Hecke Algebras and the Schlichting completion for discrete quantum groups

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Outline

Introduction

Let $\mathbb{\Lambda} \subset \mathbb{\Gamma}$ be a quantum subgroup of a discrete quantum group. The von Neumann algebra of Hecke operators is the commutant of the quasi-regular representation: $\mathscr{L}(\mathbb{\Gamma}, \mathbb{\Lambda}) = B(\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda}))^{\mathbb{\Gamma}}$.

Questions:

1. (combinatorial) description of $\mathscr{L}(\mathbb{\Gamma}, \mathbb{\Lambda})$?
2. modular properties of the canonical state $(\delta_{\mathbb{\Lambda}} \mid \cdot\, \delta_{\mathbb{\Lambda}})$?
3. analytical properties of this von Neumann algebra...

In the classical case 1 is achieved using a dense subalgebra $\mathscr{H}(\Gamma, \Lambda) \simeq c_c(\Lambda\backslash\Gamma/\Lambda)$ with convolution product. Quantum case:

4. construction and description of $\mathbb{\Gamma}/\mathbb{\Lambda}$, $\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda}$?
5. boundedness of the action of $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$ on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$?

For 3, 5 we construct the Schlichting completion $(\mathbb{G}, \mathbb{H})$ of $(\mathbb{\Gamma}, \mathbb{\Lambda})$.
→ construction of new locally compact quantum groups.

Outline

Quantum groups

An **algebraic quantum group** [Van Daele] $\mathbb{G}$ is given by

- a (non-unital) $*$-algebra $\mathscr{O}_c(\mathbb{G})$,
- a coproduct $\Delta : \mathscr{O}_c(\mathbb{G}) \to \mathscr{M}(\mathscr{O}_c(\mathbb{G}) \odot \mathscr{O}_c(\mathbb{G}))$

with some axioms, in particular:

- $\Delta(\mathscr{O}_c(\mathbb{G}))(\mathscr{O}_c(\mathbb{G}) \otimes 1) \subset \mathscr{O}_c(\mathbb{G}) \odot \mathscr{O}_c(\mathbb{G})$,
- left integral: $\varphi : \mathscr{O}_c(\mathbb{G}) \to \mathbb{C}$ s.t. $(\mathrm{id} \otimes \varphi)((a \otimes 1)\Delta(b)) = \varphi(b)a$.

There is an associated locally compact quantum group, with $\nu = 1$.

Commutative case: there exists a locally compact group G with a compact-open $H \subset G$ s.t.

$$\mathscr{O}_c(\mathbb{G}) = \{f \in C_c(G) \mid \dim \mathrm{Vect}(H \cdot f) < \infty\}.$$

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- left integral: $\varphi : \mathscr{O}_c(\mathbb{G}) \to \mathbb{C}$ s.t. $(\mathrm{id} \otimes \varphi)((a \otimes 1)\Delta(b)) = \varphi(b)a$.

Discrete case: $\mathscr{O}_c(\mathbb{\Gamma}) \simeq \bigoplus_{\alpha \in I(\mathbb{\Gamma})} B(H_\alpha)$, $\dim H_\alpha < \infty$.
The existence of integrals $\varphi := h_L$, h_R is automatic.
We put $c_c(\mathbb{\Gamma}) = \mathscr{O}_c(\mathbb{\Gamma})$, $\ell^\infty(\mathbb{\Gamma}) = M(c_c(\mathbb{\Gamma}))$.
Denote $p_\alpha = \mathrm{id}_{H_\alpha}$, $a_\alpha = p_\alpha a \in B(H_\alpha)$ for $a \in \ell^\infty(\mathbb{\Gamma})$, $\alpha \in I(\mathbb{\Gamma})$.

