On the adjoint representation of orthogonal free quantum groups

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Orthogonal free quantum groups

Consider the unital C^*-algebras defined by generators and relations:

$$C_o(n) = \langle u_i, 1 \leq i \leq n \mid u_i = u_i^*, \quad u_i \text{ unitary} \rangle,$$
$$A_o(n) = \langle u_{ij}, 1 \leq i,j \leq n \mid u_{ij} = u_{ij}^*, \quad (u_{ij}) \text{ unitary} \rangle.$$

We recognize $C_o(n) = C^*(FO_n)$ where $FO_n = (\mathbb{Z}/2\mathbb{Z})^{*n}$.
We denote $A_o(n) = C^*(\mathbb{F}O_n)$. $A_o(n)$ was introduced by S. Wang.

The full structure of FO_n is reflected by a coproduct

$$\Delta : C_o(n) \to C_o(n) \otimes C_o(n), \; u_i \mapsto u_i \otimes u_i.$$

Similarly there is a natural coproduct

$$\Delta : A_o(n) \to A_o(n) \otimes A_o(n), \; u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}.$$

→ $\mathbb{F}O_n$ is a discrete quantum group : the ortogonal free quantum group.
It is the Pontrjagin dual of the compact quantum group O_n^+.

Discrete quantum groups

A Woronowicz C^*-algebra is a unital C^*-algebra A with $*$-homomorphism $\Delta : A \to A \otimes A$ (coproduct) such that

- $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
- $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

Examples :

- G compact group, $A = C(G)$, $\Delta(f) = ((x, y) \mapsto f(xy))$, characterized by commutativity of A ;
- Γ discrete group, $A = C^*(\Gamma)$, $\Delta(g) = g \otimes g$ — but also $A = C^*_{\mathrm{red}}(\Gamma)$, characterized by co-commutativity : $\Sigma\Delta = \Delta$.

Notation : $A = C^*(\mathbb{\Gamma})$.

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General theory :

- Haar state $h \in C^*(\mathbb{\Gamma})^*$ ➔ GNS representation $\lambda : C^*(\mathbb{\Gamma}) \to B(\ell^2\mathbb{\Gamma})$,
- $C^*_{\mathrm{red}}(\mathbb{\Gamma}) = \lambda(C^*(\mathbb{\Gamma}))$ is again a Woronowicz C^*-algebra,
- $\mathscr{L}(\mathbb{\Gamma}) = C^*_{\mathrm{red}}(\mathbb{\Gamma})''$ von Neumann algebra of $\mathbb{\Gamma}$,
- right regular representation $\rho : C^*(\mathbb{\Gamma}) \to B(\ell^2\mathbb{\Gamma})$,
- adjoint representation $\mathrm{ad} = (\lambda, \rho) \circ \Delta : C^*_{\mathrm{full}}(\mathbb{\Gamma}) \to B(\ell^2\mathbb{\Gamma})$,
- trivial representation $\epsilon : C^*_{\mathrm{full}}(\mathbb{\Gamma}) \to \mathbb{C}$,

$\mathbb{\Gamma}$ is called unimodular if h is a trace, amenable if ϵ factors through λ.

Analogies with free group C^*-algebras

$\mathbb{F}O_n$ shares many (analytical) properties with usual free groups:

- diagonal quotient map $C^*(\mathbb{F}O_n) \twoheadrightarrow C^*(FO_n)$;
- we have $C^*(\mathbb{F}O_n) \twoheadrightarrow C^*(\Gamma)$ for any Γ with "self-adjoint generators";
- $\mathbb{F}O_n$ is non amenable for $n \geq 3$ [Banica 1997];
- $C^*_{\mathrm{red}}(\mathbb{F}O_n)$ is simple, $\mathscr{L}(\mathbb{F}O_n)$ is a full factor [Vaes-V. 2005];
- $\mathbb{F}O_n$ is K-amenable [Voigt 2009];
- later in this talk : rapid decay, a-T-menability, weak amenability, bi-exactness, ...

Main result of this talk [Fima-V.] :

- $\mathrm{ad} \ominus \epsilon \prec \lambda$ for $\mathbb{F}O_n$,
- deformation of the identity by automorphisms.

Applications : fullness, strong solidity, property (HH)...

Outline

Classical results

F_n : free group on n generators. $\ell(g)$: length of $g \in F_n$.

Recall the main results of **[Haagerup 1979]** :

- rapid decay : for $x \in C^*(F_n)$ supported on elements of length k, $\|\lambda(x)\| \leq (k+1)\|x\|_2$, where $\|x\|_2^2 = h(x^*x)$.
- a-T-menability : $(g \mapsto e^{-t\ell(g)}g)$ defines a completely positive map $T_t : C^*_{\mathrm{red}}(F_n) \to C^*_{\mathrm{red}}(F_n)$ for all $t > 0$.

