Furstenberg boundary for discrete quantum groups

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Outline

Motivation

Classical results:

- notion of Γ-**boundary** in topological dynamics (Furstenberg, 1950s)
- surprising connection with the structure of **reduced group** C^***-algebra** (Kalantar-Kennedy, Breuillard-Kalantar-Kennedy-Ozawa, 2010s)

Theorem (BKKO)

$C^*_{\mathrm{red}}(\Gamma)$ *simple* $\Leftrightarrow \exists$ *free* Γ*-boundary* $\Gamma \curvearrowright X$.
$C^*_{\mathrm{red}}(\Gamma)$ *has a unique trace* $\Leftrightarrow \exists$ *faithful* Γ*-boundary* $\Gamma \curvearrowright X$.

In particular simplicity ⇒ uniqueness of trace for reduced C^*-algebras of discrete groups. The converse is false.

Quantum case?

- other notions of boundary already studied (Poisson, Martin, Gromov...)
- type III examples — no trace on $C^*_{\mathrm{red}}(\mathbb{\Gamma})$ in "non Kac" cases!

Discrete quantum groups

A **discrete quantum group** $\mathbb{\Gamma}$ is given by :

- a von Neumann algebra $\ell^\infty(\mathbb{\Gamma}) = \bigoplus_{\alpha\in I}^{\ell^\infty} B(H_\alpha)$ with $\dim H_\alpha < \infty$
- a normal $*$-homomorphism $\Delta : \ell^\infty(\mathbb{\Gamma}) \to \ell^\infty(\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma})$ such that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ (coproduct)
- left and right Δ-invariant nsf weights h_L, h_R on $\ell^\infty(\mathbb{\Gamma})$

$\mathbb{\Gamma}$ is **unimodular** if $h_L = h_R$. Denote $\ell^2(\mathbb{\Gamma}) = L^2(\ell^\infty(\mathbb{\Gamma}), h_L)$.

Canonical dense subalgebra : $c_0(\mathbb{\Gamma}) \subset \ell^\infty(\mathbb{\Gamma})$ given by $\bigoplus_{\alpha\in I}^{c_0}$.
It is a multiplier Hopf C^*-algebra.

Tensor C^*-category $\mathrm{Corep}(\mathbb{\Gamma}) = \mathrm{Rep}(\ell^\infty(\mathbb{\Gamma}))$ with $\pi \otimes \rho := (\pi \otimes \rho)\Delta$.
$I = \mathrm{Irr}(\mathbb{\Gamma})$: simple objects up to equivalence.
Classical case: $\mathbb{\Gamma} = \Gamma = I$ when $\dim H_\alpha = 1$ for all α.

Actions

A $\mathbb{\Gamma}$-C^*-**algebra** is a C^*-algebra A equipped with a $*$-homomorphism $\alpha : A \to M(c_0(\mathbb{\Gamma}) \otimes A)$ such that $(\mathrm{id} \otimes \alpha)\alpha = (\Delta \otimes \mathrm{id})\alpha$ (coaction).

For $a \in A$, $\nu \in A^*$, $\mu \in c_0(\mathbb{\Gamma})^*$ we can then define

$$L_\mu(a) = (\mu \otimes \mathrm{id})\alpha(a) \in M(A),$$
$$P_\nu(a) = (\mathrm{id} \otimes \nu)\alpha(a) \in \ell^\infty(\mathbb{\Gamma}),$$
$$\mu * \nu = (\mu \otimes \nu)\alpha \in A^*.$$

A $\mathbb{\Gamma}$-map $T : A \to B$ is a linear map such that $T \circ L_\mu = L_\mu \circ T$.

Example: $A = c_0(\mathbb{\Gamma})$, $\alpha = \Delta$ "translation action".
By invariance, the maps L_μ extend to bounded operators on $\ell^2(\mathbb{\Gamma})$.
➔ C^*-algebra $C^*_{\mathrm{red}}(\mathbb{\Gamma}) = \overline{\mathrm{Span}}\ \{L_\mu\}$.
It is a Woronowicz C^*-algebra, with Haar state denoted h.

Orthogonal free quantum groups

Let $N \in \mathbb{N}$, $Q \in GL_N(\mathbb{C})$ s.t. $Q\bar{Q} = \pm I_N$.
The discrete quantum group $\Gamma = \mathbb{F}O(Q)$ can be described as follows:
- ➔ $\mathrm{Corep}(\mathbb{F}O(Q))$ is the Temperley-Lieb category with $\delta = \mathrm{Tr}(Q^*Q)$,
- ➔ $I = \mathbb{N}$ with $k \otimes 1 \simeq 1 \otimes k \simeq (k-1) \oplus (k+1)$, $\bar{k} = k$,
- ➔ $H_0 = \mathbb{C}$, $H_1 = \mathbb{C}^N$ and $\mathrm{Hom}(0, 1 \otimes 1) = \mathbb{C}t_1$ with $t_1 = \sum e_i \otimes Qe_i$.