Δ endows the C^*-category $\mathrm{Corep}(\mathbb{\Gamma}) := \mathrm{Rep}(c_c(\mathbb{\Gamma}))$ with a tensor structure: $v \otimes w := (v \otimes w) \circ \Delta$. Underlying combinatorial data: spaces $\mathrm{Hom}_{\mathbb{\Gamma}}(\alpha, \beta \otimes \gamma)$ for $\alpha, \beta, \gamma \in I(\mathbb{\Gamma})$.

Subgroups of discrete quantum groups

A (quantum) **subgroup** $\Lambda \subset \Gamma$ is given by $\ell^\infty(\Lambda) \simeq p_\Lambda \ell^\infty(\Gamma)$ for some central proj. $p_\Lambda \in \ell^\infty(\Gamma)$ s.t. $\Delta(p_\Lambda)(1 \otimes p_\Lambda) = p_\Lambda \otimes p_\Lambda = \Delta(p_\Lambda)(p_\Lambda \otimes 1)$. We have $I(\Lambda) \subset I(\Gamma)$, $\mathrm{Corep}(\Lambda) \subset \mathrm{Corep}(\Gamma)$, $p_\Lambda = \sum_{\alpha \in I(\Lambda)} p_\alpha$.

Quotient spaces.

- Quantum: $\ell^\infty(\Gamma/\Lambda) = \ell^\infty(\Gamma)^\Lambda = \{a \in \ell^\infty(\Gamma) \mid (1 \otimes p_\Lambda)\Delta(a) = a \otimes p_\Lambda\}$. We have $\Delta(\ell^\infty(\Gamma/\Lambda)) \subset \ell^\infty(\Gamma)\bar{\otimes}\ell^\infty(\Gamma/\Lambda)$.
- Categorical: $\mathrm{Corep}(\Gamma/\Lambda) := \mathrm{Rep}(\ell^\infty(\Gamma/\Lambda))$ is a left-$\mathrm{Corep}(\Gamma)$-module category with restriction functor $\mathrm{Corep}(\Gamma) \to \mathrm{Corep}(\Gamma/\Lambda)$.
- Classical: $I(\Gamma)/\Lambda = I(\Gamma)/\sim$ where $\alpha \sim \beta \Leftrightarrow \exists \lambda \in I(\Lambda) \quad \beta \subset \alpha \otimes \lambda$.

Some notation:

- $p_{[\alpha]} = \sum_{\beta \in [\alpha]} p_\beta \in \ell^\infty(\Gamma/\Lambda)$ for $[\alpha] \in I(\Gamma)/\Lambda$ (or $\Lambda \backslash I(\Gamma)$),
- $v_\Lambda \in \mathrm{Corep}(\Lambda)$, Λ-isotypical component of $v \in \mathrm{Corep}(\Gamma)$,
- $\kappa_\alpha = \dim_q(\bar{\alpha} \otimes \alpha)_\Lambda$ for $\alpha \in I(\Gamma)$.

Description of the quotient space

Theorem

- *We have* $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \ell^\infty\text{-}\bigoplus_{[\alpha]} p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$.
- *We have* $p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) \simeq B(H_\alpha)'_{\mathbb{\Lambda}} \cap B(H_\alpha)$ *canonically.*
- *More generally* $\mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{\Lambda}}(\alpha,\beta) = B(H_\alpha, H_\beta)_{\mathbb{\Lambda}}$.
- *For* $a \in \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ *we have* $a = \sum_{[\alpha]} \kappa_\alpha^{-1}(h_R \otimes \mathrm{id})[(S^{-1}(a_\alpha) \otimes 1)\Delta(p_{\mathbb{\Lambda}})]$.

Denote $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = \bigoplus p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$, $c(\mathbb{\Gamma}/\mathbb{\Lambda}) = \prod p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$.

Analogue of the counting measure on Γ/Λ :

Corollary

$c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ *admits a (unique) positive,* $\mathbb{\Gamma}$*-invariant faithful form given by*

$$\mu(a) = \sum_{[\alpha]} \kappa_\alpha^{-1} h_L(a_\alpha).$$

Hecke pairs

The subalgebra $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \subset \ell^\infty(\mathbb{\Gamma})$ can be degenerate.

