Introduction : I will present an overview on the results I have obtained in my thesis, and on the problems I have been working on since then. As a result, I won't have time to go into the details, but I hope that such an overview will nevertheless fit the aim of this workshop. ((☺))

Quantum groups (1)

No time to really introduce the theory of l.c. quantum groups. Just recall some notations: we have two dual Hopf C^*-algebras $S, \hat{S}$, ie C^*-algebras S equipped with a coproduct δ which is a non-degenerate homomorphism from S to some multiplier algebra of $S \otimes S$, and which is coassociative: $(\delta \otimes id)\delta = (id \otimes \delta)\delta$. The theory also provides us with two "regular" representations $\lambda, \hat{\lambda}$ of S and $\hat{S}$ on an Hilbert space H, and with two "trivial" rep $\varepsilon, \hat{\varepsilon}$.
There is also a notion of coaction on a C^*-algebra A: it is a homomorphism $\delta_A : A \to \tilde{M}(A \otimes S)$ compatible with the coproduct of S. To such a coaction one can associate a full and a reduced crossproduct: $A \rtimes \hat{S}$, $A \rtimes_{red} \hat{S}$. B&S also defined S-equivariant KK groups $KK_S(A,B)$ for C^*-algebras endowed with coactions of S.
Example: $S = S_{red} = C_0(G)$, $\delta(f)(r,s) = f(rs)$, $H = L^2(G)$, $\hat{S} = C^*G$, $\hat{S}_{red} = C^*_{red} G$, coactions of $S \Leftrightarrow G$-actions, $KK_S = KK_G$.

5'

Green-Julg theorem

This result relates equivariant KK-theory with the KK-theory of crossproducts, in the compact case, ie when S (and S_{red}) are unital. One assumes that the coaction on A is trivial, and that B is a S_{red}-algebra for S_B. Then one gets an isomorphism between $KK_{S_{red}}(A,B)$ and $KK(A, B \rtimes_{red} \hat{S})$ that can be described very simply. One first applies the descent morphism to $KK_{S_{red}}(A,B)$: this is a generalization of the crossproduct construction at the level of Kasparov triples, which can be extended to the quantum case without difficulty. Then one uses the map $\phi: A \to A \otimes \hat{S}_{red} = A \rtimes_{red} \hat{S}$, $a \mapsto a \otimes p_0$, where p_0 is the central support of $\hat{\varepsilon}$. I also found a simple expression of the inverse isomorphism, which wasn't known even in the case of compact groups: one first applies the evident morphism Ψ that puts trivial coactions everywhere (the index 1 points out to the fact that the C^*-algebras are equipped with trivial coactions), and then one uses the Kasparov product on the right by a certain element β which appears naturally in the following way. $B \rtimes_{red} \hat{S}$ is defined as a C^* algebra of operators on the Hilbert B-module $B \otimes H$, and in the compact case one can show that $B \rtimes_{red} \hat{S} = K_B(B \otimes H)^{S_{red}}$ for a certain coaction of S_{red} on $B \otimes H$. With $F = 0$, this gives $\beta \in KK((B \rtimes_{red} \hat{S})_1, B)$.

9'30

K-amenability

The notion of K-amenability was introduced by Cuntz: for discrete groups, it means that the regular representation induces a KK-equivalence. I have proved that the implications between the various notions of K-amenability (which are all equivalent in the discrete case) carry over to the quantum case. In particular, to prove the K-amenability of a discrete quantum group it is enough to lift the class of the trivial repr. $[\varepsilon]$ to an element $\alpha \in KK(S_{red}, \mathbb{C})$, and then full and reduced are "KK-equivalent".

11'

The rest of my work deals with that notion of K-amenability: in particular I tried to prove the K-amenability of certain classes of discrete quantum groups by means of geometrical methods, and in particular, the method of Julg-Valette for groups acting on trees. As a consequence I was led to define and study "quantum trees" in some particular situations.

