Higson's proof of Atiyah-Singer index theorem.

We want to prove, for D odd 1st order symmetric elliptic LPDO $S \to M$, hermitian bundle over a compact manifold:

$$\mathrm{Ind}(D) = (-1)^{\dim M} \int_{T^*M} \mathrm{Todd}(T^*M \otimes \mathbb{C}) \cdot \mathrm{ch}([\sigma_D])$$

The idea is to insert a new quantity between the two: the topological index $\alpha_M([\sigma_D])$. We will

1) define the topological index $\alpha_M : K(T^*M) \to \mathbb{Z}$
2) prove that $\alpha_M([\sigma_D]) = \mathrm{Ind}_a(D)$
3) prove the cohomological formula for α_M

The topological index

$$\alpha_M : K(T^*M) \to K(K(L^2M)) \simeq \mathbb{Z}$$

will be induced by an asymptotic morphism

$$\alpha_t : C_0(T^*M) \to K(L^2M),$$

ie a family of maps from $C_0(T^*M)$ to $K(L^2M)$, indexed by $t \in [1, \infty[$, and such that

1) $t \mapsto \alpha_t(a)$ is continuous & bounded
2) $\varphi_t(ab) \sim \varphi_t(a)\varphi_t(b)$ (ie $\|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| \to 0$)
$\varphi_t(a^*) \sim \varphi_t(a)^*$, $\varphi_t(\lambda a + b) \sim \lambda\varphi_t(a) + \varphi_t(b)$

We start with $M = U \subset \mathbb{R}^n$ (non-compact).
From $a \in \Gamma_c^\infty(T^*M) = C_c^\infty(U \times \mathbb{R}^n)$ we define a family of kernels

$$k_t(x,y) = \frac{1}{(2\pi)^n} \int a(x, t^{-1}\xi)\, e^{i(x-y)\xi}\, d\xi$$
$$= \frac{t^n}{(2\pi)^n} \int a(x,\xi)\, e^{it(x-y)\xi}\, d\xi$$

and the associated operators on $L^2(U)$:

$$(A_t(a)u)(x) = \int k_t(x,y)\, u(y)\, dy = \int a(x, t^{-1}\xi)\, \hat{u}(\xi)\, e^{ix\xi}\, d\xi dx$$

Lemma. Let A_t be a family of operators on $L^2(U)$ associated to L^2-kernels k_t st

$$\forall t, x, y \quad |k_t(x,y)| \leq C \frac{t^n}{(1+t|y-x|)^{n+1}}$$

Then the ops A_t are uniformly bounded $\forall t \ \|A_t\| \leq K_n \cdot C$.

Moreover putting $k_{t,L} = k_t \cdot \delta_{|x-y|<t^{-1}L}$ we have $\|A_{t,L} - A_t\| \xrightarrow[L]{} 0$ uniformly in t.

Lemma. Let $f_t \in C^\infty(U, U, \mathbb{R}^n)$ be uniformly compactly supported, uniformly bounded as well as the partial derivatives up to order N, with respect to ξ:

$$\forall t \quad \operatorname{Supp} f_t \subset U \times U \times B(0,R)$$

$$\forall t, |\alpha| \leq N \quad \operatorname{Sup} \left| \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} f_t \right| \leq C$$

Put

$$k_t(x,y) = \int f_t(x,y,t^{-1}\xi)\, e^{i(x-y)\xi}\, d\xi.$$

Then

$$|k_t(x,y)| \leq \frac{t^n}{(1+t|y-x|)^N} K_{n,N}\, C \operatorname{Vol} B(0,R)$$

Applications. $a, b \in C_c^\infty(U \times \mathbb{R}^n)$.

1) The $A_t(a)$ are uniformly bounded: take $f_t(x,y,\xi) = a(x,\xi)$. In fact the same ideas apply to show that they are bdd from L^2 to H^1, and in particular, compact.

2) $A_t(a)^* - A_t(a^*) \to 0$: apply the lemmas to

$$f_{t,L}(x,y,\xi) = t\,(a(x,\xi) - a(y,\xi))\, \delta_{|x-y|\leq t^{-1}L},$$

thanks to the Mean Value Theorem, to obtain

$$\|t\,(A_{t,L}(a)^* - A_{t,L}(a^*))\| \leq C,$$

and let $t, L \to \infty$.