Definition

The commensurator of $\mathbb{\Lambda}$ in $\mathbb{\Gamma}$ is the unique intermediate quantum subgroup $\mathbb{\Lambda} \subset \mathbb{\Gamma}' \subset \mathbb{\Gamma}$ such that $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda}) = c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.
We say that $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is a Hecke pair if $\mathbb{\Gamma}' = \mathbb{\Gamma}$.

Denote, for $\tau \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}$:

$$L(\tau) = \#\{\alpha \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma}) \mid \alpha \subset \tau\}, \quad R(\tau) = \#\{\alpha \in I(\mathbb{\Gamma})/\mathbb{\Lambda} \mid \alpha \subset \tau\}.$$

Proposition

$\mathbb{\Gamma}'$ is given by $I(\mathbb{\Gamma}') = \{\alpha \in I(\mathbb{\Gamma}) \mid L([\![\alpha]\!]), R([\![\alpha]\!]) < \infty\}$.

Trivial examples: finite subgroups, finite index subgroups.

The Hecke algebra

Definition

For $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ *define*

$$a * b = (\mathrm{id} \otimes \mu)[\Delta(a)(1 \otimes S(b))] = (\mu S \otimes \mathrm{id})[\Delta(b)(S^{-1}(a) \otimes 1)] \in c(\mathbb{\Gamma})$$

and $a^\sharp = S(a^*) \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.

Classical case: $\forall \alpha \ \dim_q(\alpha) = 1$, $\kappa_\alpha = 1$. We recover the formula

$$(a * b)(g) = \sum_{[h] \in \Gamma/\Lambda} a(gh) b(h^{-1}).$$

Proposition-Definition

$\mathscr{H}(\mathbb{\Gamma}, \mathbb{\Lambda}) := c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ *is an involutive algebra for* $*$ *and* $^\sharp$, *with unit* $p_{\mathbb{\Lambda}}$, *stable under* σ_t^R, σ_t^L *and* τ_t.

Normal case: if $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$, $\mathscr{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ is the convolution algebra of the quotient quantum group $\mathbb{\Gamma}/\mathbb{\Lambda}$.

Hecke Operators

Recall that $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ is endowed with a left $\mathbb{\Gamma}$-action.

Proposition

Let $(\mathbb{\Gamma},\mathbb{\Lambda})$ be a Hecke pair. We have an isomorphism

$$\mathscr{H}(\mathbb{\Gamma},\mathbb{\Lambda}) \to \mathrm{End}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))^{\mathbb{\Gamma}},\ a \mapsto T(a) := (\cdot * a)$$

with inverse $(T \to T(p_{\mathbb{\Lambda}}))$. Moreover $(x \mid T(a)y) = (T(a^\sharp)x \mid y)$ for the scalar product associated with μ.

Theorem

Let $(\mathbb{\Gamma},\mathbb{\Lambda})$ be a Hecke pair. $T(a)$ is bounded on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$ for all $a \in \mathscr{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ **iff** *we have (RT) : $\kappa_\gamma \leq C_\beta \kappa_\alpha$ for all $\gamma \subset \alpha \otimes \beta$.*

Hecke Operators

Theorem

Let $(\mathbb{\Gamma}, \mathbb{\Lambda})$ be a Hecke pair. $T(a)$ is bounded on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$ for all $a \in \mathscr{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ **iff** *we have (RT) : $\kappa_\gamma \leq C_\beta \, \kappa_\alpha$ for all $\gamma \subset \alpha \otimes \beta$.*

This property is not satisfied by all inclusions $\mathbb{\Lambda} \subset \mathbb{\Gamma}$.

