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Gamma-Elements For Free Quantum Groups
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1. Motivations
   - The first motivation was the study of the K-theoretic properties of the free quantum groups, and in particular of their K-amenability. The notion of K-amenability was introduced by J. Cuntz for the study of the K-theory of the free groups : a discrete group is K-amenable if its regular representation induces a KK-equivalence between its full and reduced C^*-algebras. On the other hand S. Baaj and G. Skandalis extended equivariant KK-theory to the case of coactions of, say, quantum groups, and it is not hard to generalize the notion of K-amenability to this framework.
   - More precisely, the idea is to use geometric methods to obtain this K-theoretic property, following the work of P. Julg and A. Valette on groups acting on trees. Let me present one of the important geometric ingredient of their method in the case of the free group on two generators. Consider a geometric (ie non-oriented) edge in the Cayley graph of this free group, and choose the furthest endpoint of this edge from a given origin. This defines an operator from the ℓ^2-space of edges to the ℓ^2-space of vertices. Together with the natural actions of the free group (by translation), it is very easy to see that it defines an element $\gamma \in KK_{F_2}(\mathbb{C}, \mathbb{C})$. This construction is the first step of the proof of the K-amenability of the free group, the other one being the equality $\gamma = 1$.
   - To apply such a method to the free quantum groups, the first step is to construct the needed geometric objects, ie a kind of « quantum graph ». An encouraging indication in this direction is the simpler case of amalgamated free products of amenable discrete quantum groups : in this case, one can construct a quantum analogue of the Serre tree of the amalgamated free product, and use it to prove the K-amenability of these amalgamated free products. Note that in the « classical case » this is another special case of Julg and Valette's result.
2. Compact Quantum Groups
   - The most general framework of the study is the theory of compact quantum groups, which is due to Woronowicz. I will just recall how to associate a compact qunatum group to a discrete group : as a matter of fact, this will be our « classical case », meaning that we will use Woronowicz' theory as a theory of discrete quantum groups.
   - Let Γ be a discrete group. The Hopf-C^*-algebra S associated to Γ will be the full C^*-algebra $C^*(\Gamma)$ endowed with the coproduct given by $\delta(r) = r \otimes r$ for any $r \in \Gamma \subset S$. Let's denote by $\mathcal{C}$ the category of finite dimensional corepresentations of (S, δ), and by Irr $\mathcal{C}$ the set of irreducible corepresentations (up to equivalence). In our case, any irreducible corepresentation is one-dimensional and has the form $\mathrm{id}_{\mathbb{C}} \otimes r$ for some $r \in \Gamma \subset S$ and up to equivalence. So Irr $\mathcal{C}$ is nothing but the group itself, the product and inverse of the group corresponding respectively to the tensor product and conjugation of corepresentations. In the general case it will sometimes be usefull to think of this set as of the set of points of the discrete quantum group.
   - Via the GNS representations of the Haar state, one just recovers the regular representation of Γ and its reduced C^*-algebra $C^*_{\mathrm{red}}(\Gamma)$. The ℓ^2-space H of Γ can be decomposed in lines associated to the elements of Γ : $\ell^2(\Gamma) = \bigoplus_{r \in \Gamma} \mathbb{C}\delta_r$. In the general case, one still has such a decomposition, indexed by Irr $\mathcal{C}$ (subspaces of coefficients of a corepresentation), and the corresponding projections are denoted by p_r. Finally, let me write the « classical formulas » for the Kac system of Γ : $V(\delta_r \otimes \delta_r) = \delta_r \otimes \delta_{rs}$ and $U(\delta_r) = \delta_{r^{-1}}$.
