum group, with $Q \in M_N(\mathbb{C})$ such that $Q\bar{Q} = \pm I_N$, $N \geq 2$. When $Q = I_N$, we write also $\mathbb{F}O_N$ instead $\mathbb{F}O_Q$. The discrete quantum group in question is defined

via the full Woronowicz C^* -algebra $A_o(Q) = C_u^*(\mathbb{F}O_Q) = C^u(O_Q^+)$, so that we have $\widehat{\mathbb{F}O_Q} = O_Q^+$. The corresponding reduced C^* -algebra is $C_r^*(\mathbb{F}O_Q) = C(O_Q^+)$.

Following Banica's classification of the irreducible representations of $\widehat{\mathbb{F}O_Q}$, we denote by H_n ($n \in \mathbb{N}_0$) the spaces of the irreducible representations (up to equivalence), in such a way that $H_0 = \mathbb{C}$, $H_1 = \mathbb{C}^N$ and $H_1 \otimes H_n \simeq H_{n-1} \oplus H_{n+1}$ equivariantly. Even more precisely we consider H_n as a (uniquely defined) subspace of $H_1^{\otimes n}$ and denote by $P_n^+ \in B(H_1^{\otimes n})$ the corresponding orthogonal projection. We write $t \subset r \otimes s$ if there is an equivariant embedding $H_t \subset H_r \otimes H_s$, this happens exactly for $t = r + s - 2a$ with $0 \leq a \leq \min(r, s)$.

The dual algebras are given by $A = c_0(\mathbb{F}O_Q) = \bigoplus_{n \in \mathbb{N}} A_n$ with $A_n = B(H_n)$, and $M = \ell^\infty(\mathbb{F}O_Q) = \prod_{n \in \mathbb{N}} B(H_n)$. All coproducts are denoted Δ . We denote by $p_n \in B(H_n) \subset M$ the minimal central projections and write $p_{\geq n} = \sum_{k \geq n} p_k \in M$.

Woronowicz' modular matrices are denoted $F_n \in B(H_n)_+$ and normalized by $\text{Tr}(F_n) = \text{Tr}(F_n^{-1})$; in particular $F_1 = {}^t(Q^*Q)$. We will use the standard and normalized categorical "traces" given for $a \in B(H_n)$ by $\text{qTr}_n(a) = \text{Tr}(F_n a)$ and $\text{qtr}_n(a) = \text{qTr}_n(a) / \dim_q(n)$, where $\dim_q(n) = \text{Tr}(F_n)$ is the quantum dimension of H_n . These dimensions are given by

$$\dim_q(n) = [n + 1]_q = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}$$

for a unique value $q \in]0, 1]$. Here we assume that $q < 1$, or equivalently, $Q \notin U_2$. Then there are constants $C_1, C_2 > 0$ (depending on q) such that $C_1 q^{-n} \leq \dim_q(n) \leq C_2 q^{-n}$ for all n .

Note that the identity $q + q^{-1} = \dim_q(1) = \text{Tr}(F_1)$, with $q^{-1} \geq 1$, resolves into $q^{-1} = f(\text{Tr}(F_1))$ where $f(t) = \frac{1}{2}(t + \sqrt{t^2 - 4})$. On the other hand it is known that the spectrum of F_1 is symmetric, so that $\text{Tr}(F_1) > \|F_1\| + \|F_1\|^{-1}$ when $N \geq 3$. Since f is strictly increasing, this yields $q^{-1} > \|F_1\|$. We will need later this fact under the form $q\|F_1\| < 1$, which is satisfied if and only if $N \geq 3$.

Denote by $t_n \in H_n \otimes H_n$ a fixed vector such that $\|t_n\|^2 = \dim_q(n)$. Such a vector realizes the self-duality of H_n as well as the quantum trace above, in the following sense: we have $\text{qTr}_n(a) = t_n^*(a \otimes \text{id})t_n$. According to the conjugate equation we have $(\text{id}_n \otimes \text{qTr}_n)(t_n t_n^*) = \text{id}_n$. On the other hand one can compute $(\text{qTr}_n \otimes \text{id}_n)(t_n t_n^*) = F_n^{-2}$ (to get the identity here one would need to use the right quantum trace). We shall moreover choose the maps t_n in a coherent way, by putting $t_{n+1} = (P_{n+1}^+ \otimes P_{n+1}^+)(\text{id} \otimes t_n \otimes \text{id})t_1$. We have then $t_{m+n} = (P_{m+n}^+ \otimes P_{m+n}^+)(\text{id} \otimes t_n \otimes \text{id})t_m$ for any $n, m \in \mathbb{N}_0$.

There are well-defined antilinear maps $j_n : H_n \rightarrow H_n$ such that $t_n = \sum_i e_i \otimes j_n(e_i)$ for any ONB $(e_i)_i$ of H_n . For $\xi \in H_n$ we denote also $\bar{\xi} = j_n(\xi) = (\xi^* \otimes \text{id})t_n$. Associated to t_n is the invariant form $s_n = \sum_i e_i^* \otimes (j_n^{-1} e_i)^*$ on $H_n \otimes H_n$. We have $F_n = j_n^* j_n$, $F_n^{-1} = j_n j_n^*$ and $j_n^2 = \pm \text{id}_n$, where the sign is the same as in the standing assumption $Q\bar{Q} = \pm I_N$. One can moreover check that $(F_n^p \otimes F_n^p)t_n = t_n$ for all $p \in \mathbb{R}$ and $j_{n \otimes n} t_n = \pm t_n$ where $j_{n \otimes n} = (j_n \otimes j_n)\sigma$. We will also use the identity

$$(7.1) \quad t_n = \pm \sigma(F_n^{-1} \otimes \text{id}_n)t_n.$$

Given $a \in B(H_n)$ one can consider the unique element $\tilde{a} \in B(H_n)$ satisfying $(a \otimes \text{id}_n)t_n = (\text{id} \otimes \tilde{a})t_n$. Explicitly we have $\tilde{a} = j_n a^* j_n^{-1}$ and $\tilde{a} = (s_n \otimes \text{id})(\text{id} \otimes a \otimes \text{id})(\text{id} \otimes t_n)$.

Recall that by our choice for the irreducible spaces H_n , the space H_{r+s} is a subspace of $H_r \otimes H_s$. From Banica's fusion rules we know that the orthogonal complement of H_{r+s} in $H_r \otimes H_s$ is isomorphic, as a corepresentation of $\mathbb{F}O_Q$, to $H_{r-1} \otimes H_{s-1}$. This isomorphism can

be realized by the intertwiner $(P_r^+ \otimes P_s^+)(\text{id} \otimes t_1 \otimes \text{id}) : \mathbf{H}_{r-1} \otimes \mathbf{H}_{s-1} \rightarrow \mathbf{H}_r \otimes \mathbf{H}_s$, which is injective although not isometric, see e.g. [Ver05, Proposition 2.3].

A related intertwiner is the map $(P_r^+ \otimes P_s^+)(\text{id} \otimes t_a \otimes \text{id})P_t^+ : \mathbf{H}_t \rightarrow \mathbf{H}_r \otimes \mathbf{H}_s$ for $t = r + s - 2a$, $0 \leq a \leq \min(r, s)$. We denote $\kappa_t^{r,s}$ the inverse of its norm, so that $V_{r,s}^t = \kappa_t^{r,s}(P_r^+ \otimes P_s^+)(\text{id} \otimes t_a \otimes \text{id})P_t^+$ is an isometric intertwiner. The quantity $\kappa_t^{r,s}$ has been studied by many authors: here we follow the notation of [FV16], in [BC18] it is denoted

$$\kappa_t^{r,s} = \|A_t^{r,s}\|^{-1} = \left(\frac{[t+1]_q}{\theta_q(t, r, s)} \right)^{1/2},$$

and in [Ver07] it appears in the proof of Lemma 4.8 in the form

$$(\kappa_t^{r,s})^{-2} = \frac{\dim_q(r)}{\dim_q(r-m)} N_{r-a,s-a}^t \cdots N_{r-1,s-1}^t, \quad \text{with } N_{r,s}^t = 1 - \frac{\dim_q(r-a) \dim_q(s-a-1)}{\dim_q(r+1) \dim_q(s)}.$$

Note that we have an evident lower bound $\kappa_t^{r,s} \geq \|t_a\|^{-1} = \dim_q(a)^{-1/2} \geq C_2^{-1/2} q^{a/2}$. In fact one can show that there are constants $D_1, D_2 > 0$ (depending on q) such that $D_1 q^{a/2} \leq \kappa_t^{r,s} \leq D_2 q^{a/2}$ for all $t \subset r \otimes s$: cf [Ver07, Lemma 4.8] and [BC18, Prop. 3.1].

In this article we need one more result about these quantities.

Lemma 7.1. *Assume that $q < 1$. For each $k \in \mathbb{N}$ there exists a constant E_k (depending also on q) such that*

$$\left| 1 - \left(\frac{\kappa_{t-k}^{r-k,s}}{\kappa_t^{r,s}} \right)^2 \right| \leq E_k q^{2(r-a)}$$

for all $t = r + s - 2a$, $0 \leq a \leq \min(r-k, s)$.