Corollaries :

- metric approximation property (MAP) for $C^*_{\mathrm{red}}(F_n)$: there exists $M_\alpha : C^*_{\mathrm{red}}(F_n) \to C^*_{\mathrm{red}}(F_n)$ contractive with finite rank such that $M_\alpha(x) \to x$ for all x.
- states $\varphi \in C^*(F_n)^*_+$ factor through λ **iff** $(g \mapsto \varphi(g)e^{-t\ell(g)})$ is in $\ell^2(F_n)$ for all $t > 0$.

Classical results

Application to $\mathrm{ad} : C^*(F_n) \to B(\ell^2 F_n)$.
The vector $\xi_0 = \delta_e \in \ell^2(F_n)$ is fixed ➜ $\mathrm{ad}^\circ = \mathrm{ad} \ominus \epsilon$ on $\xi_0^\perp$.

Consider $\varphi : x \mapsto (\delta_g | \mathrm{ad}(x)\delta_g)$ on $C^*(F_n)$.
We have $\varphi(h) = 1$ if $hg = gh$, $\varphi(h) = 0$ else.
But $C(g) = \{h \in F_n \mid hg = gh\}$ is cyclic for $g \neq e$:
$C(g) = \{w^k \mid k \in \mathbb{Z}\}$ with $w = uvu^{-1}$, v cyclically reduced.
➜ non-zero values of $\varphi(h)e^{-t\ell(h)}$: $e^{-t(|k|p+q)}$, for $h = w^k$.

Haagerup's characterization ➜ $\varphi \prec \lambda$ for $g \neq e$.
Conclusion : $\mathrm{ad}^\circ \prec \lambda$.

The quantum case

There is a natural "word length" on $\mathbb{F}O_n$: $\ell^2\mathbb{F}O_n = \bigoplus p_k\ell^2\mathbb{F}O_n$.
Definition : $\bigoplus_{l\leq k} p_l\ell^2\mathbb{F}O_n = \mathrm{Span}\{P(u_{ij})\xi_0 \mid \deg P \leq k\}$.

Property of Rapid Decay :

Theorem (V. 2004)

If $x \in C^*(\mathbb{F}O_n)$ is such that $\lambda(x)\xi_0 \subset p_k\ell^2\mathbb{F}O_n$, then

$$\|\lambda(x)\| \leq (2k+5)\|x\|_2.$$

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A-T-menability :
Denote U_k the Chebyshev polynomials of the second kind.

Theorem (Brannan 2011)

For all $t \in]2, n]$, the formula $T_t(x)\xi_0 = \sum_k \frac{U_k(t/2)}{U_k(n/2)} p_k x\xi_0$

defines a completely positive map $T_t : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C^*_{\mathrm{red}}(\mathbb{F}O_n)$.

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Define $\ell \in C^*(\mathbb{F}O_n)^*$ (unbounded) by:

$$\ell(x) = k\epsilon(x) \text{ if } \lambda(x)\xi_0 \in p_k\ell^2(\mathbb{F}O_n).$$

Corollary : states $\varphi \in C^*(\mathbb{F}O_n)^*_+$ factor through λ **iff** $\varphi e^{-t\ell}$ is continuous with respect to $\|\cdot\|_2$ for all $t > 0$.

On the adjoint representation

The line $\mathbb{C}\xi_0 \subset \ell^2\Gamma$ is invariant **iff** Γ is unimodular.
→ we can still consider $\mathrm{ad}^\circ = \mathrm{ad} \ominus \epsilon$ on $\xi_0^\perp \subset \ell^2(\mathbb{F}O_n)$.

Theorem (Fima-V. 2012)

We have $\mathrm{ad}^\circ \prec \lambda$ *for* $\mathbb{F}O_n$.

Proof : use the preceeding criterium.
However : no combinatorial property of centralizers as in the classical case.
Instead : growth estimates for $\varphi : x \mapsto (\xi|\mathrm{ad}(x)\xi)$, $\xi \in p_k\ell^2(\mathbb{F}O_n)$, $k \geq 1$, using computations in the category of corepresentations of $\mathbb{F}O_n$.

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First application :

Corollary (Vaes-V. 2005)

For $n \geq 3$, the representation ad° has spectral gap : $\epsilon \not\prec \mathrm{ad}^\circ$.
In particular $\mathbb{F}O_n$ is not inner amenable and $\mathscr{L}(\mathbb{F}O_n)$ is a full factor.

