Hecke Algebras and the Schlichting completion for discrete quantum groups

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Outline

Introduction

Let $\mathbb{\Lambda} \subset \mathbb{\Gamma}$ be a quantum subgroup of a discrete quantum group.
The von Neumann algebra of Hecke operators is the commutant of the quasi-regular representation: $\mathscr{L}(\mathbb{\Gamma}, \mathbb{\Lambda}) = B(\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda}))^{\mathbb{\Gamma}}$.

Questions:

1. (combinatorial) description of $\mathscr{L}(\mathbb{\Gamma}, \mathbb{\Lambda})$?
2. modular properties of the canonical state $(\delta_{\mathbb{\Lambda}} \mid \cdot \delta_{\mathbb{\Lambda}})$?
3. analytical properties of this von Neumann algebra...

In the classical case 1 is solved using a dense subalgebra $\mathscr{H}(\Gamma, \Lambda) \simeq c_c(\Lambda\backslash\Gamma/\Lambda)$ with convolution product.

4. construction and description of $\mathbb{\Gamma}/\mathbb{\Lambda}$, $\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda}$?
5. boundedness of the action of $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$ on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$?

For 3, 5 we construct the Schlichting completion $(\mathbb{G}, \mathbb{H})$ of $(\mathbb{\Gamma}, \mathbb{\Lambda})$.
→ construction of new locally compact quantum groups.

Outline

Discrete quantum groups

A **discrete quantum group** $\mathbb{\Gamma}$ is given by

- an algebra of the form $\ell^\infty(\mathbb{\Gamma}) = \ell^\infty\text{-}\bigoplus_{\alpha\in I(\mathbb{\Gamma})} B(H_\alpha)$, $\dim H_\alpha < \infty$,
- a coproduct $\Delta : \ell^\infty(\mathbb{\Gamma}) \to \ell^\infty(\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma})$.

Δ endows the category $\mathrm{Corep}(\mathbb{\Gamma}) = \mathrm{Rep}(\ell^\infty(\mathbb{\Gamma}))$ with a tensor structure: $v \otimes w := (v \otimes w) \circ \Delta$. $I(\mathbb{\Gamma})$ is the set of irreducibles (up to equiv.). Combinatorial data underlying $\mathbb{\Gamma}$: spaces $\mathrm{Hom}_{\mathbb{\Gamma}}(\alpha, \beta \otimes \gamma)$ for $\alpha, \beta, \gamma \in I(\mathbb{\Gamma})$.

Some notation:

- $p_\alpha = \mathrm{id}_{H_\alpha}$, $a_\alpha = p_\alpha a \in B(H_\alpha)$ for $a \in \ell^\infty(\mathbb{\Gamma})$, $\alpha \in I(\mathbb{\Gamma})$;
- dual corepresentation $\bar{\alpha}$;
- quantum dimension $\dim_q(\alpha)$, equals $\dim(H_\alpha)$ if $h_L = h_R$;
- Haar weights h_L, h_R on $\ell^\infty(\mathbb{\Gamma})$; antipode S.

Classical case $\mathbb{\Gamma} = \Gamma$: $\ell^\infty(\mathbb{\Gamma})$ commutative $\Leftrightarrow \forall\alpha \dim H_\alpha = 1$.
Then $I(\mathbb{\Gamma}) = \Gamma$, $\alpha \otimes \beta = \alpha\beta$, $\bar{\alpha} = \alpha^{-1}$, $h_L = h_R =$ counting measure.

Subgroups of discrete quantum groups

A (quantum) **subgroup** $\Lambda \subset \Gamma$ is given by a restriction homomorphism $\pi_\Lambda : \ell^\infty(\Gamma) \twoheadrightarrow \ell^\infty(\Lambda)$ compatible with Δ.
We have $\ell^\infty(\Lambda) \simeq p_\Lambda \ell^\infty(\Gamma)$ for some central projection $p_\Lambda \in \ell^\infty(\Gamma)$.
We have $I(\Lambda) \subset I(\Gamma)$: in fact, $\pi_\Lambda^* : \mathrm{Corep}(\Lambda) \to \mathrm{Corep}(\Gamma)$ is fully faithful.