Proof. We start from the explicit expression that can be found in [KL94, BC18]:

$$(\kappa_t^{r,s})^{-2} = [a]_q! \frac{[r-a]_q! [s-a]_q! [t+a+1]_q!}{[r]_q! [s]_q! [t+1]_q!},$$

where $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$. This yields:

$$(7.2) \quad K := \left(\frac{\kappa_{t-k}^{r-k,s}}{\kappa_t^{r,s}} \right)^2 = \frac{[r-k]_q! [r-a]_q! [t-k+1]_q! [t+a+1]_q!}{[r]_q! [r-k-a]_q! [t+1]_q! [t-k+a+1]_q!}.$$

Now we have by definition $[n]_q = \delta q^{-n} (1 + O(q^{2n}))$ where $\delta = (q^{-1} - q)^{-1}$. From this it follows

$$\frac{[n]_q!}{[n-k]_q!} = [n]_q [n-1]_q \cdots [n-k+1]_q = \delta^k q^{-nk} q^{k(k-1)/2} (1 + O(q^{2n}))$$

where the constant involved in the $O(\cdot)$ depends on q and k . We apply this, with $n = r, r-a, t+1, t+a+1$ respectively, to the 4 quotients appearing in (7.2). The factors $\delta^k, q^{k(k-1)/2}$ simplify and we are left with

$$K = q^{kr} q^{-k(r-a)} q^{k(t+1)} q^{-k(t+a+1)} \frac{(1 + O(q^{2(r-a)}))(1 + O(q^{2(t+a+1)}))}{(1 + O(q^{2r}))(1 + O(q^{2(t+1)}))}.$$

The powers of q also simplify and since $t \geq r-a$ we obtain $K = 1 + O(q^{2(r-a)})$ as claimed. \square

7.2. A \mathbb{F} -boundary. The Gromov boundary of $\mathbb{F}O_Q$ is constructed in [VV07] using the ucp maps $\psi_{m,n} : \mathbf{A}_m \rightarrow \mathbf{A}_n$ given by

$$\psi_{m,n}(a) = V_{m,n-m}^{n*}(a \otimes \text{id}_{n-m})V_{m,n-m}^n.$$

Using the embedding of \mathbf{H}_n in $\mathbf{H}_1^{\otimes n}$ one can also write $\psi_{m,n}(a) = P_n^+(a \otimes P_{n-m}^+)P_n^+ = P_n^+(a \otimes \text{id})P_n^+$. The algebra $\mathbf{B} \subset \mathbf{M}$ is then defined as the norm closure of the subspace

$$\mathbf{B}_0 = \{b = (b_n)_n \in \mathbf{M} \mid \exists m \forall n \geq m \ b_n = \psi_{m,n}(b_m)\}.$$

For $a \in \mathbf{A}_m$, put $\psi_{m,\infty}(a) = (\psi_{m,n}(a))_{n \geq m} \in \mathbf{B}_0$. One can show that \mathbf{B} is a sub- C^* -algebra of \mathbf{M} which contains \mathbf{A} [VV07, Prop. 3.4] and one defines $\mathbf{B}_\infty = C(\partial\mathbb{F}O_Q) = \mathbf{B}/\mathbf{A}$ with the canonical projection $\pi : \mathbf{B} \rightarrow \mathbf{B}_\infty$. For $b = (b_m)_{m \in \mathbb{N}_0} \in \mathbf{M}$, the norm of $\pi(b)$ in \mathbf{M}/\mathbf{A} is $\limsup_{m \rightarrow \infty} \|b_m\|$, moreover for $b = \psi_{m,\infty}(a)$ the sequence of norms is decreasing, hence by density the norm of $\pi(b) \in \mathbf{B}_\infty$ is $\lim_{m \rightarrow \infty} \|b_m\|$.

One also proves that $\Delta(\mathbf{B}) \subset M(\mathbf{A} \otimes \mathbf{B})$ [VV07, Prop. 3.6], and since $\Delta(\mathbf{A}) \subset M(\mathbf{A} \otimes \mathbf{A})$ the restriction of Δ factors to an action of the discrete quantum group $\mathbb{F}O_Q$ on \mathbf{B}_∞ , which we denote by $\beta : \mathbf{B}_\infty \rightarrow M(\mathbf{A} \otimes \mathbf{B}_\infty)$. The algebra \mathbf{B}_∞ equipped with this action is called the algebra of continuous functions on the *Gromov boundary of $\mathbb{F}O_Q$* . There is a well-defined state ω on \mathbf{B} such that $\omega(b) = \lim_{m \rightarrow \infty} \text{qtr}_m(b_m)$ [VV07, Prop. 5.5]. It vanishes on \mathbf{A} and one denotes ω_∞ its factorization to a state on \mathbf{B}_∞ . Note that for $b = \psi_{m,\infty}(a)$ one has $\omega(b) = \text{qtr}_n \psi_{m,n}(a) = \text{qtr}_m(a)$ for all $n \geq m$.

Since $(\text{qtr}_1 \otimes \text{qtr}_n)\Delta$ is a convex combination of qtr_{n-1} and qtr_{n+1} [VV07, Thm. 5.6], the state ω_∞ is *stationary* (with respect to the ‘‘nearest neighborhood’’ random walk), meaning that $(\text{qtr}_1 \otimes \omega_\infty)\beta_\infty = \omega_\infty$. In fact this property characterizes ω_∞ .

Theorem 7.2. *Let $N \geq 3$ and let $Q \in M_N(\mathbb{C})$ be such that $Q\bar{Q} = \pm I_N$. Consider the Gromov boundary algebra \mathbf{B}_∞ of a discrete quantum group $\mathbb{F}O_Q$. If a state ν on \mathbf{B}_∞ is qtr_1 -stationary, i.e. $\nu = (\text{qtr}_1 \otimes \nu)\beta_\infty$ then $\nu = \omega_\infty$.*

Proof. By continuity and density, to prove $\nu = \omega_\infty$ it is enough to prove that $\nu_k := \nu \circ \pi \circ \psi_{k,\infty}$ on $B(\mathbf{H}_k)$ coincides with qtr_k for all k . Since ν_k and qtr_k are both states, $\nu_k \leq \text{qtr}_k$ is in fact sufficient. Hence we fix $k \in \mathbb{N}$, $a \in B(\mathbf{H}_k)$ such that $a \geq 0$ and $\|a\| \leq 1$, and we shall prove that $\nu_k(a) \leq \text{qtr}_k(a)$.

Since $(\text{qtr}_n \otimes \text{qtr}_1)\Delta$ is a (non trivial) convex combination of qtr_{n-1} and qtr_{n+1} , an easy induction shows that the stationarity assumption implies $(\text{qtr}_n \otimes \nu)\beta_\infty = \nu$ for all $n \in \mathbb{N}_0$. Hence we have $\nu_k(a) = (\text{qtr}_n \otimes \nu\pi)\Delta(\psi_{k,\infty}(a))$ for all n . Moreover, given $\epsilon > 0$, we know by the proof of [VV07, Prop. 3.6] that for $r \geq k + n$ large enough

$$\|(\text{qtr}_n \otimes p_{\geq r})\Delta(\psi_{k,\infty}(a)) - (\text{qtr}_n \otimes \psi_{r,\infty})(b)\| \leq \epsilon,$$

where $b = \sum_{s \subset n \otimes r} V_{n,r}^s \psi_{k,s}(a) V_{n,r}^{s*}$. In particular this shows that $\nu_k(a) = \lim_{r \rightarrow \infty} \nu_k^{n,r}(a)$ with

$$\nu_k^{n,r}(a) = \sum_{m=0}^n (\text{qtr}_n \otimes \nu_r)(V_{n,r}^{n+r-2m} \psi_{k,n+r-2m}(a) V_{n,r}^{n+r-2m*}).$$

To compare this with $\text{qtr}_k(a)$ we will decompose $\text{qtr}_k(a)$ in a slightly unnatural way. For all $r \geq n \geq k$ we have

$$\text{qtr}_k(a) = \text{qtr}_n \psi_{k,n}(a) = \text{qtr}_n (P_n^+(a \otimes 1)P_n^+) = (\text{qtr}_n \otimes \nu_r)((P_n^+ \otimes P_r^+)(a \otimes \text{id})(P_n^+ \otimes P_r^+))$$

since $\nu_r(P_r^+) = 1$. Next we decompose $P_n^+ \otimes P_r^+ = \sum_{m=0}^n V_{n,r}^{n+r-2m} V_{n,r}^{n+r-2m*}$, so that

$$\text{qtr}_k(a) = \sum_{m=0}^n \sum_{m'=0}^n (\text{qtr}_n \otimes \nu_r)(V_{n,r}^{n+r-2m} \phi_{n+r-2m, n+r-2m'}^{n,r}(a) V_{n,r}^{n+r-2m'*}),$$

where $\phi_{s,s'}^{n,r}(a) = V_{n,r}^{s*}(a \otimes \text{id})V_{n,r}^{s'} = V_{n,r}^{s*}(a \otimes P_{n-k}^+ \otimes P_r^+)V_{n,r}^{s'}$. Our strategy will be to perform a term-by-term comparison with $\psi_{k,s}(a) = P_s^+(a \otimes P_{s-k}^+)P_s^+$.

We will take care simultaneously of the terms $s \neq s'$, but let us note already that $\phi_{s,s'}^{n,r}(a)$ vanishes if $|m - m'| > k$. Indeed, one can obtain $a \in B(\mathbf{H}_k)$ by slicing $t_k t_k^* \in B(\mathbf{H}_k) \otimes B(\mathbf{H}_k)$ on the left with some functional $\omega \in B(\mathbf{H}_k)^*$, hence one can obtain $\phi_{s,s'}^{n,r}(a)$ by slicing $(\text{id} \otimes V_{n,r}^{s*})(t_k t_k^* \otimes \text{id})(\text{id} \otimes V_{n,r}^{s'}) \in B(\mathbf{H}_k) \otimes B(\mathbf{H}_{s'}, \mathbf{H}_s)$ on the left as well. But this map is an intertwiner from $\mathbf{H}_k \otimes \mathbf{H}_{s'}$ to $\mathbf{H}_k \otimes \mathbf{H}_s$, which have no irreducible component in common if $|s - s'| = 2|m - m'| > 2k$.