Outline

The deformation

Action of O_n

By definition there is a surjective map $\pi : C^*(\mathbb{F}O_n) = C(O_n^+) \to C(O_n)$.
By Fell's absorption principle, Δ factors to

$$\Delta' : C^*_{\text{red}}(\mathbb{F}O_n) \to C^*(\mathbb{F}O_n) \otimes C^*_{\text{red}}(\mathbb{F}O_n).$$

We obtain an action of O_n on $C^*_{\text{red}}(\mathbb{F}O_n)$ by automorphisms :

$$\alpha_g = ((ev_g \circ \pi) \otimes \mathrm{id}) \circ \Delta : C^*_{\text{red}}(\mathbb{F}O_n) \to C^*_{\text{red}}(\mathbb{F}O_n).$$

Deformation of $C^*_{\text{red}}(\mathbb{F}O_n)$ inside a bigger algebra

Put $C = C^*_{\text{red}}(\mathbb{F}O_n)$ and $\tilde{C} = C^*_{\text{red}}(\mathbb{F}O_n) \otimes C^*_{\text{red}}(\mathbb{F}O_n)$

$\iota = \Delta_{\text{red}} : C \to \tilde{C}$ the natural embedding

$E : \tilde{C} \to \iota(C)$ unique trace-pres. cond. exp.

We deform ι by putting $A_g(x) = (\mathrm{id} \otimes \alpha_g)\iota : C \to \tilde{C}$ for $g \in O_n$.

The deformation

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Proposition (Fima-V. 2012)

We have $E \circ A_g = T_t : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C^*_{\mathrm{red}}(\mathbb{F}O_n)$, where $t = \mathrm{Tr}(g)$.

- recover complete positivity of Brannan's deformation.
- get deformation of $C \subset \tilde{C}$ by 1-param. group of autom. $(A_{g_s})_{s \in \mathbb{R}}$.

Application to strong solidity

Recall M is stronly solid if for every diffuse amenable $P \subset M$, the normalizer $\mathscr{N}_M(P)$ generates an amenable vN subalgebra.

strongly solid + non-amenable $\Rightarrow$ prime + no Cartan subalgebra .

[Chifan-Sinclair 2011, Popa-Vaes 2012] CBAP + AO$^+$ $\Rightarrow$ strongly solid

Theorem (V. 2004)

$\mathbb{F}O_n$ satisifies a strong Akemann-Ostrand Property (AO$^+$).

Theorem (Freslon 2012)

$\mathbb{F}O_n$ is weakly amenable (CBAP) with constant 1.

Theorem (Isono 2012)

$\mathscr{L}(\mathbb{F}O_n)$ is strongly solid.

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[Ozawa-Popa 2007, Sinclair 2010] strong solidity follows from CBAP + deformation A_t of $\iota(M) \subset \tilde{M}$ by $*$-hom. such that $E \circ A_t$ is L^2-compact and $\tilde{M} \ominus \iota(M) \prec M \otimes M$.

Theorem (Freslon 2012)

$\mathbb{F}O_n$ is weakly amenable (CBAP) with constant 1.

Take $M = C'' = \mathscr{L}(\mathbb{F}O_n)$, $\tilde{M} = \tilde{C}'' = M \otimes M$ bimodule via ι.

${}_{M\otimes 1}(M \otimes M)_{1 \otimes M}$ corresponds to λ; ${}_{\iota(M)}\tilde{M}_{\iota(M)} \ominus \iota(M)$ corresponds to ad°.

Corollary (Fima-V. 2012)

$\mathscr{L}(\mathbb{F}O_n)$ is strongly solid.

Application to cocycles

Recall : a-T-menability $\Leftrightarrow$ existence of a proper cocycle in some repr. π. Classical case F_n : natural proper cocycle c given by paths in the Cayley graph. In that case $\pi = \bigoplus_{2n} \lambda$.

[Brannan 2011] → proper cocycle for $\mathbb{F}O_n$. What can be said about π ?

Theorem (V. 2009)

For $n \geq 3$ we have $H^1(\mathbb{C}[\mathbb{F}O_n], \ell^2(\mathbb{F}O_n)) = 0$.
All cocycles in (finite sums of) λ are trivial.

Application to cocycles

[Brannan 2011] ➜ proper cocycle for $\mathbb{F}O_n$. What can be said about π ?

Concrete construction of a proper cocycle for $\mathbb{F}O_n$

Differentiate the deformation A_g : get for all $X \in \mathfrak{o}_n$

➜ a derivation $\delta_X : \mathbb{C}[\mathbb{F}O_n] \to \tilde{M} \ominus \iota(M)$

➜ a cocycle $c_X : \mathbb{C}[\mathbb{F}O_n] \to \ell^2(\mathbb{F}O_n) \ominus \mathbb{C}\xi_0$

Proposition (Fima-V. 2013)

For all $X \in \mathfrak{o}_n$, $X \neq 0$, c_X is proper. The conditionnaly negative type function associated to c_X is the one associated to Brannan's deformation.

In particular the cocycle arising from Brannan's deformation can be realized inside $\pi = \mathrm{ad}^\circ$. Since $\mathrm{ad}^\circ \prec \lambda$, this shows that $\mathbb{F}O_n$ satisfies Property strong (HH) from [Ozawa-Popa 2008].

Note : CBAP + strong (HH) $\Rightarrow$ strong solidity.