3. Quantum Cayley Graphs

- Let me recall the case of a discrete group before going to the general quantum definitions. The data we are starting from is then a discrete group Γ and a finite subset $\Delta \subset \Gamma$ playing the role of the set of directions, such that $\Delta^{-1} = \Delta$ and $1_\Gamma \notin \Delta$. In the quantum case, we take a discrete quantum group, with all the objects associated to it : the C^*-algebra S, the category $\mathcal{C}$, the Kac system (H, V, U), ..., and we choose a finite subset $\mathcal{D} \subset \mathcal{C}$ such that $\bar{\mathcal{D}} = \mathcal{D}$ and $1_\mathcal{C} \notin \mathcal{D}$. Equivalently, $\mathcal{D}$ can be replaced by a central projection $p_1 \in \hat{S}$ commuting with U and disjoint from p_0, the central support of the trivial corepresentation $1_\mathcal{C}$.
- Then one define two different objects, generalizing in different directions the Cayley graphe associated to (Γ, Δ). Consider first the simplicial picture of this Cayley graph : the set of vertices $\mathfrak{v}$ is the group itself, and the (oriented) edges are the pairs (r, r') of vertices such that $r' = rs$ for some s in the set of directions Δ. We generalize this picture to the quantum case by tacking $\mathfrak{v} = \mathrm{Irr}\ \mathcal{C}$ — recall that $\mathrm{Irr}\ \mathcal{C}$ plays the role of the set of points of the discrete quantum group —, and by replacing the condition $r' = rs$ by the inclusion $r' \subset r \otimes s$. The result is called the classical graph associated to $(\mathcal{C}, \mathcal{D})$.
- In the discrete case, the evident reversing map, sending the edge (r, r') to the edge (r', r), is well-defined because we require Δ to be symmetric. This is still the case in the quantum case thanks to the equivalence $r' \subset r \otimes s \ \Leftrightarrow r \subset r' \otimes \bar{s}$ (Jacobi duality). Hence there is a notion of geometrical edges, which amounts to identifying an oriented edge and the reversed edge, and a notion of orientation : an orientation is a subset $\mathfrak{e}_+ \subset \mathfrak{e}$ such that $\mathfrak{e} = \mathfrak{e}_+ \sqcup \theta(\mathfrak{e}_+)$, it amounts to choosing one oriented edge for each geometrical one.
- Now, there is another picture of the Cayley graph of a discrete group, were the edges are given by an origin and a direction. In this picture, one still has $\mathfrak{v} = \Gamma$, but one puts $\mathfrak{e} = \Gamma \times \Delta$. One has then to define source and target maps : the source of (r, s) is r and its target is the product rs. These maps define a bijection between this picture and the simplicial one, and in particular one can read the reversing map in the new picture : $\theta(r, s) = (rs, s^{-1})$.
- It is the generalization of this second picture which will give us the quantum graph associated to $(\mathcal{C}, \mathcal{D})$. We proceed in the spirit of non-commutative geometry, at the level of the ℓ^2-spaces. It is clear that the ℓ^2-space of quantum vertices should be H itself, the ℓ^2-space of the quantum group, and that one should take $K = H \otimes (p_1 H)$ for the ℓ^2-space of quantum edges : as a matter of fact p_1 plays the role of the projection onto the subset of directions. The way of generalizing the formula $\theta(r, s) = (rs, s^{-1})$ is less evident, the right formula being $\Theta = \Sigma(1 \otimes U) V (U \otimes U) \Sigma$. To give a meaning to this ℓ^2-quantum graph, we need source and target operators, in fact they are both contained in an « endpoints » operator which is nothing but V itself.
- The ℓ^2-quantum graph associated to $(\mathcal{C}, \mathcal{D})$ is the right geometric tool for KK-theory because H and K are naturally endowed with representations of the C^*-algebra S that intertwin all the operators defined here. But it is much more complicated than in the case of discrete groups, the main reason for this being the non-involutivity of the reversing operator Θ. On the other hand, the classical graph associated to $(\mathcal{C}, \mathcal{D})$ will help to understand the quantum one, for instance via the decomposition of H into orthogonal subspaces indexed by $\mathrm{Irr}\ \mathcal{C}$.