To compare $\phi_{s,s'}^{n,r}(a)$ and $\psi_{k,s}(a)$, for $s = n + r - 2m$ we write $V_{n,r}^s = \kappa_s^{n,r}(P_n^+ \otimes P_r^+)(\text{id} \otimes t_m \otimes \text{id})P_s^+$ as explained previously. We use moreover the estimate $\|P_n^+ - (P_{n-m}^+ \otimes P_m^+)(P_k^+ \otimes P_{n-k}^+)\| \leq Bq^{n-m-k}$ from [VV07, Lemma A.4] which yields

$$\begin{aligned} (\kappa_s^{n,r} \kappa_{s'}^{n,r})^{-1} \phi_{s,s'}^{n,r}(a) &\simeq P_s^+(\text{id} \otimes t_m^* \otimes \text{id})(P_{n-m}^+ \otimes P_m^+ \otimes P_r^+)(a \otimes P_{n-k}^+ \otimes P_r^+) \\ &\quad (P_{n-m'}^+ \otimes P_{m'}^+ \otimes P_r^+)(\text{id} \otimes t_{m'} \otimes \text{id})P_{s'}^+, \end{aligned}$$

meaning more precisely that the difference of both terms has a norm dominated by $B(q^{n-m-k} + q^{n-m'-k})\|t_m\|\|t_{m'}\|$, hence by $BC_2q^{n-k}(q^{-m} + q^{-m'})q^{-m/2}q^{-m'/2}$.

In the second term the projection $(P_{n-m}^+ \otimes P_m^+ \otimes P_r^+)$ is absorbed by other projections on the left and on the right, so that for $m, m' \leq n - k$ this term equals:

$$\begin{aligned} P_s^+(a \otimes [(\text{id} \otimes t_m^* \otimes \text{id})(P_{n-k}^+ \otimes P_r^+)(\text{id} \otimes t_{m'} \otimes \text{id})])P_{s'}^+ &= \\ &= P_s^+(a \otimes [P_{s-k}^+(\text{id} \otimes t_m^* \otimes \text{id})(P_{n-k}^+ \otimes P_r^+)(\text{id} \otimes t_{m'} \otimes \text{id})P_{s'-k}^+])P_{s'}^+ \\ &= \delta_{s,s'}(\kappa_{s-k}^{n-k,r})^{-2}P_s^+(a \otimes V_{n-k,r}^{s-k*}V_{n-k,r}^{s-k})P_s^+ = \delta_{s,s'}(\kappa_{s-k}^{n-k,r})^{-2}P_s^+(a \otimes P_{s-k}^+)P_s^+. \end{aligned}$$

Indeed $P_{s-k}^+(\text{id} \otimes t_m^* \otimes \text{id})(P_{n-k}^+ \otimes P_r^+)(\text{id} \otimes t_{m'} \otimes \text{id})P_{s'-k}^+$ is an intertwiner between irreducibles, hence a scalar. So it vanishes if $s \neq s'$, and when $s = s'$ one recognizes $(\kappa_{s-k}^{n-k,r})^{-2}V_{n-k,r}^{s-k*}V_{n-k,r}^{s-k} = (\kappa_{s-k}^{n-k,r})^{-2}P_{s-k}^+$.

Using the inequalities $\kappa_s^{n,r} \leq D_2q^{m/2}$, $\kappa_{s'}^{n,r} \leq D_2q^{m'/2}$ and $\kappa_{s-k}^{n-k,r} \leq D_2q^{m/2}$ this yields, for $0 \leq m, m' \leq n - k$ and $s = n + r - 2m$ different from $s' = n + r - 2m'$:

$$\begin{aligned} \|\phi_{s,s'}^{n,r}(a)\| &\leq BC_2\kappa_s^{n,r}\kappa_{s'}^{n,r}q^{n-k}(q^{-m} + q^{-m'})q^{-m/2}q^{-m'/2} \\ &\leq BC_2D_2^2q^{n-k}(q^{-m} + q^{-m'}). \end{aligned}$$

Further for s as above

$$\|\psi_{k,s}(a) - (\kappa_{s-k}^{n-k,r}/\kappa_s^{n,r})^2\phi_{s,s}^{n,r}(a)\| \leq 2BC_2(\kappa_{s-k}^{n-k,r})^2q^{n-2m-k} \leq 2BC_2D_2^2q^{n-m-k}.$$

Finally for the terms $s = s'$ we can apply Lemma 7.1 and we obtain, since $\|\phi_{s,s}^{n,r}(a)\| \leq \|a\| \leq 1$:

$$\|\psi_{k,s}(a) - \phi_{s,s}^{n,r}(a)\| \leq 2BC_2D_2^2q^{n-m-k} + E_kq^{2(n-m)} \leq F_kq^{n-m}$$

for some constant F_k . We can assume also that $F_k \geq BC_2D_2^2q^{-k}$ so as to control the term $s \neq s'$ at the same time. We will use these estimates later in the case $m, m' < n/2 + k$ — assuming that n is even and greater than $4k$.

For the terms corresponding to $m \geq n/2$, (and in particular for the terms corresponding to $m > n - k$), we will concentrate instead on the “exterior part” of the formulae for $\nu_k^{n,r}(a)$ and $\text{qtr}_k(a)$, that is, on the linear forms $\mu_{n,r}^{s,s'} : x \mapsto (\text{qtr}_n \otimes \nu_r)(V_{n,r}^s x V_{n,r}^{s'*})$, which in both cases are applied to elements $x \in B(\mathbf{H}_{s'}, \mathbf{H}_s)$ with $\|x\| \leq \|a\| \leq 1$.

Without loss of generality we can assume $m' =: m + p \geq m$. We compute again $V_{n,r}^s$ using t_m and we recall that $t_{m'} = (P_{m+p}^+ \otimes P_{m+p}^+)(\text{id} \otimes t_m \otimes \text{id})t_p$. Since the projections P_{m+p}^+ are absorbed by P_n^+ resp. P_r^+ (e.g. $(\text{id} \otimes P_{m+p})P_n = P_n$, i.e. \mathbf{H}_n is a subrepresentation of $\mathbf{H}_{n-m-p} \otimes \mathbf{H}_{m+p}$) this allows to write

$$\begin{aligned} V_{n,r}^s x V_{n,r}^{s'*} &= \kappa_{n,r}^s \kappa_{n,r}^{s'} (P_n^+ \otimes P_r^+) (\text{id} \otimes t_m \otimes \text{id}) P_s^+ x P_{s'}^+ (\text{id} \otimes t_p^* \otimes \text{id}) (\text{id} \otimes t_m^* \otimes \text{id}) (P_n^+ \otimes P_r^+) \\ &= \kappa_{n,r}^s \kappa_{n,r}^{s'} (P_n^+ \otimes P_r^+) (\text{id} \otimes t_m \otimes \text{id}) y (\text{id} \otimes t_m^* \otimes \text{id}) (P_n^+ \otimes P_r^+) \end{aligned}$$

where $y := P_s^+ x P_{s'}^+ (\text{id} \otimes t_p^* \otimes \text{id}) \in B(\mathbf{H}_{n-m} \otimes \mathbf{H}_{r-m})$ satisfies $\|y\| \leq \|t_p\| \|x\| \leq C_2^{1/2} q^{-p/2}$.

We see in particular that $\mu_{n,r}^{s,s'}(x)$ is a composition of a bounded map $x \mapsto y$ and a positive linear functional of y , whose norm can be computed by evaluating at $y = 1$. Since $\text{qTr}_n = (\text{qTr}_{n-m} \otimes \text{qTr}_m)(P_n^+ \cdot P_n^+) \leq (\text{qTr}_{n-m} \otimes \text{qTr}_m)$, where we view both sides of the inequality as functionals acting on $B(\mathbf{H}_{n-m} \otimes \mathbf{H}_m)$, we can write

$$\begin{aligned} \kappa_{n,r}^s \kappa_{n,r}^{s'} (\text{qtr}_n \otimes \nu_r) [(P_n^+ \otimes P_r^+) (\text{id} \otimes t_m t_m^* \otimes \text{id}) (P_n^+ \otimes P_r^+)] &\leq \\ &\leq \frac{\kappa_{n,r}^s \kappa_{n,r}^{s'}}{\dim_q(n)} (\text{qTr}_{n-m} \otimes \text{qTr}_m \otimes \tilde{\nu}_r) (P_{n-m}^+ \otimes t_m t_m^* \otimes P_{r-m}^+), \end{aligned}$$

where $\tilde{\nu}_r = \nu_r(P_r^+ \cdot P_r^+)$ on $B(\mathbf{H}_m \otimes \mathbf{H}_{r-m})$. Since $(\text{qTr}_m \otimes \text{id})(t_m t_m^*) = F_m^{-2}$ and $\text{qTr}_{n-m}(P_{n-m}^+) = \dim_q(n - m)$, the last quantity can be simplified and bounded above as follows:

$$\begin{aligned} \kappa_{n,r}^s \kappa_{n,r}^{s'} \frac{\dim_q(n - m)}{\dim_q(n)} \tilde{\nu}_r(F_m^{-2} \otimes P_{r-m}^+) &\leq \kappa_{n,r}^s \kappa_{n,r}^{s'} \frac{\dim_q(n - m)}{\dim_q(n)} \|F_m\|^2 \\ &\leq D_2^2 q^{m/2} q^{m'/2} \frac{C_2 q^{-n+m}}{C_1 q^{-n}} \|F_m^{-1}\|^2 =: G q^{2m} q^{p/2} \|F_m^{-1}\|^2. \end{aligned}$$