By restriction to $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ we have also $\alpha \in \mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda})$. Denote $\tilde{\alpha}$ is the image of α in the Grothendieck ring $\mathbb{Z}[\mathbb{\Gamma}/\mathbb{\Lambda}]$. If $\mathbb{\Gamma}$ is unimodular:

$$\kappa_\alpha = \dim_q(\bar{\alpha} \otimes \alpha)_{\mathbb{\Lambda}} = \|\tilde{\alpha}\|_2^2.$$

Exercise

Prove in $\mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda})$ that (RT) is satisfied if $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is a Hecke pair.

In the sequel we will see an analytical proof.

Modular properties

Canonical state on $\mathscr{H}(\mathbb{\Gamma},\mathbb{\Lambda})$: $\omega = \epsilon = (p_{\mathbb{\Lambda}} \mid T(\cdot)p_{\mathbb{\Lambda}})$. It is faithful.

Proposition-Definition

Let $\nabla \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$ unique such that $\mu S(a) = \mu(\nabla a)$ for all $a \in \mathscr{H}(\mathbb{\Gamma},\mathbb{\Lambda})$. Then $\theta_t : a \mapsto \sigma_t^R(\nabla^{it} a)$ is a group of $\sharp$-automorphisms of $\mathscr{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ and ω is θ-KMS.

Theorem

Asumme $\mathbb{\Lambda}$ is unimodular. Then $\nabla_\alpha = (\tilde{L}([\![\alpha]\!])/\tilde{R}([\![\alpha]\!]))\, F_\alpha^2$ where the F_α are Woronowicz' modular matrices, and for $\tau = [\![\alpha]\!] \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}$:

$$\tilde{L}(\tau) = \textstyle\sum_{[\delta]\in\mathbb{\Lambda}\backslash I(\mathbb{\Gamma}),[\delta]\subset\tau} \ (\dim_q\delta)^2/\kappa_{\bar\delta}$$
$$\tilde{R}(\tau) = \textstyle\sum_{[\delta]\in I(\mathbb{\Gamma})/\mathbb{\Lambda},[\delta]\subset\tau} \ (\dim_q\delta)^2/\kappa_{\delta}.$$

There is also a more involved formula when $\mathbb{\Lambda}$ is not unimodular...

Outline

The C^*-Hopf algebra

Classical reminder. Consider $\pi : \Gamma \to \mathrm{Bij}(\Gamma/\Lambda)$ by left translations. Define $G = \overline{\pi(\Gamma)}$. If (Γ, Λ) is a Hecke pair, it is a locally compact group and $H = \overline{\pi(\Lambda)}$ is compact open.

But $\mathrm{Bij}(\Gamma/\Lambda)$ has no good quantum analogue...

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But $\mathrm{Bij}(\Gamma/\Lambda)$ has no good quantum analogue...

Definition

We put $\mathscr{O}_c(\mathbb{G}) = \mathrm{alg}\text{–}\langle a * b, a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}), b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\rangle \subset c(\mathbb{\Gamma})$.

$C_0(\mathbb{G}) = C^*\text{–}\langle a * b, a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}), b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\rangle \subset \ell^\infty(\mathbb{\Gamma})$.

Classical case: $\Gamma \to G$ induces $C_0(G) \subset \ell^\infty(\Gamma)$.
If $a = \mathbb{1}_{[r]}$, $b = \mathbb{1}_{[s]}$ then $a * b = \mathbb{1}_{\{g \mid g[s]^{-1}=[r]\}}$.

The C^*-Hopf algebra

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We put $\mathcal{O}_c(\mathbb{G}) = \mathrm{alg}\text{–}\langle a * b, a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}), b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\rangle \subset c(\mathbb{\Gamma})$.

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If $a = \mathbb{1}_{[r]}$, $b = \mathbb{1}_{[s]}$ then $a * b = \mathbb{1}_{\{g \mid g[s]^{-1} = [r]\}}$.