Classical graphs (no slide)

A classical graph is given by a set of vertices V, a set of (oriented) edges E, two target & source maps $s,t : E \to V$ and a reversing map $\theta : E \to E$. The set of oriented edges is $E_g = E/\langle x = \theta(x)\rangle$. An orientation is a subset $E_o \subset E$ s.t. the proj. $E_o \to E_g$ is a bijection. If (E,V) is a tree and if we fix an origin $x_0 \in V$, the ascending orientation E_+ is the subset of edges going away from x_0.

We identify $\ell^2(E_g)$ with the subspace of θ-antisymmetric vectors of $\ell^2(E)$. Then the Julg-Valette operator is given by

$$F : \ell^2(E_g) \xrightarrow{\perp} \ell^2(E_+) \xrightarrow{t} \ell^2(V).$$

Given a chosen geometric edge, F selects the ascending orientation and picks its target, ie it selects the end-point of the geometrical edge which is the furthest one from the origin.

F defines an element $\gamma \in KK_G(\mathbb{C},\mathbb{C})$, and if $\gamma = 1$ then G is K-amenable.

16'05

~~The method of Julg & Valette can be applied to the Baum-Sl...~~

Quantum groups (2)
We will interpret the case when S (and S_{red}) are unital as the discrete case: this is a dual point of view on Woronowicz' original theory of compact quantum groups. An important object of the theory is the category $\mathcal{C}$ of corepresentations of S, it induces a decomposition of $\hat{S}$ in a direct sum of matrix algebras indexed by $\operatorname{Irr}\mathcal{C}$. We will denote by p_λ the corresponding minimal central projections of $\hat{S}$.
In the classical case we have $S = C^*\Gamma$ and $\hat{S} = c_0(\Gamma)$, $\operatorname{Irr}\mathcal{C}$ can be identified with Γ and the p_λ's correspond to the lines in $c_0(\Gamma)$ generated by the elements of Γ. 18'30

Amalgamated free products
The classical method of Julg & Valette can be applied to amalgamated free products $G = G_1 *_H G_2$ of amenable groups, using the associated Bass-Serre tree: $V = G/G_1 \sqcup G/G_2$, $E_g \simeq E_o = G/H$ (G-inv orientation), $t(gH) = gG_1$, $s(gH) = gG_2$.
Given two amenable Woronowicz C^*-algebras S_1, S_2 with a common Woronowicz sub-C^*-algebra, the amalgamated free product $S = S_1 *_T S_2$ is naturally a Woronowicz C^*-algebra (Wang) with canonical conditional expectations P, R_1, R_2 onto T, S_1, S_2. Using these conditional expectations and the trivial representa°s of T, S_1, S_2, one can introduce the spaces of "ℓ^2-functions" on the amalgamated product which are invariant wrt the "subgroups" S_1, S_2, T and hence, the generalizations of the ℓ^2-spaces of the classical Bass-Serre tree. Together with natural target and source operators and with the GNS representations of S, this defines the Hilbertian quantum Bass Serre tree of $S_1 *_T S_2$. Considering

the irreducible coreps of S_1, S_2, T as points of the corresponding discrete quantum groups, one can also define a classical Bass-Serre tree associated to $S_1 *_T S_2$. It doesn't carry any representation of the quantum amalgamated product, but provides some geometrical insight on the quantum tree — more precisely, it gives orthogonal decompositions of the Hilbert spaces of the quantum tree. Using this geometrical point of view, one can a define a Julg-Valette operator for the quantum Bass-Serre tree. This operator turns out to have a simple interpretation in Voiculescu's theory of reduced amalgamated free products, as pointed out on the slide.
Finally, one can prove that this Julg-Valette operator indeed defines an element $\gamma \in KK(S_{red}, \mathbb{C})$ and moreover, that $J^*(\gamma) = [\varepsilon] \in KK(S, \mathbb{C})$. Hence an amalgamated free product of amenable discrete quantum groups is K-amenable. 25'

