Hypercontractivity of the heat semigroup on free orthogonal quantum groups

Roland Vergnioux

joint work with M. Brannan, S.-G. Youn

University of Normandy

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Outline

Rapid Decay and applications

Property of Rapid Decay (RD) / Haagerup's inequality

- Classical case: [Haagerup 1978] free groups
- Quantum case (unimodular): [V. 2007]
- Non-unimodular case: [Vaes-V. 2007], [Bhowmick–Voigt–Zacharias 2015]
- **Update**: [Brannan–V.–Youn 2021]

Applications

- Structure of operator algebras [Haagerup 1978, ...]
- K-theory [Jolissaint 1989, Lafforgue 2002]
- Compact quantum metric spaces [Antonescu–Christensen 2004]
- Quantum information theory [Brannan–Collins 2018]
- **Hypercontractivity** results [Biane 1997, ...]
- ...

Quantum groups (1)

A compact quantum group $\mathbb{G}$ and its discrete dual $\mathbb{\Gamma}$ are given by a regular multiplicative unitary $V \in B(H \otimes H)$ with a unique fixed line $\mathbb{C}\xi_0 \subset H$ — i.e. we have $V_{12}V_{13}V_{23} = V_{23}V_{12}$ and $V(\xi_0 \otimes \zeta) = (\xi_0 \otimes \zeta)$ for all $\zeta \in H$.

The associated Hopf-C^*-algebras are

$$C(\mathbb{G}) = C^*_{\text{red}}(\mathbb{\Gamma}) = \{(\omega \otimes \text{id})(V) \mid \omega \in B(H)_*\}^-$$
$$C^*(\mathbb{G}) = c_0(\mathbb{\Gamma}) = \{(\text{id} \otimes \omega)(V) \mid \omega \in B(H)_*\}^-$$

with coproducts induced by V. If ξ_0 is a normed fixed vector, it is cyclic and separating for $C(\mathbb{G})$ and $h = \omega_{\xi_0}$ is the Haar state of $C(\mathbb{G})$. We denote $H = L^2(\mathbb{G})$.

The (inverse) Fourier transform is

$$\mathscr{F} : C(\mathbb{G}) \to M(C^*(\mathbb{G})), x \mapsto (\text{id} \otimes h)(V^*(1 \otimes x)) ,$$

it is isometric wrt the 2-norms associated with h and the normalized left Haar weight $\hat{h}_L$. We write $H = L^2(\mathbb{G}) = \ell^2(\mathbb{\Gamma})$.

Quantum groups (2)

The interesting C^*-algebra is $C(\mathbb{G}) = C^*_{\text{red}}(\mathbb{\Gamma})$.

The dual algebra is very simple:

$$C^*(\mathbb{G}) = c_0(\mathbb{\Gamma}) \simeq \bigoplus_{\alpha\in I}^{c_0} B(H_\alpha) \text{ with } \dim H_\alpha < \infty.$$

We denote $I = \mathrm{Irr}(\mathbb{\Gamma}) = \mathrm{Irr}(\mathbb{G})$.

The left and right invariant weights on $c_0(\mathbb{\Gamma})$ are

$$\hat{h}_L(a) = \sum \mathrm{qd}(\alpha)\mathrm{Tr}(F_\alpha a_\alpha),\ \hat{h}_R(a) = \sum \mathrm{qd}(\alpha)\mathrm{Tr}(F_\alpha^{-1} a_\alpha)$$

where the F_α are Woronowicz' modular matrices and $\mathrm{qd}(\alpha) = \mathrm{Tr}(F_\alpha) = \mathrm{Tr}(F_\alpha^{-1})$ is the quantum dimension of α.

$\mathbb{\Gamma}$ unimodular / $\mathbb{G}$ of Kac type: h tracial $\Leftrightarrow \forall\alpha\ F_\alpha = \mathrm{id}_\alpha \Leftrightarrow \hat{h}_L = \hat{h}_R$.