3) The kernel of $A_t(a)A_t(b)$ is associated to

$$f_t(x,y,\xi) = \iint a(x, \xi+\eta)\, b(x+z, \xi)\, e^{-i t \eta z}\, dz\, d\eta$$

According to the Stationary Phase Theorem, when $t \to \infty$ the only important part of the integral is the one where $d\eta\xi = 0$, ie $\eta = \xi = 0$. More precisely one has
$$t \,|f_t(x,y,\xi) - a(x,\xi)\, b(x,\xi)| \leq C$$
and similarly for the partial derivatives wrt ξ.
Hence the lemmas show that $\|A_t(a)A_t(b) - A_t(ab)\| = O(t^{-1})$.

4) $M_\varphi A_t(a) - A_t(a) M_\varphi \to 0$, where M_φ is the pointwise multiplication by $\varphi \in C^\infty(U)$. applying the lemmas with $f_t(x,y,\xi) = (f(x) - f(y))\, a(x,\xi)\, t\, \delta_{|x-y| \leq t^{-1}L}$ one obtains $\| t (M_f A_{t,L} - A_{t,L} M_f)\| \leq C$ ...

To apply this to the general case of a smooth manifold M, we have to check covariance under the action of a diffeomorphism $\Phi : U \to V$. Note that Φ induces a map $\tilde{\Phi} : T^*U \to T^*V$ as well as a unitary $U_\Phi : L^2(U) \to L^2(V)$
<u>Prop</u> For $a \in C_c^\infty(T^*V)$, $A_t(a \circ \tilde{\Phi}) \sim_t U_\Phi^* A_t(a)\, U_\Phi$. $(*)$

We proceed then in the following way, fixing a partition of unity $\{\varphi\}$ subordinate to a locally finite atlas of M. For $a \in C_c^\infty(T^*M)$ we define $A_t(\pi^*\varphi . a)$ by $(*)$ and we put $\tilde{\alpha}_t(a) = \sum A_t(\pi^*\varphi . a)$. This does not depend on $\{\varphi\}$ in the limit $t \to \infty$ according to the proposition. Moreover we have $\tilde{\alpha}_t(\pi^*\varphi . a) = M_\varphi \tilde{\alpha}_t(a)$ for any $\varphi \in C^\infty(M)$, since this is evidently true in the charts.

This defines a family of maps $\tilde{\alpha}_t : C_c^\infty(T^*M) \to K(L^2 M)$ which has the properties 1) 2) of asymptotic morphisms (continuity in t is immediate, linearity is satisfied for each t). To extend it to $C_0(T^*M)$ we proceed in the following way:

The asymptotic properties we have established show that $(\tilde{\alpha}_t)$ defines a morphism of $*$-algebras

$$\tilde{\alpha} : C_c^\infty(T^*M) \to C_b([1,\infty[, K) / C_0([1,\infty[, K)$$

This extends to the enveloping C^* alg of $C_c^\infty(T^*M)$ which identifies with $C_0(T^*M)$. Composing with a section $\sigma : C_b([1,\infty[, K) / C_0([1,\infty[, K) \to C_b([1,\infty[, K)$ we get a new family of maps

$$\alpha_t : C_0(T^*M) \longrightarrow K(L^2M),$$

which forms by definition an asymptotic morphism, which does not depend on σ up to equivalence.

By definition we have the following link with the "concrete maps" A_t:

$$\forall a \in C_c^\infty(T^*M) \quad \alpha_t(a) \sim \sum A_t(\pi^*\varphi . a)$$

where $\{\varphi\}$ is a partition of unity subordinate to some atlas, and the A_t are computed in the associated charts.

Application to the index problem.

Let D be a 1^{st} order elliptic LPDO, symmetric & odd on $S \to M$. The (principal) symbol of D is an element $\sigma_D \in \Gamma^\infty(T^*M, \pi^* \mathrm{End}\, S)$. We want to apply the preceding construction to $f(\sigma_D)$ (fiberwise functional calculus: in the case when $S = M \times \{0\}$ this is just $f \circ \sigma_D$) which is in $\Gamma_0(T^*M, \pi^* \mathrm{End}\, S)$ if $f \in C_0(\mathbb{R})$, thanks to the ellipticity.