Guantum Cayley Graphs
Julg-Valette's method also applies to the free groups acting on their Cayley graph. One is thus led to introduce a notion of Cayley Graph for discrete quantum groups. The data are: a discrete quantum group given by its Woronowicz C^*-algebra, and a finite set of directions $\mathcal{D} \subset \mathrm{Irr}\,\mathcal{C}$ st $\overline{\mathcal{D}} = \mathcal{D}$ and $1 \notin \mathcal{D}$.
In the classical case $\mathcal{D}$ is a subset of Γ and the Cayley graph is defined by $V = \Gamma$, $E = \{(\gamma, \gamma') \in \Gamma^2 / \exists s \in \mathcal{D}\ \gamma' = \gamma s\} \simeq \Gamma \times \mathcal{D}$ (simplicial and directional pictures).
The Hilbertian Quantum Cayley Graph of $(S, \mathcal{D})$ is the natural generalization of the directional picture at the level of the ℓ^2-spaces. The formula for the reversing operator is a bit awkward, in the classical case it just reads $\Theta(f)(\gamma, s) = f(\gamma s, s^{-1})$. The ℓ^2-space of geometrical edges can then be identified with $\mathrm{Ker}(\Theta + id)$: Θ -

antisymmetric vectors in the ℓ^2-space of oriented edges. The discrete quantum groups acts by its GNS repr on H, and the trivial one on $p_1 H$: this commutes to the reversing and target operators.
There is also a classical Cayley graph, which generalizes the simplicial picture — but one has to keep track of the direction α, which is not determined by r & r' in the quantum case, and of the multiplicity of the inclusion $r \subset r'\alpha$. It induces a decomposition of the ℓ^2-spaces of the quantum graph, via the projections p_r.   32'30

Free Quantum Groups
The Cayley graph of a discrete group is a tree iff the group is free. One has a corresponding result in the quantum case, with free products of Wang-Banica free quantum groups playing the role of free groups, and using the classical Cayley graph. Then the ascending orientation of the classical Cayley graph induces a projection p_{*+} on the ℓ^2-space of quantum edges, that should allow to define a quantum JV operator.
However, many problems arise, mainly caused by the non-involutivity of the reversing operator. For instance, reversing the projection p_{*+} should give a projection $\Theta p_{*+} \Theta^*$ on descending edges, whose complement $1 - \Theta p_{*+} \Theta^*$ should be again the projection on ascending edges. However, one gets a different projection p_{+*} and one has to consider the product $p_{++} = p_{*+} p_{+*}$.
Besides, the commutation properties are not so good as in the classical case: one gets "real" compact operators instead of finite rank ones. Finally, the naive JV operator is not Fredholm.
A more carefull study show that one has to add to the naive

JV operator a spaces of "edges at infinity" which appears very naturally in the quantum case. One can then construct an element $Y \in KK(S_{red}, \mathbb{C})$ that should be helpfull to prove the K-amenability of the free quantum groups. 38'30

Negative Type Function
In Julg & Valette's method, the main geometrical ingredient in the construction of the homotopy between γ and 1 is the conditionnally negative type function on the group given by the distance to the origin. In the quantum case, it is not clear how to define the corresponding form on the dense sub*-Hopf-algebra of S: the naive one given by the distance to the origin in the classical Cayley graph isn't of (conditional) negative type.
Instead, one prefers to define the corresponding cocycle in the sense of the Proposition: as a matter of fact, this Proposition show that the "square of the norm", in an appropriate sense, of such a cocycle, is of negative type. In the classical case, the interesting cocycle is given by the paths to the origin seen as vectors in the l^2-space of geometrical edges. Again, this notion of path is not evident in the quantum case, but one uses the following remark: applying the operator $(T-S)$ to the interesting cocycle gives a trivial cocycle in H (a coboundary of the regular repr) which is easy to define in the quantum case. The last theorem shows that one can use $(T-S)$ to lift this trivial cocycle to an interesting one in K_g, in the case of $A_o(Q)$ with $\mathrm{Tr}\, Q^*Q > 2$.