Quotient spaces.

- Quantum: $\ell^\infty(\Gamma/\Lambda) = \ell^\infty(\Gamma)^\Lambda = \{a \in \ell^\infty(\Gamma) \mid (1 \otimes p_\Lambda)\Delta(a) = a \otimes p_\Lambda\}$.
We have $\Delta(\ell^\infty(\Gamma/\Lambda)) \subset \ell^\infty(\Gamma)\bar{\otimes}\ell^\infty(\Gamma/\Lambda)$.
- Classical: $I(\Gamma)/\Lambda = I(\Gamma)/\sim$ where $\alpha \sim \beta \Leftrightarrow \exists \lambda \in I(\Lambda) \quad \beta \subset \alpha \otimes \lambda$.

Some notation:

- $v_\Lambda \in \mathrm{Corep}(\Lambda)$, Λ-isotypical component of $v \in \mathrm{Corep}(\Gamma)$;
- $\kappa_\alpha = \dim_q(\bar{\alpha} \otimes \alpha)_\Lambda$ for $\alpha \in I(\Gamma)$;
- $[\alpha]$ the class of α in $I(\Gamma)/\Lambda$ (or $\Lambda\backslash I(\Gamma)$);
- $p_{[\alpha]} = \sum_{\beta\in[\alpha]} p_\beta \in \ell^\infty(\Gamma/\Lambda)$.

Description of the quotient space

Theorem

- *We have* $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \ell^\infty\text{-}\bigoplus_{[\alpha]} p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$.
- *We have* $p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) \simeq B(H_\alpha)'_{\mathbb{\Lambda}} \cap B(H_\alpha)$ *canonically.*
- *More generally* $\mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{\Lambda}}(\alpha,\beta) = B(H_\alpha, H_\beta)_{\mathbb{\Lambda}}$.
- *For* $a \in \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ *we have* $a = \sum_{[\alpha]} \kappa_\alpha^{-1}(h_R \otimes \mathrm{id})[(S^{-1}(a_\alpha) \otimes 1)\Delta(p_{\mathbb{\Lambda}})]$.

Denote $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = \bigoplus p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$, $c(\mathbb{\Gamma}/\mathbb{\Lambda}) = \prod p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$.

Analogue of the counting measure on Γ/Λ :

Corollary

$c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ *admits a (unique) positive,* $\mathbb{\Gamma}$*-invariant faithful form given by*

$$\mu(a) = \sum_{[\alpha]} \kappa_\alpha^{-1} h_L(a_\alpha).$$

The Hecke algebra

Definition

For $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ *define*

$$a * b = (\mathrm{id} \otimes \mu)[\Delta(a)(1 \otimes S(b))] = (\mu S \otimes \mathrm{id})[\Delta(b)(S^{-1}(a) \otimes 1)] \in c(\mathbb{\Gamma})$$

and $a^\sharp = S(a^*) \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.

Classical case: $\forall\alpha \ \dim_q(\alpha) = 1$, $\kappa_\alpha = 1$. We recover the formula

$$(a * b)(g) = \sum_{[h]\in\Gamma/\Lambda} a(gh)b(h^{-1}).$$

Proposition-Definition

$\mathscr{H}(\mathbb{\Gamma}, \mathbb{\Lambda}) := c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ *is an involutive algebra for* $*$ *and* $^\sharp$, *with unit* $p_{\mathbb{\Lambda}}$, *stable under* σ_t^R, σ_t^L *and* τ_t.

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NB. $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ is also a (possibly degenerate) sub-$*$-algebra of $\ell^\infty(\mathbb{\Gamma})$. Denote, for $\tau \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}$:

$$L(\tau) = \#\{\alpha \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma}) \mid \alpha \subset \tau\}, \quad R(\tau) = \#\{\alpha \in I(\mathbb{\Gamma})/\mathbb{\Lambda} \mid \alpha \subset \tau\}.$$

Proposition-Definition

We say that $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is a Hecke pair if $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \subset \ell^\infty(\mathbb{\Gamma})$ is non degenerate $\Leftrightarrow \forall\tau \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda} \quad L(\tau) < \infty$.