We can then construct H_k by induction, $\ell^\infty(\mathbb{F}O_Q)$ and compute Δ.

Then $C^*_{\mathrm{red}}(\mathbb{F}O_Q)$ is the reduced version of Wang's algebra:

$$A_o(Q) = C^*\langle 1, u_{ij} \mid uu^* = u^*u = I_n, Q\bar{u}Q^{-1} = u\rangle,$$
$$\Delta(u_{ij}) = \textstyle\sum_k u_{ik} \otimes u_{kj}.$$

The terminology comes from the following "classical" quotients of $A_o(I_N)$:

$$A_o(I_N)/(u_{ij}, i \neq j) \simeq C^*(FO_N), \qquad A_o(I_N)/([u_{ij}, u_{kl}]) \simeq C(O_N).$$

where $FO_N = (\mathbb{Z}/2\mathbb{Z})^{*N}$ and O_N is the classical orthogonal group.

Outline

Γ-boundaries

Classical case: $\Gamma \curvearrowright X$ compact.
We have $X \subset \mathrm{Prob}(X)$ via Dirac measures and $\Gamma \curvearrowright \mathrm{Prob}(X)$.

The action $\Gamma \curvearrowright X$ is:

- minimal if $\forall x, y \in X\ \exists g_n \in \Gamma$ s.t. $\lim g_n \cdot x = y$,
  in other words: $\forall x \in X \quad \overline{\Gamma \cdot x} = X$;
- proximal if $\forall x, y \in X\ \exists g_n \in \Gamma$ s.t. $\lim g_n \cdot x = \lim g_n \cdot y$;
- strongly proximal if $\Gamma \curvearrowright \mathrm{Prob}(X)$ proximal,
  or equivalently: $\forall \nu \in \mathrm{Prob}(X) \quad \overline{\Gamma \cdot \nu} \cap X \neq \emptyset$.

X is a Γ-**boundary** if it is minimal and strongly proximal.

Classical examples:

- G connected simple Lie group, $H < G$ maximal amenable, $X = G/H$
- Γ non elementary hyperbolic, $X = \partial_G \Gamma$ Gromov boundary

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Equivalently:

i) $\forall \nu \in \mathrm{Prob}(X) \quad X \subset \overline{\Gamma \cdot \nu}$
ii) $\forall \nu \in \mathrm{Prob}(X) \quad \mathrm{Prob}(X) = \overline{\mathrm{Prob}(\Gamma) * \nu}$
iii) $\forall \nu \in \mathrm{Prob}(X) \quad P_\nu$ is an isometry
iv) all UCP Γ-maps $T : C(X) \to B$ are complete isometries

UCP: unital completely positive, "complete": remains true for $T \otimes \mathrm{id}$, automatic if $A = C(X)$ is commutative but not in general.

Quantum case:

- take $\mathbb{\Gamma} \curvearrowright A$ unital, replace $\mathrm{Prob}(X)/\,\mathrm{Prob}(\Gamma)$ with $S(A)/S(c_0(\mathbb{\Gamma}))$
- i) has no meaning, only ii) $\Leftarrow$ iii) $\Leftarrow$ iv)

Boundaries and unique stationarity

Definition

A unital Γ-C^*-algebra A is a Γ-boundary if every UCP Γ-map $T : A \to B$ is automatically UCI.

This has good categorical properties : $\mathbb{C} \hookrightarrow A$ is an "essential extension" in the category of unital Γ-C^*-algebras with UCP Γ-maps as morphisms and UCI Γ-maps as embeddings.

Choose $\mu \in S(c_0(\Gamma))$. A state $\nu \in S(A)$ is μ-stationary if $\mu * \nu = \nu$.

Proposition (Kalantar)

Assume that A admits a unique μ-stationary state ν and that P_ν is completely isometric. Then A is a Γ-boundary.

Boundaries and unique stationarity

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Proposition (Kalantar)

Assume that A admits a unique μ-stationary state ν and that P_ν is completely isometric. Then A is a $\mathbb{\Gamma}$-boundary.

Proof. ν is stationary **iff** $P_\nu(A) \subset H^\infty_\mu(\mathbb{\Gamma}) := \{f \in \ell^\infty(\mathbb{\Gamma}) \mid L_\mu(f) = f\}$. Then P_ν is the unique UCP $\mathbb{\Gamma}$-map $A \to H^\infty_\mu(\mathbb{\Gamma})$. Moreover we know that $H^\infty_\mu(\mathbb{\Gamma})$ is $\mathbb{\Gamma}$-injective. Thus it suffices to apply:

Exercise. Let $X \hookrightarrow Y$ be an embedding, Z an injective object. Assume that there exists a unique morphism $Y \to Z$, which is moreover an embedding. Then $X \hookrightarrow Y$ is essential.