Altogether we have obtained the estimate $\|\mu_{n,r}^{s,s'}\| \leq G C_2^{1/2} q^{2m} \|F_m\|^2$. Since $\|F_m^{-1}\|^2 = \|F_m\|^2 = \|F_1\|^{2m}$ this can also be written, without any assumptions on m, m' , as

$$\|\mu_{n,r}^{s,s'}\| \leq G C_2^{1/2} (q \|F_1\|)^{2 \min(m, m')}.$$

We finally put all terms together as follows, for $r \geq n \geq 4k$, n even, dropping the terms $\phi_{s,s}^{n,r}(a)$ for $m \geq n/2$ which are positive:

$$\begin{aligned} \nu_k^{n,r}(a) - \text{qtr}_k(a) &\leq \sum_{m=0}^{n/2-1} (\text{qtr}_n \otimes \nu_r)(V_{n,r}^s [\psi_{k,s}(a) - \phi_{s,s}^{n,r}(a)] V_{n,r}^{s*}) \\ &\quad + \sum_{n/2}^n (\text{qtr}_n \otimes \nu_r)(V_{n,r}^s \psi_{k,s}(a) V_{n,r}^{s*}) \\ &\quad + \sum_{m \neq m'} |(\text{qtr}_n \otimes \nu_r)(V_{n,r}^s \phi_{s,s'}^{n,r}(a) V_{n,r}^{s'*})|. \end{aligned}$$

Using the fact that the terms $m \neq m'$ vanish if $|m - m'| > k$, we can split the last sum according to whether $m, m' \geq n/2$ or $m, m' < n/2 + k$, up to adding some of the terms twice. Using the estimates obtained previously we arrive at

$$\begin{aligned} \nu_k^{n,r}(a) - \text{qtr}_k(a) &\leq F_k \sum_{m=0}^{n/2-1} q^{n-m} + G C_2^{1/2} \sum_{n/2}^n (q \|F_1\|)^{2m} + \\ &\quad + F_k q^n \sum_{m, m' < n/2+k} (q^{-m} + q^{-m'}) \end{aligned}$$

$$\begin{aligned}
& + GC_2^{1/2} \sum_{m,m' \geq n/2} (q \|F_1\|)^{2 \min(m,m')} \\
& \leq F_k \frac{n}{2} q^{n/2} + GC_2^{1/2} (\frac{n}{2} + 1) (q \|F_1\|)^n + \\
& \quad + 2F_k (\frac{n}{2} + k)^2 q^{n/2-k} + GC_2^{1/2} (\frac{n}{2} + 1)^2 (q \|F_1\|)^n,
\end{aligned}$$

where we have used the fact that $q \|F_1\| < 1$. Letting $r \rightarrow \infty$ we obtain the same upper bound for $\nu_k(a) - \text{qtr}_k(a)$. Then we let $n \rightarrow \infty$ and obtain $\nu_k(a) \leq \text{qtr}_k(a)$. \square

Note that the assumption $N \geq 3$ in the previous theorem is optimal, since for $N = 2$ (and still $q < 1$) the discrete quantum group $\mathbb{F}O_Q$ is amenable whereas the algebra \mathbb{B}_∞ is still infinite-dimensional. When $q = 1$ the construction of \mathbb{B}_∞ in [VV07] does not apply.

As advertised before, the concept of boundary actions allows us to confirm Ozawa's conjecture for the exact C^* -algebras $C_r^*(\mathbb{F}O_Q) = C(O_Q^+)$, and to obtain a C^* -simplicity result for the crossed product of the Gromov boundary actions.

Corollary 7.3. *Let $N \geq 3$ and let $Q \in M_N(\mathbb{C})$ be such that $Q\bar{Q} = \pm I_N$. The action of the discrete quantum group $\mathbb{F}O_Q$ on its Gromov boundary \mathbb{B}_∞ , introduced in [VV07], is a boundary action. Moreover we have the embedding*

$$C(O_Q^+) \subset \mathbb{F}O_Q \rtimes_r \mathbb{B}_\infty \subset I(C(O_Q^+)),$$

and the C^* -algebra in the middle is nuclear.

If in addition $\|Q\|^8 \leq \frac{3}{8} \text{Tr}(QQ^*)$, then the crossed product $\mathbb{F}O_Q \rtimes_r \mathbb{B}_\infty$ is simple.

Proof. The last theorem, Theorem 4.19 and Theorem 5.6 of [VV07] imply that the Gromov boundary action is a boundary action in the sense of Definition 4.1.

The displayed inclusion is then a consequence of Theorem 6.3. Corollary 6.2 of [VV07] implies that $\mathbb{F}O_Q$ is exact; Theorem 4.5 of that paper (amenability of the Gromov boundary action), together with the results in Section 3 of [VV07] (nuclearity of \mathbb{B}_∞) imply via Proposition 4.4 of that paper that the crossed product $\mathbb{F}O_Q \rtimes_r \mathbb{B}_\infty$ is nuclear.

Finally the extra condition on Q is shown in Theorem of [VV07] to guarantee C^* -simplicity of $\mathbb{F}O_Q$, and the last statement follows now from Theorem 6.3. \square

7.3. Faithfulness. Our next aim is to prove that the action β_∞ of $\mathbb{F}O_Q$ on \mathbb{B}_∞ is faithful. Let $\mathbf{N}_\infty = \{(\text{id} \otimes \varphi)\beta_\infty(x) \mid x \in \mathbb{B}_\infty, \varphi \in \mathbb{B}_\infty^*\}'' \subset \ell^\infty(\mathbb{F}O_Q)$ be the cokernel of β_∞ . Using Proposition 2.10 we see that showing that β_∞ is faithful is equivalent with showing that $p_0 \in \mathbf{N}_\infty$. Before proving this we first fix some further notation. Recall that for $a \in B(\mathbf{H}_n)$ we denote $\tilde{a} \in B(\mathbf{H}_n)$ the unique element satisfying $(a \otimes \text{id}_n)t_n = (\text{id} \otimes \tilde{a})t_n$.

Definition 7.4. *We consider $z_n = H_n(A_n) \in \mathbf{N}_\infty$, where $A_n \in B(\mathbf{H}_n) \otimes B(\mathbf{H}_n)$ is such that $\sum \tilde{A}_{n(1)} \otimes A_{n(2)} = t_n t_n^*$, and $H_n : B(\mathbf{H}_n) \otimes B(\mathbf{H}_n) \rightarrow \mathbf{N}_\infty$ is defined by*

$$H_n(a \otimes b) = (\text{id} \otimes \omega_\infty)(\beta_\infty(\pi\psi_{n,\infty}(a))(1 \otimes \pi\psi_{n,\infty}(b))).$$

Recalling that $t_n = \sum_i e_i \otimes j(e_i) = \sum j^*(e_i) \otimes e_i$ if $(e_i)_i$ is an ONB of \mathbf{H}_n one can check that

$$(7.3) \quad A_n = \sum e_j (F_n^{-1} e_i)^* \otimes e_i e_j^*.$$

In particular for every $g \in B(\mathbf{H}_n)$ we have

$$\begin{aligned}
\sum A_{n(1)} g A_{n(2)} &= \sum_{i,j} e_i e_i^* \langle e_j | F_n^{-1} g e_j \rangle \\
&= \text{qTr}'_n(g) \sum_i e_i e_i^* = \mathbf{1} \text{qTr}'_n(g)
\end{aligned}$$

where $\text{qTr}'_n(a) = t_n^*(\text{id}_n \otimes a)t_n = \text{Tr}_n(F_n^{-1}a)$ is the ‘‘right’’ quantum trace. In particular we have the following property:

$$(7.4) \quad \sum \text{qTr}_n(fA_{n(1)}gA_{n(2)}) = \text{qTr}_n(f) \text{qTr}'_n(g), \quad f, g \in B(\mathbf{H}_n).$$

Our aim is now to show that $q^n z_n$ converges σ -weakly to p_0 , thus establishing that $p_0 \in \mathbf{N}_\infty$. This will require a few lemmas. Let us formulate first two remarks about our strategy.

Remark 7.5. *In the case of a classical free group F_N , the analogue of $q^n z_n$ is the function $f_n \in \ell^\infty(F_N)$ defined as follows. Let ∂F_N be the Gromov boundary of F_N identified with the space of infinite reduced words on the generators and their inverses. For $g \in F_N$, denote $X_g \subset \partial F_N$ the subspace of infinite reduced words starting by g , and $\chi_g \in C(\partial F_N)$ its characteristic function. The function f_n is then given by*

$$f_n(g) = \sum_{|h|=n} (\chi_h \mid g \cdot \chi_h)$$

where we use the hermitian structure on $C(\partial F_N)$ given by the canonical ‘‘uniform harmonic’’ probability measure on ∂F_N . We have $f_n(e) = 1$ and one can check that $f_n(g) \rightarrow 0$ as $n \rightarrow \infty$ for fixed $g \neq e$. This is of course not the easiest way to establish faithfulness of the boundary action for F_N .

Remark 7.6. *Imagine that instead of working in \mathbf{B}_∞ we remain in $B(\mathbf{H}_n)$ and consider $G_n(a \otimes b) = (\text{id} \otimes \text{qtr}_n)(\Delta(a)(1 \otimes b))$, $y_n = G_n(A_n) \in \ell^\infty(\mathbb{F}O_Q)$ with the same element A_n as above. One can then compute directly for any k , using the equality (7.4):*

$$\begin{aligned} p_k y_n &= \sum (p_k \otimes \text{qtr}_n)(V_{k,n}^n A_{n(1)} V_{k,n}^{n*} (1 \otimes A_{n(2)})) \\ &= \sum (\text{id} \otimes \text{qtr}_n)(V_{k,n}^n) (\text{id} \otimes \text{qTr}'_n)(V_{k,n}^{n*}). \end{aligned}$$

Moreover $(\text{id} \otimes \text{qTr}_n)(V_{k,n}^n) = (\text{id}_k \otimes t_n^*)(V_{k,n}^n \otimes \text{id}_n)t_n$ vanishes for all $k > 0$ because it is an invariant vector in \mathbf{H}_k . As a result $y_n = \dim_q(n)p_0$ for all n . However the connection of y_n to our element z_n is not clear.