Thank to the coproduct, the category of f.-d. $*$-representations of $c_0(\mathbb{\Gamma})$ is in fact a tensor C^*-category, and it is *rigid*. We have e.g. fusion rules $\alpha \otimes \beta \simeq \bigoplus m_{\alpha,\beta}^\gamma \gamma$ for α, $\beta \in \mathrm{Irr}(\mathbb{\Gamma})$. We write $\gamma \subset \alpha\otimes\beta$ if $m_{\alpha,\beta}^\gamma \neq 0$.

Orthogonal free quantum groups

$N \in \mathbb{N}^*$, $Q \in GL_N(\mathbb{C})$, $Q\bar{Q} = \pm I_N$ → discrete $\mathbb{F}O(Q)$, compact $O^+(Q)$.

- The *full* C^*-algebra $C^*(\mathbb{F}O(Q))$ can be defined by generators u_{ij} and matricial relations $uu^* = u^*u = 1$, $u = Q\bar{u}Q^{-1}$. [Van Daele–Wang]
- The tensor category of representations of $c_0(\mathbb{F}O(Q))$ is the Temperly-Lieb category TL_q, with $q + q^{-1} = \mathrm{Tr}(Q^*Q)$, together with a specific tensor functor $TL_q \to \mathrm{Hilb}$. [Banica]

$\mathrm{Irr}(\Gamma) = \{\alpha_n \mid n \in \mathbb{N}\}$, $\alpha_0 = 1$, $\alpha_1 = u$, $\alpha_n \otimes \alpha_1 = \alpha_{n-1} \oplus \alpha_{n+1}$.
We have $\mathrm{qd}(\alpha_n) = U_n(q + q^{-1})$, where $(U_n)_n$ are type-II Chebychev.

Connection to classical groups for $\mathbb{F}O(I_N) =: \mathbb{F}O_N$ and $O^+(I_N) =: O_N^+$

- $C^*(\mathbb{F}O_N) / \langle u_{ij}; i \neq j \rangle \simeq C^*((\mathbb{Z}/2\mathbb{Z})^{*N})$,
- $C(O_N^+) / \langle [u_{ij}, u_{kl}] \rangle \simeq C(O_N)$.

Outline

Rapid Decay

Fix a discrete quantum group $\mathbb{\Gamma}$. Length function: $\ell : \mathrm{Irr}(\mathbb{\Gamma}) \to \mathbb{N}$ s.t.

$$\ell(1) = 0,\ \ell(\bar{\alpha}) = \ell(\alpha),\ \ell(\gamma) \leq \ell(\alpha) + \ell(\beta) \text{ if } \gamma \subset \alpha \otimes \beta.$$

Example: if $\mathrm{Irr}(\mathbb{\Gamma})$ is generated by $\mathcal{D} = \bar{\mathcal{D}}$, word length

$$\ell(\alpha) = \min\{k \mid \exists \beta_1, \ldots, \beta_k \in \mathcal{D}\ \alpha \subset \beta_1 \otimes \cdots \otimes \beta_k\}.$$

Denote $p_\alpha = \mathrm{id}_\alpha \in c_0(\mathbb{\Gamma})$, $p_n = \sum\{p_\alpha \mid \ell(\alpha) = n\} \in M(c_0(\mathbb{\Gamma}))$.

Definition (V. 2007)

$\mathbb{\Gamma}$ has Property RD (with respect to ℓ) if there exists $P \in \mathbb{R}[X]$ s.t.

$$\forall n \in \mathbb{N}\ \forall x \in C^*_{\mathrm{red}}(\mathbb{\Gamma})\ \mathscr{F}(x) \in p_n c_0(\mathbb{\Gamma}) \Rightarrow \|x\| \leq P(n)\|\mathscr{F}(x)\|_2.$$

Characterization using Sobolev norms or the Fréchet space of functions of rapid decay, as in the classical case.

When $\mathrm{Irr}(\mathbb{\Gamma})$ is finitely generated, Γ has RD wrt some ℓ **iff** it has RD wrt to some word length.