The Gromov boundary of $\mathbb{F}O_Q$

Classical case: free group $\mathbb{\Gamma} = \Gamma = F_N$.
Word length: $|g|$, spheres: $S_n = \{g \in F_N; |g| = n\}$.
"Gromov" boundary $\partial_G F_N$: set of infinite reduced words.
Compactification $\beta_G F_N = F_N \sqcup \partial_G F_N$ with topology of projective limit:
$\partial_G F_N = \varprojlim (S_k, \rho_k)$ where $\rho_k : S_{k+1} \to S_k$ "forgets last letter".

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At the level of function algebras:
$\ell^\infty(F_N) = \bigoplus_k^{\ell^\infty} C(S_k)$ and $C(\beta_G F_N) = \overline{\bigcup_m C(\beta_G F_N)_m} \subset \ell^\infty(F_N)$ with

$$C(\beta_G F_N)_m = \{(f_k)_k \in \bigoplus_k^{\ell^\infty} C(S_k) \mid \forall k \geq m \; f_{k+1} = f_k \circ \rho_k\}$$

Quantum case: $\mathbb{\Gamma} = \mathbb{F}O_Q$, $N \geq 3$.
Recall $\ell^\infty(\mathbb{\Gamma}) = \bigoplus_{k \geq 0}^{\ell^\infty} B(H_k)$ and we have canonical isometries
$V_k : H_{k+1} \to H_k \otimes H_1$ from the Temperley-Lieb category.
Replace: $C(S_k) \leftrightarrow B(H_k)$, $f_k \circ \rho_k \leftrightarrow V_k^*(f_k \otimes \mathrm{id})V_k$.

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Theorem (Vaes-Vergnioux '05)

$C(\beta_G \mathbb{F}O_Q)$ is a sub-$\mathbb{F}O_Q$-C^*-algebra of $\ell^\infty(\mathbb{F}O_Q)$.

We also denote $C(\partial_G \mathbb{F}O_Q) = C(\beta_G \mathbb{F}O_Q)/c_0(\mathbb{F}O_Q)$, which is still a unital $\mathbb{F}O_Q$-C^*-algebra.

An $\mathbb{F}O_Q$-boundary

We have "quantum traces" $\mathrm{qtr}_k : B(H_k) \to \mathbb{C}$.
They satisfy $\mathrm{qtr}_{k+1}(V_k^*(f_k \otimes \mathrm{id})V_k) = \mathrm{qtr}_k(f_k)$
→ we get a state $\omega = \varinjlim \mathrm{qtr}_k$ on $C(\partial_G \mathbb{F}O_Q)$.
One checks that ω is μ-stationary for $\mu = \mathrm{qtr}_1 \in B(H_1)^* \subset c_0(\mathbb{F}O_Q)^*$.

Denote $C_r(\partial_G \mathbb{F}O_Q)$ the image of the GNS representation of ω.

Theorem (Vaes-Vergnioux '05)

Assume $N \geq 3$. Then P_ω extends to a normal $*$-isomorphism $P_\omega : C_r(\partial \mathbb{F}O_Q)'' \to H_\mu^\infty(\mathbb{F}O_Q)$.

Theorem (KKSV '20)

For $N \geq 3$, ω is the unique μ-stationary state on $C(\partial_G \mathbb{F}O_Q)$. Hence $C_r(\partial_G \mathbb{F}O_Q)$ is an $\mathbb{F}O_Q$-boundary. proof

For $N = 2$, $\mathbb{F}O_Q$ is amenable, the only $\mathbb{F}O_Q$-boundary is $\mathbb{C}$.

Outline

Uniqueness of trace

Definition

The cokernel $N_\alpha \subset \ell^\infty(\mathbb{\Gamma})$ of α is the weak closure of $\{P_\nu(a), a \in A, \nu \in A^*\}$. We say that α is faithful if $N_\alpha = \ell^\infty(\mathbb{\Gamma})$.

We have $\Delta(N_\alpha) \subset N_\alpha \bar{\otimes} N_\alpha$: "Baaj-Vaes" subalgebra. In the classical case this implies $N_\alpha = \ell^\infty(\Gamma)^\Lambda$ with $\Lambda \lhd \Gamma$, and we have $\Lambda = \mathrm{Ker}\,\alpha$ in this case.

Recall : h is a trace $\Leftrightarrow$ $\mathbb{\Gamma}$ unimodular.