Examples: $\mathbb{\Lambda}$ finite or finite index. Normal case: if $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$, $\mathscr{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ is the convolution algebra of the quotient quantum group $\mathbb{\Gamma}/\mathbb{\Lambda}$.

Hecke Operators

Recall that $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ is endowed with a left $\mathbb{\Gamma}$-action.

Proposition

Let $(\mathbb{\Gamma},\mathbb{\Lambda})$ be a Hecke pair. We have isomorphisms

$$\begin{array}{rcl} \mathscr{H}(\mathbb{\Gamma},\mathbb{\Lambda}) & \longrightarrow & \mathrm{End}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))^{\mathbb{\Gamma}} \\ a & \longmapsto & T(a) := (\cdot * a) \\ F(p_{\mathbb{\Lambda}}) & \longleftarrow & F \end{array}$$

Moreover $(x \mid T(a)y) = (T(a^\sharp)x \mid y)$ for the scalar product assoc'd with μ.

Theorem

Let $(\mathbb{\Gamma},\mathbb{\Lambda})$ be a Hecke pair. $T(a)$ is bounded on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$ for all $a \in \mathscr{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ **iff** *$\kappa_\gamma \le C_\beta\, \kappa_\alpha$ for all $\gamma \subset \alpha \otimes \beta$.*

This property is not satisfied by all inclusions $\mathbb{\Lambda} \subset \mathbb{\Gamma}$. I don't know how to prove *directly* that it is satisfied by Hecke pairs.

Modular properties

Canonical state on $\mathscr{H}(\Gamma,\Lambda)$: $\omega = \epsilon = (p_\Lambda \mid T(\cdot)p_\Lambda)$. It is faithful.

Proposition-Definition

Let $\nabla \in c(\Lambda\backslash\Gamma/\Lambda)$ unique such that $\mu S(a) = \mu(\nabla a)$ for all $a \in \mathscr{H}(\Gamma,\Lambda)$. Then $\theta_t : a \mapsto \sigma_t^R(\nabla^{it} a)$ is a group of $\sharp$-automorphisms of $\mathscr{H}(\Gamma,\Lambda)$ and ω is θ-KMS.

Theorem

Asumme Λ is unimodular. Then $\nabla_\alpha = (\tilde{L}([\![\alpha]\!])/\tilde{R}([\![\alpha]\!]))\, F_\alpha^2$ where the F_α are Woronowicz' modular matrices, and for $\tau = [\![\alpha]\!] \in \Lambda\backslash I(\Gamma)/\Lambda$:

$$\tilde{L}(\tau) = \sum_{[\delta]\in\Lambda\backslash I(\Gamma),[\delta]\subset\tau} (\mathrm{dim}_q\,\delta)^2/\kappa_{\bar{\delta}}$$
$$\tilde{R}(\tau) = \sum_{[\delta]\in I(\Gamma)/\Lambda,[\delta]\subset\tau} (\mathrm{dim}_q\,\delta)^2/\kappa_{\delta}.$$

There is also a more involved formula when Λ is not unimodular...

Outline

The C^*-Hopf algebra

Classical reminder. Consider $\lambda : \Gamma \to \mathrm{Sym}(\Gamma/\Lambda)$ by left translations. Define $G = \overline{\lambda(\Gamma)}$. If (Γ, Λ) is a Hecke pair, it is a locally compact group and $H = \overline{\lambda(\Lambda)}$ is compact open.

But $\mathrm{Sym}(\Gamma/\Lambda)$ has no good quantum analogue...

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But $\mathrm{Sym}(\Gamma/\Lambda)$ has no good quantum analogue...

Definition

We put $\mathscr{O}_c(\mathbb{G}) = \mathrm{alg}\text{–}\langle a * b, a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}), b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\rangle \subset c(\mathbb{\Gamma})$.

$$C_0(\mathbb{G}) = C^*\text{–}\langle a * b, a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}), b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\rangle \subset \ell^\infty(\mathbb{\Gamma}).$$

Classical case: $\Gamma \to G$ induces $C_0(G) \subset \ell^\infty(G)$.
If $a = \mathbb{1}_{[r]}$, $b = \mathbb{1}_{[s]}$ then $a * b = \mathbb{1}_{\{g \mid g[s]^{-1}=[r]\}}$.