Let us get back to the 0-dim. bundle on $U \subset \mathbb{R}^n$ and consider the case of a constant coeffs operator $D = \sum_{|\alpha| \leq 1} a_\alpha \frac{\partial}{\partial \xi^\alpha}$, $a_\alpha \in \mathbb{R}$. Then $\sigma_D(x,\xi) = \sum_{|\alpha|\leq 1} a_\alpha \xi^\alpha$.
For $f \in C_c^\infty(\mathbb{R})$ $f(\sigma_D)$ is compactly supported (at least with resp to ξ) and smooth, one can apply A_t to it:

$$\left(A_t(f(\sigma_D))u\right)(x) = \int f\Big(t^{-1}\sum_{|\alpha|\leq 1} a_\alpha \xi^\alpha\Big) e^{ix\xi}\, \hat{u}(\xi)\, d\xi$$
$$\Rightarrow \underline{A_t(f(\sigma_D))} = \mathcal{F}^{-1} f\big(M_{t^{-1}\sum a_\alpha \xi^\alpha}\big)\mathcal{F} = f\big(\mathcal{F}^{-1} M_{t^{-1}\sum a_\alpha \xi^\alpha}\mathcal{F}\big)$$
$$= f\Big(t^{-1}\sum a_\alpha \frac{\partial}{\partial \xi^\alpha}\Big) = \underline{f(t^{-1}D_t)} \sim f(t^{-1}D)$$

<u>Thm A</u> D op on $S \to M$, $f \in C_0(\mathbb{R})$. Then (M compact)

$$(\mathrm{id} \otimes \alpha_t)(f(\sigma_D)) \sim f(t^{-1}D)$$

(One can consider S as a subbundle of some $\mathbb{C}^N \times M$, this induces

$$\Gamma_0(T^*M, \pi^* \mathrm{End}\, S) \hookrightarrow M_N \otimes C_0(T^*M),$$
$$\Gamma^2(M,S) \hookrightarrow \mathbb{C}^N \times L^2 M,$$
$$K(\Gamma^2(M,S)) \hookrightarrow M_N \otimes K(L^2M).)$$

The K-theoretic translation of this result yields almost immediately the "left equality" of the index theorem. To understand this one needs the following facts about K-theory:

1) Let G be an odd, sa regular op on a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert A-module st $(1+G^2)^{-1} \in K_A(E)$. The functional calculus yields an odd $*$-hom $\Phi_G : C_0(\mathbb{R}) \to K_A(E)$. Conversely any such $*$-hom arises from such a G_Φ on E. We will denote by $[G] = [\Phi]$ the associated elt of $K(A)$ in the Baaj-Julg picture, it is given by the Kasparov

cycle $(E, G(1+G^2)^{-1/2})$.

2) Let $\varphi : A \to B$ be a $*$-hom and $[\Phi] \in K_A$. One can assume that $E = H \otimes A$ so that $K_A(E) = K(H) \otimes A$. Then $\varphi_*([\Phi]) = [(\mathrm{id} \otimes \varphi) \circ \Phi]$

3) An asymptotic morphism $\alpha_t : A \to B$ induces a morphism $\alpha : K(A) \to K(B)$ in the following way. By def α_t induces a $*$-hom

$$\tilde{\alpha} : A \to Q(B) = C_b([1,\infty[, B) / C_0([1,\infty[, B).$$

Since $C_0([1,\infty[,B)$ is contractible we have

$$K(Q(B)) \simeq K(C_b([1,\infty[,B)) \xrightarrow{ev_{1*}} K(B).$$

One puts $\alpha = ev_{1*} \circ \tilde{\alpha}_*$.