Theorem

If $(\mathbb{\Gamma}, \mathbb{\Lambda})$ *is a Hecke pair we have*

$$\Delta(\mathcal{O}_c(\mathbb{G}))(1 \otimes \mathcal{O}_c(\mathbb{G})) = \mathcal{O}_c(\mathbb{G}) \odot \mathcal{O}_c(\mathbb{G}) = \Delta(\mathcal{O}_c(\mathbb{G}))(\mathcal{O}_c(\mathbb{G}) \otimes 1).$$

$\mathcal{O}_c(\mathbb{G})$ *is a multiplier Hopf algebra.*

$C_0(\mathbb{G})$ *is a bisimplifiable Hopf* C^**-algebra.*

The Haar weights

Recall: $C_0(\mathbb{G})$ is a bisimplifiable Hopf C^*-algebra.
To define a quantum group we need Haar weights.

Corollary-Definition

*$C(\mathbb{H}) := p_{\mathbb{\Lambda}} C_0(\mathbb{G})$ is a Hopf C^*-algebra (with unit $p_{\mathbb{\Lambda}}$).
Hence it admits a Haar state h.*

Moreover by definition of $(\mathbb{G}, \mathbb{H})$ we have $\mathscr{O}_c(\mathbb{G})^{\mathbb{H}} = c_c(\Gamma/\mathbb{\Lambda})$.
One can project onto $c_c(\Gamma/\mathbb{\Lambda})$ using h.

Corollary-Definition

$\varphi := \mu(\mathrm{id} \otimes hp_{\mathbb{\Lambda}})\Delta$ is an integral on $\mathscr{O}_c(\mathbb{G})$. $\mathbb{G}$ is an algebraic quantum group, hence a locally compact quantum group.

Reduced pairs

If $\delta : C_0(\mathbb{X}) \to M(c_0(\mathbb{\Gamma}) \otimes C_0(\mathbb{X}))$ is an action of $\mathbb{\Gamma}$ on $\mathbb{X}$, consider the cokernel $N(\mathbb{X}) = \{(\mathrm{id} \otimes \varphi)\delta(a) \mid a \in C_0(\mathbb{X}), \varphi \in C_0(\mathbb{X})^*\}'' \subset \ell^\infty(\mathbb{\Gamma})$.

We have $N(\mathbb{X}) = \ell^\infty(\mathbb{\Sigma})$ for a discrete quantum group $\mathbb{\Sigma}$ which in general is not a quotient of $\mathbb{\Gamma}$. We say that the action is faithful if $N(\mathbb{X}) = \ell^\infty(\mathbb{\Gamma})$.

Definition

We call the pair $(\mathbb{\Gamma}, \mathbb{\Lambda})$ reduced if the action of $\mathbb{\Gamma}$ on $\mathbb{X} = \mathbb{\Gamma}/\mathbb{\Lambda}$ is faithful.

By definition of $a * b = (\mathrm{id} \otimes \mu)[\Delta(a)(1 \otimes S(b))]$, we have

$$N(\mathbb{\Gamma}/\mathbb{\Lambda}) = C_0(\mathbb{G})'' \subset \ell^\infty(\mathbb{\Gamma}).$$

$(\mathbb{\Gamma}, \mathbb{\Lambda})$ is reduced **iff** "$\mathbb{\Gamma}$ embeds into $\mathbb{G}$".

Proposition

Assume $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is reduced. Then $\mathbb{G}$ is discrete **iff** *$\mathbb{\Lambda}$ is finite.*

The Hecke Algebra

Let $(\mathbb{G}, \mathbb{H})$ be the Schlichting completion of a Hecke pair (Γ, Λ).
Since $\mathbb{H}$ is compact, it makes sense to consider $\mathscr{O}_c(\mathbb{G})^{\mathbb{H}}$.

Proposition

We have $^{\mathbb{H}}\mathscr{O}_c(\mathbb{G})^{\mathbb{H}} \simeq \mathrm{End}(\mathscr{O}_c(\mathbb{G})^{\mathbb{H}})^{\mathbb{G}}$, $b \mapsto T'(b) := (\cdot * b)$,
using the convolution product of $\mathscr{O}_c(\mathbb{G})$.