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Examples: [V. 2007, Brannan 2013]

- *unimodular* groups with polynomial growth: $\sum\{\dim(H_\alpha) \mid \ell(\alpha) \le n\} \le P(n)$, in particular duals of connected compact Lie groups
- *unimodular* orthogonal and unitary free quantum groups, duals of quantum permutation groups
- Non-unimodular DQG do not have RD!

The non-unimodular case

Idea: twist the 2-norm $\|a\|_2 = \hat{h}_L(a^*a)^{1/2}$ [BVZ 2015].

For $D \in c_0(\mathbb{\Gamma})^\eta$ s.t. $D_\alpha F_\alpha \geq 0$ for all α, put $\|a\|_{2,D} = \|aD\|_2$.
These are still "easily computable" norms.

Definition (BVZ 2015)

$\mathbb{\Gamma}$ has Property RD_D with respect to ℓ if there exists $P \in \mathbb{R}[X]$ s.t.

$$\forall n \in \mathbb{N}\ \forall x \in C^*_{\mathrm{red}}(\mathbb{\Gamma})\ \mathscr{F}(x) \in p_n c_0(\mathbb{\Gamma})) \Rightarrow \|x\| \leq P(n)\|\mathscr{F}(x)\|_{2,D}.$$

$\mathbb{\Gamma}$ has the twisted Property RD with respect to ℓ if it has $\mathrm{RD}_{\sqrt{C}}$ with

$$C_\alpha = \frac{\mathrm{qd}(\alpha)}{\dim(\alpha)} F_\alpha.$$

Note: RD $\Leftrightarrow$ RD_1 ; $C = 1 \Leftrightarrow \mathbb{\Gamma}$ unimodular.

New examples: *all* groups with polynomial growth have twisted RD, e.g. duals of q-deformation of connected compact Lie groups [BVZ 2015].

Non-unimodular free quantum groups

We consider the case of $\Gamma = \mathbb{F}O(Q)$, $Q \in GL_N(\mathbb{C})$, $Q\bar{Q} = \pm I_N$, $N \geq 2$.
We use the word length $\ell(\alpha_n) = n$.

For $N = 2$ we recover the duals of $SU_q(2)$, $q \in [-1, 1]$.
For $N \geq 3$ $\mathbb{F}O(Q)$ has exponential growth.

Proposition (BVY 2021)

- *We have* $\forall n \in \mathbb{N}\ \forall x \in C^*_{red}(\mathbb{F}O(Q))\ \mathscr{F}(x) \in p_n c_0(\mathbb{F}O(Q)) \Rightarrow \|p_0 x p_n\| = \|\mathscr{F}(x)\|_{2,D} O(n^k)$ **iff** $\|F_n^{-1/2} D_n^{-1} F_n\| \|F_n^{1/2}\| = O(n^k)$. $\quad (*)$
- *For* $N \geq 3$, *the non-unimodular* $\mathbb{F}O(Q)$ *do not satisfy* $RD_{\sqrt{C}}$.

Questions:

- For $\mathbb{F}O(Q)$, is RD_D equivalent to $(*)$? At least if $[D_n, Q_n] = 0$?
- Find minimal D's such that $\mathbb{F}O_Q$ satisfies RD_D.

Exponential estimates

One can always achieve RD_D by taking a sufficiently rapidly growing D...
Consider the canonical central element $B = \sum \|F_\alpha\| p_\alpha \in c_0(\mathbb{\Gamma})^\eta$.
We have $\sqrt{C} \leq B$, hence $\mathrm{RD}_{\sqrt{C}} \Rightarrow \mathrm{RD}_B$.

Proposition (Vaes-V. 2007)

$\mathbb{F}O(Q)$ *satisfies* RD_B *for all* Q *and* N.

If $\mathbb{\Gamma}$ is non unimodular, B is exponentially growing (as well as C).
However it is still usefull:

- in [Vaes-V. 2007] to prove simplicity of $C^*_{\mathrm{red}}(\mathbb{F}O(Q))$,
- in [V. 2012] to prove the vanishing of (twisted) L^2-cocycles,
- for hypercontractivity results...