Now consider $\sigma_D \in M_N \otimes C_0(T^*M)$ and the topological index α. By definition

$$\tilde{\alpha}_*([\sigma_D]) = [f \mapsto (\mathrm{id} \otimes \tilde{\alpha})(f(\sigma_D))]$$
$$= [f \mapsto q(t \mapsto (\mathrm{id} \otimes \alpha_t)(f(\sigma_D)))]$$

The theorem gives a lift in $K(C_b([1,\infty[,B))$:

$$[f \mapsto (t \mapsto f(t^{-1}D))]$$

Now it is clear that $\alpha([\sigma_D]) = [f \mapsto f(D)] = [D]$. In the canonical identification $K(M_N \otimes K(L^2M)) \simeq \mathbb{Z}$ this is the analytical index of D.

The cohomological formula for the topological index.

It will result from "functorial" results concerning $\{\alpha_M\}_M$. For instance, consider an open inclusion $i : M_1 \to M_2$ of manifolds. It doesn't induce a hom from $C_0(M_2)$ to $C_0(M_1)$ because it is not proper, but it induces $i_! : C_0(M_1) \to C_0(M_2)$: extension by 0. This is an exple of wrong-way functoriality. This also applies to $i : T^*M_1 \to T^*M_2$, and it is completely evident that $\alpha_{M_2} \circ i_! = \alpha_{M_1}$.

Another, less trivial, wrong-way functoriality result we will need concerns the Thom homomorphisms: if $S \to M$ is a hermitian bundle over a cpct mfld, one defines $b_S \in \Gamma(S, \pi^* \mathrm{End}\, \Lambda^* S)$ by

$$b_{S|v}(s) = v \wedge s + v \lrcorner s.$$

It is elliptic, hence it defines an elt $[b] \in K(V)$. Now the proj $\pi : V \to M$ doesn't induce a hom from $C(M)$ to $C_0(S)$, but a $C(M)$-module structure $C(M) \otimes C_0(S) \to C_0(S)$. This yields

$$\Phi_S : K(M) \to K(S), \quad x \mapsto x \cdot [b_S].$$

When M is non-compact, b_S does not define a K-theory elt anymore, but Φ_S can still be defined from the $*$-hom $C_0(\mathbb{R}) \otimes C_0(M) \xrightarrow{\Phi} \Gamma_0(S, \pi^* \mathrm{End}\, \Lambda^* S)$,
$\Phi(f \otimes h)(v) = f(b(v))\, h(\pi(v))$.

If $V \to M$ is a euclidean vector bundle, then there exists a (non-canonical) diffeomorphism

$$\Psi : T^*V \simeq \pi^*(V \oplus V) \qquad (\pi : T^*M \to M)$$

which corresponds to the choice of horizontal & vertical subbundles (this is only canonical in a chart). T^*V is then endowed with the structure of a hermitian vector bundle over T^*M: write $V \oplus V \simeq V \oplus iV = V \otimes_{\mathbb{R}} \mathbb{C}$.

<u>Thm B</u> We have $\alpha_V \circ \Phi_{T^*V} = \alpha_M$.

We will explain how to prove this when $M = \{pt\}$: this is the fiberwise point of view and the general proof relies on it (but one has to deal with the fact that Ψ is non canonical). Since $\alpha_{pt} = id : K(pt) \to \mathbb{Z}$, it amounts to the identity $\alpha_V([b_V]) = 1$ for a euclidean vector space V. Here we use the canonical identifications $T^*V \simeq V \oplus V \simeq V \otimes_{\mathbb{R}} \mathbb{C}$ so that

$$b_{(x,\xi)}(s) = x \wedge s + x \lrcorner s + i\xi \wedge s + i\xi \lrcorner s$$

for $s \in \Lambda^* V \otimes \mathbb{C}$. The idea is to consider the diff operator on $L^2(V, \Lambda^* V \otimes \mathbb{C})$ associated to the (total) symbol b:

$$Du(x) = \int b(x,\xi)\, e^{ix\xi}\, \hat{u}(\xi)\, d\xi \quad (u \in C_c(V, \Lambda^* V \otimes \mathbb{C}))$$
$$= x \wedge u(x) + x \lrcorner u(x) + du(x) + d^* u(x),$$

and to use Thm A to compute $\alpha([b])$ via the index of D.

