Maximal amenability of the radial subalgebra of free quantum groups

Roland Vergnioux
joint work with Xumin Wang (Seoul)

University of Normandy (France)

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Outline

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Orthogonal free quantum groups

Definition (Wang)

S. Wang's algebra $A_o(N)$ is defined by generators and relations:

$$A_o(N) = \langle v_{ij}, 1 \leq i,j \leq N \mid v_{ij}^* = v_{ij}, v = (v_{ij})_{ij} \text{ unitary} \rangle$$

It is connected to classical groups via two natural quotient algebras:

$$A_o(N)/\langle v_{ij}, i \neq j \rangle \simeq C^*(\mathbb{Z}_2^{*N}),$$
$$A_o(N)/\langle [v_{ij}, v_{kl}] \rangle \simeq C(O_N).$$

Moreover the formula $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$ defines a coproduct Δ which turns $A_o(N)$ into a Woronowicz C^*-algebra.

We denote $A_o(N) = C(O_N^+) = C^*(\mathbb{F}O_N)$.
O_N^+ is a compact quantum group and $\mathbb{F}O_N$ is a discrete quantum group.

There are "Q-deformations" O_Q^+, $\mathbb{F}O_Q$ where $Q \in GL_N(\mathbb{C})$, $Q\bar{Q} = \pm I_N$.
For $N = 2$ we have $\{O_Q^+, Q\bar{Q} = \pm I_2\} = \{SU_q(2), q \in [-1, 1]\}$.

Made with Goodnotes

The von Neumann algebra

As a Woronowicz C^*-algebra, $C^*(\mathbb{F}O_N)$ has a canonical "Haar" state h.
→ GNS representation $\lambda : C^*(\mathbb{F}O_N) \to B(\ell^2\mathbb{F}O_N)$
→ von Neumann algebra $\mathscr{L}(\mathbb{F}O_N) = \lambda(C^*\mathbb{F}O_N)''$.

For $N \geq 3$, $\mathscr{L}(\mathbb{F}O_N)$ shares many properties with the **free group factors:**

- it is a full II_1 factor with Property AO, [V., Vaes-V.]
- it has the HAP and the CBAP, [Brannan, Freslon] [De Commer-Freslon-Yamashita]
- it is stronly solid hence has no regular MASA, [Isono, Fima-V.]
- it embeds in R^ω. [Brannan-Collins-V.]

On the other hand:

- $\beta_1^{(2)}(\mathbb{F}O_N) = 0$ for all N, [V., Kyed-Raum-Vaes-Valvekens]
- and in fact $\mathscr{L}(\mathbb{F}O_N) \not\simeq \mathscr{L}(F_M)$. [Brannan-V. 2018]

Representation theory

Corepresentation of $\mathbb{F}O_N$:

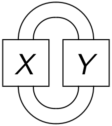
$$u \in \mathscr{U}(B(H_u) \otimes \mathscr{L}(\mathbb{F}O_N)) \text{ s.t. } (\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}.$$

They form a rigid tensor C^*-category $\mathrm{Corep}(\mathbb{F}O_N)$ with a canonical fiber functor to Hilbert spaces ($u \mapsto H_u$).

[Banica 1996]: This category is the **Temperley-Lieb category** TL_N with generating object $\bullet$ and generating morphism $\cap : 1 \to \bullet \otimes \bullet$.
The fiber functor is determined by $F(\bullet) = H_\bullet = \mathbb{C}^N$, $F(\cap) = \sum_i e_i \otimes e_i$.
In particular $\mathrm{Irr}(\mathbb{F}O_N) = \{v_k, k \in \mathbb{N}\}$, with $v_0 = 1$, $v_1 = \bullet = (v_{ij})_{ij}$ and

$$\forall k \geq 1 \quad v_k \otimes v_1 \simeq v_1 \otimes v_k \simeq v_{k-1} \oplus v_{k+1}.$$

From the category to the algebra: **coefficients**
$u \in \mathrm{Corep}(\Gamma)$, $X \in B(H_u) \to u(X) = (\mathrm{Tr} \otimes \mathrm{id})[(X \otimes 1)u] \in \mathscr{L}(\Gamma)$.
$\to$ computations in $\mathscr{L}(\Gamma)$ using TL_N:

if $x = v_2(X), y = v_2(Y)$ then $h(xy) =$ 

$/(N^2-1)$

Outline

The radial subalgebra

Definition

The radial subalgebra is $A = \chi_1'' \subset \mathscr{L}(\mathbb{F}O_N)$ where $\chi_1 = \sum_k \lambda(v_{kk})$.