Outline

The heat semigroup on $O^+(Q)$

Write $L^\infty(O^+(Q)) = \mathcal{L}(\mathbb{F}O(Q)) = C^*_{\text{red}}(\mathbb{F}O(Q))'' \subset B(H)$.

Central symmetric quantum Markov semigroups $T_t : L^\infty(O_N^+) \to L^\infty(O_N^+)$ were classified in [Cipriani–Franz–Kula 2014] by an analogue of Hunt's formula. The generators decompose into a "jump part", and a "gaussian part" which corresponds to

$$T_t(x) = e^{-t\lambda_n} x \quad \text{if} \quad \mathscr{F}(x) \in p_n c_0(\mathbb{F}O_N),$$
$$\text{with} \quad \lambda_n = U_n'(q+q^{-1})/U_n(q+q^{-1}).$$

This justifies the analogy with the classical heat semigroup on O_N.

We have $\lambda_n = C_q n + D_q + o(1)$, thus $(T_t)_t$ is also analogous to the free Poisson semigroup on $\mathscr{L}(F_N)$.

Ultracontractivity

Put $L^P(\mathbb{G}) = L^P(L^\infty(\mathbb{G}), h)$. Markov semigroups $(T_t)_t$ on $L^\infty(\mathbb{G})$ extend to $T_t : L^P(\mathbb{G}) \to L^p(\mathbb{G})$ for all $p \geq 1$.

Denote $t_\infty = \inf\{t > 0 \mid T_t(L^2(\mathbb{G})) \subset L^\infty(\mathbb{G})\}$.
The semigroup $(T_t)_t$ is **ultracontractive** if $t_\infty < \infty$.

[FHLUZ 2017] In the unimodular case, the heat semigroup on $O^+(Q)$ is ultracontractive at all times ($t_\infty = 0$).

Proposition (BVY 2021)

In the non-unimodular case we have

$$(2q + 2q^{-1} - 4) \log \|Q\| \leq t_\infty \leq (2q + 2q^{-1}) \log \|Q\|$$

The upper bound results from RD_B. The lower bounds follows from the fact that $(x \mapsto x^*)$ is isometric for $\| \cdot \|_\infty$ but not for $\| \cdot \|_2$.

Hypercontractivity

Denote $t_p = \inf\{t > 0 \mid \forall x \in L^\infty(\mathbb{G}) \ \ \|T_t(x)\|_p \leq \|T_t(x)\|_2\}$.
The semigroup $(T_t)_t$ is **hypercontractive** if $t_p < \infty$ for all $p > 2$.

Theorem (BVY 2021)

The heat semigroup on O_N^+ is hypercontractive and for all $p \geq 4$ we have

$$\frac{N}{2}\log(p-1) \leq t_p \leq c_p \frac{N}{2}\log(p-1) + \epsilon_N$$

with $\lim_{N\infty} \epsilon_N = 0$, $\lim_{p\infty} c_p = 1$, $c_p \leq 1.78$.

Note: [FHLUZ 2017] already proves the estimate $c_p \leq 1.83$.

Tools. Khintchine inequalities for $p \geq 4$: if $\mathscr{F}(x) \in p_n c_0(\mathbb{F}O_N)$,

$$\|x\|_p \leq \left(C_q^2(n+1)\right)^{1-\frac{3}{p}} \|x\|_2.$$

Follows from complex interpolation between $p = \infty$ (RD) and $p = 4$.
We also use the NC martingale convexity inequality of [Ricard–Xu 2016].

Hypercontractivity

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Conjecture

We have $t_p = \frac{N}{2}\log(p-1) + o_N(1)$.

Note: $t_p = \frac{1}{2}\log(p-1)$ for the Poisson semigroup on $\mathscr{L}(F_N)$, $\mathscr{L}(\mathbb{Z}^N)$.