The C^*-Hopf algebra

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We put $\mathscr{O}_c(\mathbb{G}) = \mathrm{alg}\text{–}\langle a * b, a \in c_c(\Gamma/\Lambda), b \in c_c(\Lambda\backslash\Gamma)\rangle \subset c(\Gamma)$.

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Theorem

If (Γ, Λ) *is a Hecke pair we have*

$$\Delta(\mathscr{O}_c(\mathbb{G}))(1 \otimes \mathscr{O}_c(\mathbb{G})) = \mathscr{O}_c(\mathbb{G}) \odot \mathscr{O}_c(\mathbb{G}) = \Delta(\mathscr{O}_c(\mathbb{G}))(\mathscr{O}_c(\mathbb{G}) \otimes 1).$$

$\mathscr{O}_c(\mathbb{G})$ *is a multiplier Hopf algebra.*

$C_0(\mathbb{G})$ *is a bisimplifiable Hopf* C^**-algebra.*

The Haar weights

Corollary-Definition

$C(\mathbb{H}) := p_\wedge C_0(\mathbb{G})$ is a Hopf C^*-algebra (with unit $p_\wedge$).
Hence it admits a Haar state h.
We have $c_c(\Gamma/\Lambda) = \mathscr{O}_c(\mathbb{G})^{\mathbb{H}}$ in $\ell^\infty(\Gamma)$.

Corollary-Definition

$\varphi := \mu(\mathrm{id} \otimes hp_\wedge)\Delta$ is an integral on $\mathscr{O}_c(\mathbb{G})$. $\mathbb{G}$ is an algebraic quantum group, hence a locally compact quantum group.

NB. If the action of Γ on Γ/Λ is faithful and Λ is infinite, $\mathbb{G}$ is non-discrete.

NB. The algebraic quantum groups $\mathbb{G}$ with commutative $\mathscr{O}_c(\mathbb{G})$ are exactly the locally compact groups G admitting a compact-open subgroup H, and

$$\mathscr{O}_c(G) = \{f \in C_c(G) \mid \dim \mathrm{Vect}(H \cdot f) < \infty\}.$$

The Hecke Algebra

Let $(\mathbb{G}, \mathbb{H})$ be the Schlichting completion of a Hecke pair $(\mathbb{\Gamma}, \mathbb{\Lambda})$.

Since $\mathbb{H}$ is compact, we can consider

- $c_c(\mathbb{G}/\mathbb{H}) := \mathscr{O}_c(\mathbb{G})^{\mathbb{H}} \subset \mathscr{O}_c(\mathbb{G})$ and $\ell^2(\mathbb{G}/\mathbb{H}) = \overline{c_c(\mathbb{G}/\mathbb{H})} \subset L^2(\mathbb{G})$,
- $\mathscr{H}(\mathbb{G}, \mathbb{H}) := {}^{\mathbb{H}}\mathscr{O}_c(\mathbb{G})^{\mathbb{H}}$ with the convolution product of $\mathscr{O}_c(\mathbb{G})$.

Proposition

We have $\mathscr{H}(\mathbb{G}, \mathbb{H}) \simeq \mathrm{End}(c_c(\mathbb{G}/\mathbb{H}))^{\mathbb{G}}$, $b \mapsto T'(b) := (\,\cdot * b)$.

By construction of $(\mathbb{G}, \mathbb{H})$ we have

$$\mathrm{End}(c_c(\mathbb{G}/\mathbb{H}))^{\mathbb{G}} = \mathrm{End}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))^{\mathbb{\Gamma}} \text{ and } \ell^2(\mathbb{G}/\mathbb{H}) = \ell^2(\mathbb{\Gamma}/\mathbb{\Lambda}).$$

The operators $T'(b)$ arise from the right regular repr. of $\mathbb{G}$, hence:

Corollary

The Hecke operators $T(a)$, for $a \in \mathscr{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$, are bounded on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$.