Denote $\ell^2(\mathbb{G}/\mathbb{H}) = \overline{\mathscr{O}_c(\mathbb{G})^{\mathbb{H}}} \subset L^2(\mathbb{G})$. By construction of $(\mathbb{G}, \mathbb{H})$ we have $\mathscr{O}_c(\mathbb{G})^{\mathbb{H}} = c_c(\Gamma/\Lambda)$ and $\ell^2(\mathbb{G}/\mathbb{H}) \simeq \ell^2(\Gamma/\Lambda)$.

Since the operators $T'(b)$ arise from the right regular repr. of $\mathbb{G}$ we get:

Corollary

The Hecke operators $T(a)$, *for* $a \in \mathscr{H}(\Gamma, \Lambda)$, *are bounded on* $\ell^2(\Gamma/\Lambda)$.

HNN Extensions

Fixe $\Lambda_{\pm 1} \subset \Gamma_0$ with an isomorphism $\theta : \Lambda_1 \to \Lambda_{-1}$.
Consider $\Gamma = HNN(\Gamma_0, \theta)$ [Fima 2013]. $\mathrm{Corep}(\Gamma)$ is generated by $\mathrm{Corep}(\Gamma_0)$ and a 1-dimensional w such that $w^\epsilon \otimes v \otimes w^{-\epsilon} = \theta_*^\epsilon(v)$ for $v \in \mathrm{Corep}(\Lambda_\epsilon)$.

Proposition

- *If $\Lambda_{\pm 1}$ have finite index in Γ_0 and are different from Γ_0, then $\Gamma_0 \subset \Gamma$ is almost normal, not normal, of infinite index.*
- *If $\bigcap_{k\in\mathbb{Z}} \mathrm{Dom}\,\theta^k = \{1\}$ the action of Γ on Γ/Γ_0 is faithful.*
- *We have $\nabla_w = (\tilde{L}([\![w]\!])/\tilde{R}([\![w]\!]))\, p_w$ with*

$$\tilde{L}([\![w]\!]) = \sum_{[\delta]\in\Lambda_1\backslash I(\Gamma_0)} \dim_q(\delta\otimes\bar\delta)/\dim_q(\delta\otimes\bar\delta)_{\Lambda_1},$$
$$\tilde{R}([\![w]\!]) = \sum_{[\delta]\in I(\Gamma_0)/\Lambda_{-1}} \dim_q(\bar\delta\otimes\delta)/\dim_q(\bar\delta\otimes\delta)_{\Lambda_{-1}}.$$

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  $\tilde{L}([\![w]\!]) = \sum_{[\delta]\in\Lambda_1\backslash I(\Gamma_0)} \dim_q(\delta\otimes\bar{\delta})/\dim_q(\delta\otimes\bar{\delta})_{\Lambda_1}$,
  $\tilde{R}([\![w]\!]) = \sum_{[\delta]\in I(\Gamma_0)/\Lambda_{-1}} \dim_q(\bar{\delta}\otimes\delta)/\dim_q(\bar{\delta}\otimes\delta)_{\Lambda_{-1}}$.

Example. $\Sigma_{\pm 1}$ non classical finite quantum groups.
Take $\Gamma_0 = \prod'_{k\in\mathbb{Z}^*} \Sigma_{\mathrm{sgn}(k)}$ and $\Lambda_\epsilon = \prod'_{k\in\mathbb{Z}^*, k\neq\epsilon} \Sigma_{\mathrm{sgn}(k)} \subset \Gamma_0$.
Then the Proposition applies, $\tilde{L}([\![w]\!]) = \#\Sigma_1$, $\tilde{R}([\![w]\!]) = \#\Sigma_{-1}$.

$\mathbb{G}$ is a non-discrete, non-classical, non-co-classical locally compact quantum group, with non-trivial modular group if $\#\Sigma_1 \neq \#\Sigma_{-1}$.