4. Ascending Edges
   - Now we can try to apply the method of Julg and Valette to this notion of Cayley graph. For that we need to restrict ourselves to the case of trees, and because it is not clear what this means for the quantum graph, we require instead the classical graph associated to $(\mathcal{C}, \mathcal{D})$ to be a tree. There is a first proposition telling us that this amounts to considering discrete quantum groups that are free products of free quantum groups.

– The key notion for the construction of the gamma element is the notion of ascending orientation. In the classical graph, this corresponds to the choice of the oriented edges that go away from the origin. We use this classical orientation to construct a projection on ascending edges in the ℓ^2-space of quantum edges : we just look at the sum of the $p_r \otimes p_{r'}$ on the classical ascending edges $(r, r') \in \mathfrak{e}_+$, and we conjugate by the endpoints operator V to come back to the « origin + direction » picture. The result is called $p_{\star+}$.
– Now we make a simple check : the completary projection, $1 - p_{\star+}$, should select the descending edges, and if we conjugate it by the reversing operator Θ we should get back the ascending edges. Unfortunately, this gives a new projection (except of course in the classical case), wich commutes with the first one, and which we call $p_{+\star}$. So we have to candidates for the projection onto ascending edges, and we choose in fact to consider their product p_{++}. We put $K_{++} = p_{++}K$, and similarly $p_{+-} = p_{+\star}p_{\star-}$, $K_{+-} = p_{+-}K$, ...
– Then the definition of the Julg-Valette operator is quite obvious. Starting from the space of geometrical edges we first choose the ascending orientation of the edges by applying p_{++}, and then we take the target of the result : this should effectively give us the furthest endpoint from the origin.

5. Space of Edges at Infinity
   – Now we have to investigate this Julg-Valette operator to see if it can define an element of KK-theory : first of all, it should be a Fredholm operator. We start with the second step of the Julg-Valette operator (applying the target operator to ascending edges), because this is the easiest one : like in the classical case, this second step is injective and its image has codimension one (corresponding to the origin). This indicates the subspace of ascending edges K_{++} is not so bad.
   – Now we turn to the projection from K_g onto K_{++} : geometrically, this amounts to asking wether we have a good notion of orientation, that is, exactly one ascending edge for each geometrical edge. The first theorem of the slide gives us the injectivity of this projection, but we only get a quite complicated expression for its image. In the classical case the condition in this expression is empty because p_{+-} is zero, but in the quantum case it is more complicated and in particular one has to study the operator $p_{+-}\Theta p_{+-}$.
   – Here we restrict ourselves to the case of $A_o(Q)$ in order to make computations — in fact this is not a too severe restriction. In particular one studies the behaviour of $p_{+-}\Theta p_{+-}$ in the decomposition of K given by the distance to the origin in the classical gaph : $K = \bigoplus_k \big(\bigoplus_{\mathrm{d}(r,1)=k}(p_r \otimes \mathrm{id})(K)\big)$. It turns out that $p_{+-}\Theta p_{+-}$ looks like a shift in this decomposition, and it is then natural to introduce the associated inductive limit, H_∞, which is an infinite dimensional Hilbert space.
   – Now we can precisely describe $p_{++}K_g$, and hence the image of the Julg-Valette operator : the second theorem states that this image is closed and that its orthogonal is isomorphic to H_∞, and in particular, infinite dimensional. Finally, to get a gamma element, we have to add a purely quantum part to the naive Julg-Valette operator. Of course to get an element of KK-theory one must also define a « natural » action of the discrete quantum group on H_∞ and check some commutation properties : this is beyond the scope of my talk.
   – Let me end with a few geometrical remarks. The fact that $p_{+-}\Theta p_{+-}$ acts like a shift on the edges is very strange : it means that when you iterate the reversing process, some pieces of your starting edge goes to the infinity of the quantum graph. On the ohter hand, the non-surjectivity of the projection from K_g onto K_{++} means that the space of geometrical edges is too small : this is of course an effect of the non-involutivity of the reversing operator. What we have done is just to enlarge this space of geometrical edges by pieces of edges located at the infinity of the graph, which don't exist in the classical case.