Our first lemma shows that $q^n z_n$ has the correct value ‘‘at the unit’’.

Lemma 7.7. *We have $p_0 z_n = q^{-n} p_0$.*

Proof. Since p_0 is the support of the co-unit ε of $\ell^\infty(\Gamma)$, and $(\varepsilon \otimes \text{id})\alpha = \text{id}$ for any continuous action α , we have $p_0 H_n(a \otimes b) = \omega(\psi_{n,\infty}(a)\psi_{n,\infty}(b))p_0 = \lim_{t \rightarrow \infty} \text{qtr}_t(\psi_{n,t}(a)\psi_{n,t}(b))p_0$, where we are using the approximate multiplicativity of the maps $\psi_{n,t}$.

Notice that we have $\text{qTr}_t(f) = (\text{qTr}_n \otimes \text{qTr}_{t-n})(f)$ for $f \in B(\mathbf{H}_t)$. Applying the equality (7.4) we can write

$$\begin{aligned} \sum \text{qTr}_t(\psi_{n,t}(A_{n(1)})\psi_{n,t}(A_{n(2)})) &= \sum \text{qTr}_{t-n}(\text{qTr}_n \otimes \text{id}_{t-n})[P_t^+(A_{n(1)} \otimes \text{id}_{t-n})P_t^+(A_{n(2)} \otimes \text{id}_{t-n})] \\ &= \text{qTr}_{t-n}[(\text{qTr}_n \otimes \text{id})(P_t^+)(\text{qTr}'_n \otimes \text{id})(P_t^+)]. \end{aligned}$$

Now $(\text{qTr}'_n \otimes \text{id})(P_t^+)$ is an intertwiner of H_{t-n} , hence a multiple of the P_{t-n}^+ , and by applying qTr'_{t-n} one finds $(\text{qTr}'_n \otimes \text{id})(P_t^+) = \dim_q(t)/\dim_q(t-n)P_{t-n}^+$. As a result we have

$$\sum \text{qtr}_t(\psi_{n,t}(A_{n(1)})\psi_{n,t}(A_{n(2)})) = \dim_q(t-n)^{-1}(\text{qTr}_n \otimes \text{qTr}_{t-n})(P_t^+) = \dim_q(t)/\dim_q(t-n).$$

Since $\dim_q(t) \sim Cq^{-t}$ when $t \rightarrow \infty$, with a constant $C > 0$ depending only on q , the limit of the above ratio is q^{-n} as stated. \square

Lemma 7.8. *Let $B \in B(\mathbf{H}_t)$ and let $e \in B(\mathbf{H}_n)$ be a rank one matrix. Then we have*

$$|\text{qtr}_t(B\psi_{n,t}(e))| \leq \frac{\dim_q(t-n)}{\dim_q(t)} \|B\| \|eF_n\|.$$

Proof. Recall that $\text{qTr}_t(\cdot) = (\text{qTr}_n \otimes \text{qTr}_{t-n})(P_t^+ \cdot P_t^+)$. Since P_t^+ commutes with $F_n \otimes F_{t-n}$ we have

$$\text{qtr}_t(B\psi_{n,t}(e)) = (\dim_q t)^{-1} (\text{qTr}_n \otimes \text{qTr}_{t-n})(P_t^+ B P_t^+(e \otimes \text{id}_{t-n})).$$

The norm of the functional qTr_{t-n} is $\dim_q(t-n)$. If e has rank one, so does eF_n and the norm of $\text{qTr}_n(\cdot e) = \text{Tr}_n(\cdot eF_n)$ equals $\|eF_n\|$. \square

Lemma 7.9. *There exists a constant $M > 0$, depending only on q , such that $\|z_n\| \leq Mq^{-n}$ for all n .*

Proof. Let $\eta \in \ell^\infty(\Gamma)_*$. Let us decompose $A_n = \sum A_{n,ij} \otimes e_i e_j^*$, where $(e_i)_i$ is an ONB of \mathbf{H}_n . We have by definition

$$\begin{aligned} \eta(z_n) &= \sum (\eta \otimes \omega_\infty \pi)(\Delta(\psi_{n,\infty}(A_{n,ij}))(1 \otimes \psi_{n,\infty}(e_i e_j^*))) \\ &= \sum \lim_{t \rightarrow \infty} (\eta \otimes \text{qtr}_t)(\Delta(\psi_{n,\infty}(A_{n,ij}))(1 \otimes \psi_{n,t}(e_i e_j^*))) \\ &= \sum \lim_{t \rightarrow \infty} \text{qtr}_t(B_{ij}^{nt} \psi_{n,t}(e_i e_j^*)), \end{aligned}$$

where $B_{ij}^{nt} = (\eta \otimes p_t) \Delta(\psi_{n,\infty}(A_{n,ij}))$ satisfies $\|B_{ij}^{nt}\| \leq \|\eta\| \|A_{n,ij}\|$. Applying Lemma 7.8 and taking the limit we obtain $|\eta(z_n)| \leq q^n \|\eta\| \sum \|A_{n,ij}\| \|F_n e_j\|$.

Now according to the explicit formula for A_n (c.f. Eq. (7.3)) we have $\|A_{n,ij}\| = \|e_j\| \|F_n^{-1} e_i\|$. Taking for $(e_i)_i$ an ONB of eigenvectors for F_n we obtain $|\eta(z_n)| \leq q^n \|\eta\| \text{Tr}(F_n^{-1}) \text{Tr}(F_n)$. Since $\text{Tr}(F_n^{-1}) = \text{Tr}(F_n) = \dim_q(n) \leq \sqrt{M} q^{-n}$ for some constant $M > 0$ this yields the result. \square

Now we need to prove that $q^n p_k z_n \rightarrow 0$, for fixed k , as $n \rightarrow \infty$ which requires a much more technical argument. We start by rewriting $p_k z_n$ in a convenient way.

Lemma 7.10. *Let ζ, ξ be vectors in \mathbf{H}_k . Then we have*

$$(\zeta \mid p_k z_n \xi) = \sum_{l=0}^k \lim_{t \rightarrow \infty} (\kappa_s^{k,t})^2 \text{qTr}_{t-n}(w_{n,t}^{k,l}(\zeta)^* w_{n,t}'^{k,l}(\xi))$$

where $s = t + k - 2l$ and $w_{n,t}^{k,l}(\xi), w_{n,t}'^{k,l}(\xi) \in B(\mathbf{H}_{t-n}, \mathbf{H}_{s-n})$, for $\xi = \sum \xi^{(1)} \otimes \xi^{(2)} \in \mathbf{H}_{k-l} \otimes \mathbf{H}_l$, are given by:

$$\begin{aligned} w_{n,t}^{k,l}(\xi) &= \dim_q(t)^{-1/2} \sum (\text{qTr}_n \otimes \text{id})(P_s^+(\xi^{(1)} \bar{\xi}^{(2)*} \otimes \text{id}_{t-l}) P_t^+) \\ w_{n,t}'^{k,l}(\xi) &= \dim_q(t)^{-1/2} \sum (\text{qTr}'_n \otimes \text{id})(P_s^+(\xi^{(1)} \bar{\xi}^{(2)*} \otimes \text{id}_{t-l}) P_t^+). \end{aligned}$$

Proof. For a linear form $\eta \in B(\mathbf{H}_k)^* \subset \ell^\infty(\Gamma)_*$ we have, as in the proof of Lemma 7.9:

$$\eta(z_n) = \sum \lim_{t \rightarrow \infty} (\eta \otimes \text{qtr}_t)(\Delta(\psi_{n,\infty}(A_{n(1)}))(1 \otimes \psi_{n,t}(A_{n(2)}))).$$

We then compute the coproduct as $(p_k \otimes p_t) \Delta(x) = \sum_{l=0}^k V_{k,t}^s x V_{k,t}^{s*}$, where $s = t + k - 2l$:

$$\begin{aligned} \eta(z_n) &= \lim_{t \rightarrow \infty} (\dim_q t)^{-1} \sum \sum_{l=0}^k (\eta \otimes \text{qTr}_t)[V_{k,t}^s P_s^+(A_{n(1)} \otimes \text{id}_{s-n}) P_s^+ V_{k,t}^{s*} \\ &\quad (\text{id}_k \otimes P_t^+)(\text{id}_k \otimes A_{n(2)} \otimes \text{id}_{t-n})(\text{id}_k \otimes P_t^+)]. \end{aligned}$$

For $\eta = (\zeta \mid \cdot \xi)$ this yields

$$(\zeta \mid p_k z_n \xi) = \lim_{t \rightarrow \infty} (\dim_q t)^{-1} \sum \sum_{l=0}^k (\text{qTr}_n \otimes \text{qTr}_{t-n})[(\zeta^* \otimes P_t^+) V_{k,t}^s P_s^+$$

$$(A_{n(1)} \otimes \text{id}_{s-n})P_s^+V_{k,t}^{s*}(\xi \otimes P_t^+)(A_{n(2)} \otimes \text{id}_{t-n}).$$

Now we apply again the equality (7.4), with f and g corresponding to the left legs of $(\zeta^* \otimes P_t^+)V_{k,t}^s P_s^+ \in B(H_n) \otimes B(H_{s-n}, H_{t-n})$ and $P_s^+V_{k,t}^{s*}(\xi \otimes P_t^+) \in B(H_n) \otimes B(H_{t-n}, H_{s-n})$ respectively:

$$\begin{aligned} (\zeta \mid p_k z_n \xi) &= \lim_{t \rightarrow \infty} (\dim_q t)^{-1} \sum_{l=0}^k \text{qTr}_{t-n}[(\text{qTr}_n \otimes \text{id})((\zeta^* \otimes P_t^+)V_{k,t}^s P_s^+ \\ &\quad (\text{qTr}'_n \otimes \text{id})(P_s^+V_{k,t}^{s*}(\xi \otimes P_t^+))]. \end{aligned}$$