Theorem (KKSV '20)

Assume that $\mathbb{\Gamma}$ acts faithfully on some $\mathbb{\Gamma}$-boundary A. Then:

- *if $\mathbb{\Gamma}$ is unimodular, h is the unique trace on $C^*_{\mathrm{red}}(\mathbb{\Gamma})$;*
- *else $C^*_{\mathrm{red}}(\mathbb{\Gamma})$ does not admit any KMS state wrt the scaling group.*

Question: in the unimodular case, does uniqueness of trace imply the existence of a faithful boundary action?

Uniqueness of trace

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Question: in the unimodular case, does uniqueness of trace imply the existence of a faithful boundary action?

Theorem (KKSV '20)

For $N \geq 3$, $\mathbb{F}O_Q$ acts faithfully on $\partial_G \mathbb{F}O_Q$.

Note: in this case, uniqueness of trace was already proved in [VV '05]. In the non-unimodular case, the absence of τ-KMS state is new.

Universal boundary and the amenable radical

Recall that an injective envelope is an injective and essential extension.

Theorem (Hamana, KKSV '20)

$\mathbb{C}$ admits an injective envelope $C(\partial_F \mathbb{\Gamma}) := I_{\mathbb{\Gamma}}(\mathbb{C})$, which is unique up to unique isomorphism. We call it the Furstenberg boundary of $\mathbb{\Gamma}$.

Then any $\mathbb{\Gamma}$-boundary embeds in a unique way in $C(\partial_F \mathbb{\Gamma})$.
There exists a faithful $\mathbb{\Gamma}$-boundary **iff** $\mathbb{\Gamma} \curvearrowright \partial_F \mathbb{\Gamma}$ is faithful.
In the classical case the kernel of this action is the maximal amenable normal subgroup of Γ (amenable radical).

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A $\mathbb{\Gamma}$-invariant subalgebra $M \subset \ell^\infty(\mathbb{\Gamma})$ is called *relatively amenable* if there exists a UCP $\mathbb{\Gamma}$-map $T : \ell^\infty(\mathbb{\Gamma}) \to M$.

Theorem (KKSV '20)

The cokernel N_F of $\mathbb{\Gamma} \curvearrowright \partial_F \mathbb{\Gamma}$ is the unique minimal relatively amenable Baaj-Vaes subalgebra of $\ell^\infty(\mathbb{\Gamma})$.

Hence there exists a faithful $\mathbb{\Gamma}$-boundary **iff** $\ell^\infty(\mathbb{\Gamma})$ has no proper relatively amenable Baaj-Vaes subalgebra.

Proof of unique stationarity for F_N

$S_n \subset F_N$: reduced words of length n. μ_n: uniform proba measure on S_n. Gromov boundary: $\partial_G F_N \simeq S_\infty$. Put $X_g = \{g \cdots \text{ reduced}\} \subset S_\infty$.

Proposition

Let ω be a proba measure on S_∞ such that $\mu_1 * \omega = \omega$. Then for any $g \in F_N$ we have $\omega(X_g) = (\# S_{|g|})^{-1}$.

It is sufficient to prove $\omega(X_g) \leq (\# S_k)^{-1}$ for $|g| = k$.
Observe that the assumption implies $\mu_n * \omega = \omega$ for all n.

Let me show that $(\mu_n * \omega)(X_g) \leq (\# S_k)^{-1} + o(\frac{1}{n})$.
We have $(\mu_n * \omega)(X_g) = (\# S_n)^{-1} \sum_{|h|=n} \omega(hX_g)$.

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Case 1: the last letter of g is not simplified in the product hg, i.e. $|hg| = k + n - 2l$ with $0 \leq l \leq k-1$. Then $hX_g = X_{hg}$ and when l is fixed these subsets are pairwise disjoint. Hence for fixed l:

$$\sum \{\omega(hX_g); |h| = n, |hg| = k + n - 2l\} \leq 1.$$

Altogether
$$\begin{aligned}(\mu_n * \omega)(X_g) &\leq (\#S_n)^{-1} \sum_{l=0}^{k-1} 1 + \text{case } 2 \\ &= (\#S_n)^{-1} k + \text{case } 2.\end{aligned}$$

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$$\sum\{\omega(hX_g); |h| = n, |hg| = k + n - 2l\} \leq 1.$$

Case 2: use the trivial estimate $\omega(hX_g) \leq 1$. In this case the last k letters of h are fixed, equal to g^{-1}, so we have $(2N-1)^{n-k}$ such elements h.

Altogether
$$(\mu_n * \omega)(X_g) \leq (\#S_n)^{-1} \sum_{l=0}^{k-1} 1 + (\#S_n)^{-1}(2N-1)^{n-k} = (\#S_n)^{-1} k + (\#S_k)^{-1} \to_{n\infty} (\#S_k)^{-1}.$$

Indeed $\#S_n = 2N(2N-1)^{n-1}$.

back

A real-world application

Remark

Thank you for your attention!