HNN Extensions

Fixe $\Lambda_{\pm 1} \subset \Gamma_0$ with an isomorphism $\theta : \Lambda_1 \to \Lambda_{-1}$.
Consider $\Gamma = HNN(\Gamma_0, \theta)$ [Fima 2013]. $\mathbb{C}[\Gamma]$ is generated by $\mathbb{C}[\Gamma_0]$ and a *group-like unitary* w such that $w^\epsilon x w^{-\epsilon} = \theta^\epsilon(x)$ for $x \in \mathbb{C}[\Lambda_\epsilon]$.

Proposition

- *If $\Lambda_{\pm 1}$ have finite index in Γ_0 and are different from Γ_0, then $\Gamma_0 \subset \Gamma$ is almost normal, not normal, of infinite index.*
- *If $\bigcap_{k \in \mathbb{Z}} \mathrm{Dom}\, \theta^k = \{1\}$ the action of Γ on Γ / Γ_0 is faithful.*
- *We have $\nabla_w = (\tilde{L}([\![w]\!]) / \tilde{R}([\![w]\!]))\, p_w$ with*
  $$\tilde{L}([\![w]\!]) = \sum_{[\delta] \in \Lambda_1 \backslash I(\Gamma_0)} \dim_q(\delta \otimes \bar{\delta}) / \dim_q(\delta \otimes \bar{\delta})_{\Lambda_1},$$
  $$\tilde{R}([\![w]\!]) = \sum_{[\delta] \in I(\Gamma_0)/\Lambda_{-1}} \dim_q(\bar{\delta} \otimes \delta) / \dim_q(\bar{\delta} \otimes \delta)_{\Lambda_{-1}}.$$

HNN Extensions

Proposition

- *If* $\mathbb{\Lambda}_{\pm 1}$ *have finite index in* $\mathbb{\Gamma}_0$ *and are different from* $\mathbb{\Gamma}_0$*, then* $\mathbb{\Gamma}_0 \subset \mathbb{\Gamma}$ *is almost normal, not normal, of infinite index.*
- *If* $\bigcap_{k\in\mathbb{Z}} \mathrm{Dom}\,\theta^k = \{1\}$ *the action of* $\mathbb{\Gamma}$ *on* $\mathbb{\Gamma}/\mathbb{\Gamma}_0$ *is faithful.*
- *We have* $\nabla_w = (\tilde{L}([\![w]\!])/\tilde{R}([\![w]\!]))\, p_w$ *with*
  $$\tilde{L}([\![w]\!]) = \sum_{[\delta]\in\mathbb{\Lambda}_1\backslash I(\mathbb{\Gamma}_0)} \dim_q(\delta\otimes\bar\delta)/\dim_q(\delta\otimes\bar\delta)_{\mathbb{\Lambda}_1},$$
  $$\tilde{R}([\![w]\!]) = \sum_{[\delta]\in I(\mathbb{\Gamma}_0)/\mathbb{\Lambda}_{-1}} \dim_q(\bar\delta\otimes\delta)/\dim_q(\bar\delta\otimes\delta)_{\mathbb{\Lambda}_{-1}}.$$

Example. $\Sigma_{\pm 1}$ non abelian finite classical groups.
Take $\mathbb{\Gamma}_0 = \prod'_{k\in\mathbb{Z}^*} \hat{\Sigma}_{\mathrm{sgn}(k)}$ and $\mathbb{\Lambda}_\epsilon = \prod'_{k\in\mathbb{Z}^*, k\neq\epsilon} \hat{\Sigma}_{\mathrm{sgn}(k)} \subset \mathbb{\Gamma}_0$.
Then the Proposition applies, $\tilde{L}([\![w]\!]) = \#\Sigma_1$, $\tilde{R}([\![w]\!]) = \#\Sigma_{-1}$.

$\mathbb{G}$ is a non-discrete, non-classical, non-co-classical locally compact quantum group, with non-trivial modular group if $\#\Sigma_1 \neq \#\Sigma_{-1}$.