This has the required form if we put

$$\begin{aligned} w_{n,t}^{k,l}(\zeta) &= \dim_q(t)^{-1/2} (\kappa_s^{k,t})^{-1} (\text{qTr}_n \otimes \text{id})(P_s^+V_{k,t}^{s*}(\zeta \otimes P_t^+)) \quad \text{and} \\ w'_{n,t}{}^{k,l}(\xi) &= \dim_q(t)^{-1/2} (\kappa_s^{k,t})^{-1} (\text{qTr}'_n \otimes \text{id})(P_s^+V_{k,t}^{s*}(\xi \otimes P_t^+)). \end{aligned}$$

We have moreover, by definition of $\kappa_s^{k,t}$:

$$\begin{aligned} w_{n,t}^{k,l}(\zeta) &= \dim_q(t)^{-1/2} (\text{qTr}_n \otimes \text{id})(P_s^+(\text{id}_{k-l} \otimes t_l^* \otimes \text{id}_{t-l})(\zeta \otimes P_t^+)) \\ &= \dim_q(t)^{-1/2} (\text{qTr}_n \otimes \text{id})(P_s^+(\zeta^{(1)} \bar{\zeta}^{(2)*} \otimes \text{id}_{t-l})P_t^+), \end{aligned}$$

and similarly for $w'_{n,t}{}^{k,l}(\xi)$. Here $\zeta^{(1)} \bar{\zeta}^{(2)*} \in B(\mathbf{H}_l, \mathbf{H}_{k-l})$ arises from the decomposition $\zeta = \sum \zeta^{(1)} \otimes \zeta^{(2)} \in \mathbf{H}_{k-l} \otimes \mathbf{H}_l$. \square

Note that the element $w_{n,t}^{k,l}(\xi) \in B(\mathbf{H}_{t-n}, \mathbf{H}_{s-n})$ is defined for any vector $\xi \in \mathbf{H}_{k-l} \otimes \mathbf{H}_l$. In the next lemma we use twisted Hilbert-Schmidt norms given for $f \in B(\mathbf{H}_p, \mathbf{H}_q)$ by $\|f\|_{\text{HS}}^2 = \text{qTr}_p(f^* f)$.

Lemma 7.11. *Let $\xi \in \mathbf{H}_{k-l} \otimes \mathbf{H}_l$.*

- We have $\dim_q(t)^{1/2} \|w_{n,t}^{k,l}(\xi)\|_{\text{HS}} = \dim_q(s)^{1/2} \|w_{n,s}^{k,k-l}((F_l^{1/2} \otimes F_{k-l}^{1/2})\bar{\xi})\|_{\text{HS}}$ and $\|\xi\| = \|(F_l^{1/2} \otimes F_{k-l}^{1/2})\bar{\xi}\|$.
- We have $\|w_{n,t}^{k,l}(\xi)\|_{\text{HS}} \leq Cq^{-n/2}q^{-l/2}\|\xi\|$ for a constant $C > 0$ depending only on q .

The same properties hold for $w'_{n,t}{}^{k,l}(\xi)$.

Proof. We clearly have $\dim_q(t)^{1/2} w_{n,t}^{k,l}(\xi)^* = \dim_q(s)^{1/2} w_{n,s}^{k,k-l}(\bar{\xi})$, since the conjugate of $\xi = \sum \xi^{(1)} \otimes \xi^{(2)} \in \mathbf{H}_{k-l} \otimes \mathbf{H}_l$ is given by $\bar{\xi} = \sum \bar{\xi}^{(2)} \otimes \bar{\xi}^{(1)} \in \mathbf{H}_l \otimes \mathbf{H}_{k-l}$. On the other hand we have $\|f^*\|_{\text{HS}} = \|F_q^{1/2} f F_p^{-1/2}\|_{\text{HS}}$. Finally, since F -matrices commute with intertwiners, $\text{qTr}_n(F_n^{1/2} \cdot F_n^{-1/2}) = \text{qTr}_n$, and $F_l^{-1/2} j_l = j_l F_l^{1/2}$, it follows that

$$F_{s-n}^{1/2} w_{n,t}^{k,l}(\xi) F_{t-n}^{-1/2} = w_{n,t}^{k,l}((F_{k-l} \otimes F_l)^{1/2} \xi).$$

For the norm estimate, first note that $\sum \|\xi^{(1)}\| \|\bar{\xi}^{(2)}\| = \sum \|\xi^{(1)}\| \|F_l^{1/2} \xi^{(2)}\| \leq \|\xi\| \dim_q(l)^{1/2}$, by taking for $(\xi^{(2)})$ an ONB of eigenvectors of F_l and using Cauchy-Schwarz. Hence we have

$$\begin{aligned} \|w_{n,t}^{k,l}(\xi)\|_{\text{HS}} &\leq \|\text{qTr}_{t-n}\|^{1/2} \|w_{n,t}^{k,l}(\xi)\| \leq (\dim_q t)^{-1/2} \|\text{qTr}_{t-n}\|^{1/2} \|\text{qTr}_n\| \sum \|\xi^{(1)}\| \|\bar{\xi}^{(2)}\| \\ &\leq (\dim_q t)^{-1/2} \dim_q(t-n)^{1/2} \dim_q(n) \dim_q(l)^{1/2} \|\xi\| \leq Cq^{-n/2}q^{-l/2}\|\xi\|. \end{aligned}$$

\square

The norm estimate above is of the same order as the one of Lemma 7.9 as $n \rightarrow \infty$, which is not enough for our purposes. Using the next lemma we will obtain a better estimate in the proof of Theorem 7.13 — in the Kac case even a much better one since we will deduce that $\|w_{n,t}^{k,l}(\xi)\|_{\text{HS}}$ is bounded with respect to n , for fixed k . In the proof we use the “higher weight” Wenzl recursion relation [FV16, Lemma 3.4]: for $\zeta \in H_{p+q}$ decomposed as $\zeta = \sum \zeta^{(1)} \otimes \zeta^{(2)} \in H_p \otimes H_q$ and $\zeta = \sum \zeta_{(1)} \otimes \zeta_{(2)} \in H_q \otimes H_p$ we have

$$\sum (\bar{\zeta}^{(1)*} \otimes \text{id}_{n-p}) P_n^+(\zeta^{(2)} \otimes \text{id}_{n-q}) = \alpha_{p,q}^n \sum P_{n-p}^+(\zeta_{(1)} \bar{\zeta}_{(2)}^* \otimes \text{id}_{n-p-q}) P_{n-q}^+,$$

where the coefficient $\alpha_{p,q}^n$ satisfies $|\alpha_{p,q}^n| \leq 1$. This formula is stated in [FV16] only in the Kac case — since the methods used in that article to study MASAs only hold in the tracial case. However a rapid inspection shows that its statement and proof hold without modification in the general, non-Kac case.

Lemma 7.12 (The Ice-Swimming Lemma). *For any $1 \leq l \leq k \leq n \leq t$ and $\xi \in \mathbf{H}_{k-l} \otimes \mathbf{H}_l$ there are vectors $\xi' \in \mathbf{H}_{k-l} \otimes \mathbf{H}_l$, $\xi'' \in \mathbf{H}_{k-l-1} \otimes \mathbf{H}_{l-1}$ with $\|\xi'\|$, $\|\xi''\| \leq \|\xi\|$ such that*

$$(7.5) \quad \|w_{n,t}^{k,l}(\xi)\|_{\text{HS}} \leq \|F_l\| \left(\frac{\dim_q(t-l)}{\dim_q(t)} \right)^{1/2} \|w_{n-l,t-l}^{k,l}(\xi')\|_{\text{HS}} + \\ + C \|F_l\|^2 \|w_{n-1,t-1}^{k-2,l-1}(\xi'')\|_{\text{HS}} + C q^{-3l/2} \|F_l\|^2 q^{n/2} \|\xi\|,$$

where C is a constant depending only on q . In the case $k = l$ we can delete the term involving ξ'' . In the case $k = 2l$, if $t_l^*(\xi) = 0$ then we have also $t_{l-1}^*(\xi'') = 0$.

Proof. In this proof $C > 0$ is a “generic constant”, depending only q , that we will modify only a finite number of times.

We first use the estimate $P_t^+ = (\text{id}_l \otimes P_{t-l}^+)(P_n^+ \otimes \text{id}_{t-n}) + D$ from [VV07, Lemma A.4], with “generic” error term $D \in B(\mathbf{H}_l \otimes \mathbf{H}_{n-l} \otimes \mathbf{H}_{t-n})$ controlled by $\|D\| \leq C q^{n-l}$, and we permute through the trace according to $\text{qTr}_l(\zeta \bar{\xi}^{(2)*}) = (F_l \bar{\xi}^{(2)})^* \zeta$:

$$(7.6) \quad w_{n,t}^{k,l}(\xi) \simeq (\dim_q t)^{-1/2} \sum (\text{qTr}_l \otimes \text{qTr}_{n-l} \otimes \text{id}) [P_s^+(\xi^{(1)} \otimes \text{id}_{t-l}) ((\bar{\xi}^{(2)*} \otimes \text{id}_{n-l}) P_n^+) \otimes \text{id}_{t-n}] \\ = (\dim_q t)^{-1/2} \sum (\text{qTr}_{n-l} \otimes \text{id}) [(F_l \bar{\xi}^{(2)})^* \otimes \text{id}_{s-l}] P_s^+(\xi^{(1)} \otimes \text{id}_{t-l}).$$

To control the error term $E \in B(\mathbf{H}_{t-n}, \mathbf{H}_{s-n})$ let us recall that $\sum \|\xi^{(1)}\| \|\bar{\xi}^{(2)}\| \leq \|\xi\| \dim_q(l)^{1/2}$, see the proof of Lemma 7.11. We shall also use a similar formula

$$(7.7) \quad \sum \|\xi^{(1)}\| \|j^* \xi^{(2)}\| \leq \|\xi\| \dim_q(l)^{1/2}$$

which is proved as the previous one by noting that $\|F_l\| = \|F_l^{-1}\|$. In particular

$$\sum \|\xi^{(1)}\| \|F_l \bar{\xi}^{(2)}\| = \sum \|\xi^{(1)}\| \|j^* \xi^{(2)}\| \leq \|\xi\| \dim_q(l)^{1/2}.$$

Hence we have

$$\|E\|_{\text{HS}} \leq \|\text{qTr}_{t-n}\|^{1/2} \|E\| \leq C q^{n-l} (\dim_q t)^{-1/2} \|\text{qTr}_{t-n}\|^{1/2} \|\text{qTr}_n\| \sum \|\xi^{(1)}\| \|F_l \bar{\xi}^{(2)}\| \\ \leq C q^{n-l} (\dim_q t)^{-1/2} \dim_q(t-n)^{1/2} \dim_q(n) \dim_q(l)^{1/2} \|\xi\| \leq C q^{-3l/2} q^{n/2} \|\xi\|.$$

This will go into the third term on the right-hand side of (7.5).