However we have seen that in the limit $f(t^{-1}D)$ forgets about the 0-order part of D: $f(t^{-1}D) \sim \alpha_t(f(b_0))$ where $b_0 = i\xi \wedge + i\xi \lrcorner$ is the (principal) symbol of D. The solution to recover the whole of b is to look at $f(D_0 + t^{-1}D_1)$ instead — the terms D_i of order i are well-defined because we are working in the canonical chart $V \to V$. As we have seen during the proof, the method of Thm A carries over:

<u>Prop</u> For any $f \in C_0(\mathbb{R})$, $f(D_0 + t^{-1}D_1) \sim \alpha_t(f(b))$. As a result, $\alpha([b]) = [D] = \mathrm{Ind}(D)$.

Now it remains to compute the index of D. Standard diff-geometric calculations show that

<u>Lemma</u> $D^2 = -\Delta + \|x\|^2 + (2k - n)$ on $L^2(V, \Lambda^k V \otimes \mathbb{C})$.

In particular degrees & variables separate and the study of D^2 boils down to the one of $\frac{\partial^2}{\partial x^2} - x^2 + 1$, which is well known.

<u>Prop</u> D is self-adjoint, elliptic and $\mathrm{Ker}\, D = \mathbb{C}\, e^{-\|x\|^2/2}$.
In particular $\mathrm{Ind}\, D = 1$.

Comparison of Thom homomorphisms.

Recall that, for $V \to M$ a complex hermitian vector bundle, $K(V)$ & $H^*(V)$ are freely generated as $K(M)$ & $H^*(M)$-modules, by the Thom elt $[\lambda_V]$ & the Thom class u_V.

Consider the associated Thom isomorphisms

$$\Phi_V : K(M) \to K(V), \; x \mapsto x \cdot [\lambda_V]$$
$$\Psi_V : H^*(M) \to H^*(V), \; x \mapsto x \cdot u_V$$

Let $\tau(V) \in H^*(M)$ be the elt st $ch_V([\lambda_V]) = \tau(V) \cdot u_V$: we have for all $x \in K(M)$

$$ch_V(\Phi_V(x)) = \tau(V) \cdot \Psi_V(ch(x)).$$

By functoriality, τ is a characteristic class, moreover τ is multiplicative according to:

<u>Lemma</u> $\Phi_{\pi_1^* V_2} \circ \Phi_{V_1} = \Phi_{V_1 \oplus V_2}$, $\Psi_{\pi_1^* V_2} \circ \Psi_{V_1} = \Psi_{V_1 \oplus V_2}$.

As a result it is determined by a power series.

<u>Prop</u> τ is associated to $(1-e^x)/x$.

<u>Def</u> Let Todd be the genus associated to $x/(1-e^{-x})$

From the general theory we easily see that

- $\tau(V) = (-1)^k$, $\mathrm{Todd}(V) = 1$ for V, k-$\dim_{\mathbb{C}}$ trivial bundle over M compact.
- $\tau(V)\,\mathrm{Todd}(\bar V) = (-1)^k$ for V k-$\dim_{\mathbb{C}}$
- $\mathrm{Todd}(V)\,\mathrm{Todd}(W) = 1$ if $V \oplus W$ trivial

<u>Thm</u> Let $\alpha_M : K(T^*M) \to \mathbb{Z}$ be a family of morphisms st

i) for $M_1 \xhookrightarrow{i} M_2$ open we have

$$\begin{array}{ccc} K(T^*M_1) & \xrightarrow{\alpha} & \mathbb{Z} \\ \downarrow i_! & \circlearrowleft & \| \\ K(T^*M_2) & \xrightarrow{\alpha} & \mathbb{Z} \end{array}$$

ii) for $V \to M$ a real vector bundle we have

./.

$$\begin{tikzcd} K(T^*M) \arrow[r, "\alpha"] \arrow[d, "\phi_{T^*V}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z} \arrow[d, equal] \\ K(T^*V) \arrow[r, "\alpha"] & \mathbb{Z} \end{tikzcd}$$

iii) $\alpha_{pt} = id : K(pt) \to \mathbb{Z}$.

Then we have, for any M and $x \in K(T^*M)$:

$$\alpha(x) = (-1)^{\dim M} \int_{T^*M} \mathrm{Todd}(TM \otimes \mathbb{C}) \cdot ch(x)$$