Now we use [FV16, Lemma 3.4]. We want to apply it to $\tilde{\sigma}(\xi) := \sum F_l^{-1} \xi^{(2)} \otimes \xi^{(1)}$ which is *not* in \mathbf{H}_k (even if ξ was), so we first decompose:

$$\tilde{\sigma}(\xi) = \xi' + (P_l^+ \otimes P_{k-l}^+)(\text{id}_{l-1} \otimes t_1 \otimes \text{id}_{k-l-1})(\xi_0'')$$

with $\xi' = \sum \xi'^{(1)} \otimes \xi'^{(2)} \in \mathbf{H}_k \subset \mathbf{H}_l \otimes \mathbf{H}_{k-l}$ given by $\xi' = P_k^+ \tilde{\sigma}(\xi)$ — in particular $\|\xi'\| \leq \|F_l\| \|\xi\|$ —, and with $\xi'' = \sum \xi''^{(1)} \otimes \xi''^{(2)} \in \mathbf{H}_{l-1} \otimes \mathbf{H}_{k-l-1}$ which we map into the orthogonal complement of \mathbf{H}_k in $\mathbf{H}_l \otimes \mathbf{H}_{k-l}$ as explained in subsection 7.1. We denote by W' , W'' the corresponding terms of the right-hand side of (7.6).

The statement of [FV16, Lemma 3.4] involves a rearrangement of the tensor product decomposition: $\xi' = \sum \xi'_{(1)} \otimes \xi'_{(2)}$ with $\xi'_{(1)} \in \mathbf{H}_{k-l}$, $\xi'_{(2)} \in \mathbf{H}_l$. We can then write

$$\begin{aligned} W' &= (\dim_q t)^{-1/2} \sum (\text{qTr}_{n-l} \otimes \text{id}) [\bar{\xi}'^{(1)*} \otimes \text{id}_{s-l}] P_s^+ (\xi'^{(2)} \otimes \text{id}_{t-l}) \\ &= (\dim_q t)^{-1/2} \alpha_{l,k-l}^s \sum (\text{qTr}_{n-l} \otimes \text{id}) [P_{s-l}^+ (\xi'_{(1)} \bar{\xi}'_{(2)}^* \otimes \text{id}_{t-2l}) P_{t-l}^+] \\ &= \left(\frac{\dim_q(t-l)}{\dim_q(t)} \right)^{1/2} \alpha_{l,k-l}^s w_{n-l,t-l}^{k,l}(\xi'). \end{aligned}$$

We recall that $\alpha_{l,k-l}^s \leq 1$. Replacing ξ' with $\xi'/\|F_l\|$ we obtain the first term on the right-hand side of (7.5).

Now we take ξ'' into account. Observe that we used $(P_l^+ \otimes P_{k-l}^+)(\text{id}_{l-1} \otimes t_1 \otimes \text{id}_{k-l-1})$ to embed $\mathbf{H}_{l-1} \otimes \mathbf{H}_{k-l-1}$ into $\mathbf{H}_l \otimes \mathbf{H}_{k-l}$ as the orthogonal of \mathbf{H}_k , and on each irreducible subspace $P_m^{l-1,k-l-1}(\mathbf{H}_{l-1} \otimes \mathbf{H}_{k-l-1})$ equivalent to \mathbf{H}_m , with $m = k - 2 - 2b$, this morphism has norm controlled as follows [Ver05, Proposition 2.3]:

$$\begin{aligned} \|(P_l^+ \otimes P_{k-l}^+)(\text{id}_{l-1} \otimes t_1 \otimes \text{id}_{k-l-1}) P_m^{l-1,k-l-1}\|^2 &= \frac{\dim_q(l) \dim_q(k-l-1) - \dim_q(l-b-1) \dim_q(k-l-b-2)}{\dim_q(l-1) \dim_q(k-l-1)} \\ &= \frac{\dim_q(b) \dim_q(k-b-1)}{\dim_q(l-1) \dim_q(k-l-1)} \geq C > 0. \end{aligned}$$

In particular we have $\|\xi''_0\| \leq C^{-1} \|\tilde{\sigma}(\xi)\| \leq C \|F_l\| \|\xi\|$.

Then we compute the term corresponding to ξ'' in $w_{n,t}^{k,l}(\xi)$, that is, we replace $F_l \bar{\xi}^{(2)}$ with $P_l^+(\bar{e}_i \otimes \bar{\xi}_0^{(1)})$ and $\xi^{(1)}$ with $P_{k-l}^+(\bar{e}_i \otimes \xi''^{(2)})$ in (7.6), where $(e_i)_i$ is a ONB of \mathbf{H}_1 . The projections P_l^+ , P_{k-l}^+ are absorbed in P_s^+ and we moreover observe that we have, for $f \in B(\mathbf{H}_1)$: $\sum \bar{e}_i^* f \bar{e}_i = \text{qTr}'_1(f)$. Since $\dim_q(s-1)(\text{qTr}'_1 \otimes \text{id})(P_s^+) = \dim_q(s) P_{s-1}^+$ we obtain

$$\begin{aligned} W'' &= (\dim_q t)^{-1/2} \sum (\text{qTr}_{n-l} \otimes \text{id}) [(\bar{\xi}_0^{(1)*} \otimes \text{id}_{s-l})(\text{qTr}'_1 \otimes \text{id})(P_s^+)(\xi''^{(2)} \otimes \text{id}_{t-l})] \\ &= (\dim_q t)^{-1/2} \frac{\dim_q(s)}{\dim_q(s-1)} \sum (\text{qTr}_{n-l} \otimes \text{id}) [(\bar{\xi}_0^{(1)*} \otimes \text{id}_{s-l}) P_{s-1}^+(\xi''^{(2)} \otimes \text{id}_{t-l})]. \end{aligned}$$

Applying backwards the first step of this proof, with $\tilde{\sigma}(\xi)$ replaced by ξ''_0 , we recognize

$$W'' \simeq \left(\frac{\dim_q(t-1)}{\dim_q(t)} \right)^{1/2} \frac{\dim_q(s)}{\dim_q(s-1)} w_{n-1,t-1}^{k-2,l-1}(\tilde{\sigma}^{-1}(\xi''_0)),$$

with a new error term E with the same control as previously — except that $\|\xi\|$ is replaced by $\|\tilde{\sigma}^{-1}(\xi''_0)\| \leq \|F_{l-1}\| \|\xi''_0\| \leq C \|F_l\|^2 \|\xi\|$ — and which goes into the third term on the right-hand side of (7.5) as well. We finally put $\xi'' = \tilde{\sigma}^{-1}(\xi''_0)/C \|F_l\|^2$ which satisfies $\|\xi''\| \leq \|F_{l-1}\| \|\xi''_0\|/C \|F_l\|^2 \leq \|\xi\|$. Since $\dim_q(t-1)/\dim_q(t) \leq 1$ and $\dim_q(s)/\dim_q(s-1) \leq C$ this yields the second term on the right-hand side of (7.5).

In the case $k = l$ we have already $\tilde{\sigma}(\xi) = F_k^{-1} \xi \in \mathbf{H}_k$ so that $\xi'' = 0$. In the case $k = 2l$, we have $\tilde{\sigma}^* t_l = \pm t_l$ (c.f. (7.1)), in particular if $t_l^*(\xi) = 0$ we get $t_l^* \tilde{\sigma}(\xi) = 0$. Since $t_l^*(\mathbf{H}_{2l}) = \{0\}$ this implies $t_{l-1}^*(\xi''_0) = 0$ and finally $t_{l-1}^*(\xi'')$ vanishes because it is proportional to $t_{l-1}^* \tilde{\sigma}^{-1}(\xi''_0) = t_{l-1}^*(\xi''_0) = 0$. This crucial property allows in the proof of the next Theorem to initialize the induction at $k = 2$ and not $k = 0$ (where the required estimate is false). \square

We are ready to state and prove the second main result of this section: the proof will exploit the expression obtained in Lemma 7.10 and the induction based on Lemma 7.12.

Theorem 7.13. *For $N \geq 3$ the boundary action β_∞ of $\mathbb{F}O_Q$ is faithful.*

Proof. From Lemma 7.7 and Lemma 7.9 we know that the elements $q^n z_n$ of \mathbf{N}_∞ are uniformly bounded and satisfy $p_0 q^n z_n = p_0$. It remains to prove that $p_k q^n z_n \rightarrow 0$ for $k > 0$ fixed as $n \rightarrow \infty$. For this we use Lemma 7.12. We put $\rho = \|F_1\|$, so that $\|F_l\| = \rho^l$, and we recall that $\rho q < 1$.

More precisely we perform an induction over k , with induction assumption (H_k) : “there exists a constant $M_k > 0$ such that, for any $0 \leq l \leq k$, $t \geq n \geq k$ and for any $\xi \in \mathbf{H}_{k-l} \otimes \mathbf{H}_l$ such that $t_l^*(\xi) = 0$ (when $k = 2l$), we have $\|w_{n,t}^{k,l}(\xi)\|_{\text{HS}} \leq M_k \rho^{n/2} \|\xi\|$ ”. We prove that (H_{k-2}) implies (H_k) , and at the same time initialize the induction at $k = 1$ and $k = 2$.

First of all we ensure that $l \geq 1$ by replacing if necessary $w_{n,t}^{k,l}(\xi)$ with $w_{n,t}^{k,k-l}((F_l^{1/2} \otimes F_{k-l}^{1/2})\bar{\xi})$, c.f. Lemma 7.11. Since $s \subset t \otimes k$ the resulting factor $\dim_q(s)^{1/2} / \dim_q(t)^{1/2}$ is controlled from above by $\dim_q(k)^{1/2}$, which we incorporate into M_k . Then we apply Lemma 7.12. When $k = 2l$ we assume that $t_l^*(\xi) = 0$, hence we have also $t_{l-1}^*(\xi'') = 0$ and we can apply (H_{k-2}) to $w_{n-1,t-1}^{k-2,l-1}(\xi'')$, obtaining for all $t \geq n \geq k$:

$$\begin{aligned} \|w_{n,t}^{k,l}(\xi)\|_{\text{HS}} &\leq \rho^l \left(\frac{\dim_q(t-l)}{\dim_q(t)} \right)^{1/2} \|w_{n-l,t-l}^{k,l}(\xi')\|_{\text{HS}} + C \rho^{2l} M_{k-2} \rho^{n/2} \|\xi''\| + C q^{-3l/2} \rho^{2l} q^{n/2} \|\xi\| \\ &\leq \rho^l \left(\frac{\dim_q(t-l)}{\dim_q(t)} \right)^{1/2} \|w_{n-l,t-l}^{k,l}(\xi')\|_{\text{HS}} + M'_k \rho^{n/2} \|\xi\|, \end{aligned}$$

where $M'_k = C \rho^{2k} M_{k-2} + C \rho^{2k} q^{-3k/2}$. Note that if $k = 1$ we are in the case $k = l = 1$ hence $\xi'' = 0$. The same applies if $k = 2$ and $l = 2$. If $k = 2$ and $l = 1$, we have $\xi'' \in \mathbf{H}_0 \otimes \mathbf{H}_0 = \mathbb{C}$ and $t_0^*(\xi'') = 0$ hence $\xi'' = 0$ as well. As a result we do not need to use (H_0) (which is false by Lemma 7.7).

Then we apply the above inequality repeatedly until $n < l$. The index n takes the values $n - pl$ with p ranging from 0 to $r := \lfloor n/l \rfloor$. Using the inequality $\dim_q(t - pl) / \dim_q(t) \leq C q^{pl}$ we obtain

$$\begin{aligned} \|w_{n,t}^{k,l}(\xi)\|_{\text{HS}} &\leq \sum_{p=0}^r \rho^{pl} \left(\frac{\dim_q(t-pl)}{\dim_q(t)} \right)^{1/2} M'_k \rho^{(n-pl)/2} \|\xi\| + \rho^{l(r+1)} q^{l(r+1)/2} \|w_{n-r,l,t-r,l}^{k,l}(\xi)\|_{\text{HS}} \\ &\leq C M'_k \left(\sum_{p=0}^\infty (\rho q)^{pl/2} \right) \rho^{n/2} \|\xi\| + C \rho^{n/2} \rho^{k/2} q^{-k} \|\xi\|. \end{aligned}$$

For the second term we used the inequalities $\rho q < 1$, $lr \leq n$ and the “easy” upper bound from Lemma 7.11: $\|w_{n-ql,t-ql}^{k,l}(\xi)\|_{\text{HS}} \leq C q^{-(n-ql)/2} q^{-l/2} \|\xi\| \leq C q^{-k} \|\xi\|$. We have now proved (H_k) .

Now we can come back to $p_k z_n$, for $k \geq 1$. By Cauchy-Schwarz we have:

$$|\text{qTr}_{t-n}(w_{n,t}^{k,l}(\zeta)^* w'_{n,t}{}^{k,l}(\xi))| \leq \|w_{n,t}^{k,l}(\zeta)\|_{\text{HS}} \|w'_{n,t}{}^{k,l}(\xi)\|_{\text{HS}} \leq C M_k q^{-k/2} \rho^{n/2} q^{-n/2} \|\zeta\| \|\xi\|.$$

We used Lemma 7.11 for the term in w' , and the property (H_k) for the term in w . Now we apply Lemma 7.10 and recall that $\kappa_s^{k,t} \leq D_2 q^{l/2} \leq D_2$. Taking the limit $t \rightarrow \infty$, the sum over $0 \leq l \leq k$ and the sup over $\zeta, \xi \in H_k$ of norm 1 this yields $\|p_k q^n z_n\| \leq C D_2^2 (k+1) M_k q^{-k/2} \rho^{n/2} q^{n/2}$. Since $\rho q < 1$ we have proved that $p_k q^n z_n \rightarrow 0$ as $n \rightarrow \infty$. \square

Theorem 5.3 now yields immediately the following corollary. In the unimodular case it already appeared in [VV07], whereas the non-unimodular case of [VV07, Theorem 7.2] deals

with the modular group of the Haar state instead of the scaling group. Recall that when Q is normalized by $QQ = \pm I_N$, $\mathbb{F}O_Q$ is unimodular **iff** Q is unitary.

Corollary 7.14. *Let $N \geq 3$ and let $Q \in M_N(\mathbb{C})$ be such that $QQ = \pm I_N$. If Q is unitary the C^* -algebra $C_r^*(\mathbb{F}O_Q) = C(O_Q^+)$ admits a unique trace, and otherwise it does not admit any KMS state for the scaling automorphism group at the inverse temperature 1.*

8. OPEN QUESTIONS

In this short section we gather and discuss some of the open question arising from our work. We begin with some questions related to Section 3, and concepts related to relative amenability/coamenability.

Question 8.1. *Suppose that \mathbb{F} is a compact quantum group with a closed quantum subgroup \mathbb{H} . Is \mathbb{G}/\mathbb{H} coamenable if and only if $\ell^\infty(\widehat{\mathbb{H}})$ is relatively amenable in $\ell^\infty(\widehat{\mathbb{G}})$?*

We know the equivalence above holds for *normal* quantum subgroups (so for example if \mathbb{G} is a dual of a classical discrete group) – see Theorem 3.10. The positive answer would generalize the duality between (non-relative) amenability and coamenability from [Tom06] and would provide another description of the co-kernel of the Furstenberg boundary action for any discrete quantum group.

Question 8.2. *Does every compact quantum group \mathbb{G} admits a smallest closed quantum subgroup \mathbb{H} such that \mathbb{G}/\mathbb{H} is coamenable?*

One possible approach to this question is the following: let \mathbb{G} be a compact quantum group and $\mathbb{H}_i \subset \mathbb{G}$, $i = 1, 2$ be a pair of quantum subgroups such that $\pi_i \in \text{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}_i))$ admit the reduced versions $\tilde{\pi}_i \in \text{Mor}(C(\mathbb{G}), C(\mathbb{H}_i))$. Denote by $\tilde{I}_i = \ker \tilde{\pi}_i$, $i = 1, 2$. Let I be the ideal in $C(\mathbb{G})$ generated by \tilde{I}_1 and \tilde{I}_2 . Is it then true that $\mathbf{1} \notin I$? If the answer is positive, so is the answer to the last displayed question, and we would have a well-defined notion of the ‘co-amenable co-radical’ of a compact quantum group.

Question 8.3. *Must the co-kernel of the Furstenberg boundary action of a discrete quantum group \mathbb{G} be a normal coideal in $\ell^\infty(\widehat{\mathbb{G}})$?*

A positive answer to the above question would yield a simple description of the co-kernel of the Furstenberg boundary action for any discrete quantum group and would in particular imply that the Furstenberg boundary action of \mathbb{G} is faithful if and only if the amenable radical of \mathbb{G} is trivial.

We now pass to a series of questions regarding the operator algebras associated to discrete quantum groups. Note that for classical discrete groups all the answers below are positive, but some have been obtained only recently.

Question 8.4. *Let \mathbb{F} be a discrete quantum group and let $\mathbb{A} \subset \mathbb{F}$ be a normal quantum subgroup. Is there any relation between C^* -simplicity or the unique trace property (in the unimodular case) of \mathbb{F} and \mathbb{A} ?*

Question 8.5. *Does the converse of Theorem 5.3 hold, at least in the unimodular case? That is, does the unique trace property of a unimodular discrete quantum group \mathbb{F} imply that the action $\mathbb{F} \curvearrowright C(\partial_F(\mathbb{F}))$ is faithful?*

Question 8.6. *Suppose that \mathbb{F} is unimodular. Is C^* -simplicity stronger than the unique trace property?*

The first two questions are naturally related to the following one, which ends our list.

Question 8.7. *Let \mathbb{F} be a discrete quantum group and let $\mathbb{A} \subset \mathbb{F}$ be a normal quantum subgroup. Does the action $\mathbb{A} \curvearrowright C(\partial_F \mathbb{A})$ extends to an action $\mathbb{F} \curvearrowright C(\partial_F \mathbb{A})$? (In that case it would be automatically a \mathbb{F} -boundary by Proposition 4.5.)*

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