One can also consider $\underline{\chi}_1 = \sum_k v_{kk}$, $\underline{A} = C^*(\underline{\chi}_1) \subset C^*(\mathbb{F}O_N)$.

Known facts:

- $\mathrm{Sp}(\underline{\chi}_1) = [-N, N]$ (the image of $\underline{\chi}_1$ in $C(O_N)$ is Tr_N).
- [HAP, Brannan] There is a cond. expectation $E : C^*(\mathbb{F}O_N) \to \underline{A}$. The positive forms $\mathrm{ev}_t \circ E$ are c_0 and converge to ε as $t \to N$.
- [Banica 1996] $\mathrm{Sp}(\chi_1) = [-2, 2]$ $\Rightarrow \mathbb{F}O_N$ not amenable, $\mathscr{L}(\mathbb{F}O_N)$ non injective.
- [Freslon-V. 2016] $A \subset \mathscr{L}(\mathbb{F}O_N)$ is maximal abelian and singular.
- [Krajczok-Wasilewski 2022] If Q is not unitary, $A \subset \mathscr{L}(\mathbb{F}O_Q)$ is not maximal abelian (and the inclusion is quasi-split).

A classical analogy

Why radial?

Analogy

$$\begin{array}{rcl} \mathbb{F}O_N & \longleftrightarrow & F_N = \langle a_i \rangle \\ v = (v_{ij}) & \longleftrightarrow & a = \mathrm{diag}(a_i, a_i^{-1}) \\ \underline{\chi}_1 = \mathrm{Tr}(v) & \longleftrightarrow & \underline{\chi}_1 = \sum_i (a_i + a_i^{-1}) \end{array}$$

In $\mathscr{L}(F_N)$, $A = \{\sum f(|g|)\lambda(g) \in \mathscr{L}(F_N)\}$ where $|\cdot|$ is the word length.

Known facts for $A \subset \mathscr{L}(F_N) =: M$.

- A is maximal abelian: $A' \cap M = A$,
- A is a singular MASA: $u \in \mathscr{U}(\mathscr{L}(F_N))$, $uAu^* \subset A \Rightarrow u \in A$,
- $\mathrm{Puk}(A) = \{\infty\}$: $\lambda(A)' \cap \rho(A)' \cap B(L^2(A)^\perp)$ is of type I_∞,
- A is maximal amenable: $A \subset B \subset \mathscr{L}(F_N)$, B amenable $\Rightarrow B = A$,
- A is absorbing amenable: B amenable, $A \cap B$ diffuse $\Rightarrow B \subset A$.

[Pytlik, Radulescu, Cameron-Fang-Ravichandran-White, Wen]

The main Result

Theorem

There exists $N_0 \geq 3$ such that, for all $N \geq N_0$, the radial subalgebra $A \subset \mathscr{L}(\mathbb{F}O_N)$ is absorbing amenable.

By work of Popa and Houdayer, it suffices to prove the following (strong) **Asymptotic Orthogonality Property** for $A \subset M$:

for every $y \in A^\perp \cap M$ and
every bdd sequence $(z_r)_r \subset A^\perp \cap M$ s.t. $\|[a, z_r]\|_2 \to_\omega 0 \ \forall a \in A$,
we have $(yz_r \mid z_r y) \to_\omega 0$.

For this we follow the strategy of **[Popa 1983]** which dealt with the case of the generator subalgebra $a_1'' \subset \mathscr{L}(F_N)$.

Open question: what about $v_{11}'' \subset \mathscr{L}(\mathbb{F}O_N)$?

Outline

Strategy of the proof

Denote $A \subset \mathscr{L}(\mathbb{F}O_N) \subset B(H)$ the quantum radial subalgebra,

$A_0 = a_1'' \subset \mathscr{L}(F_N) \subset B(H)$ the classical generator subalgebra.

Work in the A, A-bimodule $H^\circ = A^\perp \cap H$.

Step 1. We find a convenient basis W of the A, A-bimodule H°.

For each $x \in W$, we construct a basis $(x_{ij})_{ij}$ of AxA over $\mathbb{C}$.

Case of A_0: $W = \{$words not starting, nor ending, with a_1 or $a_1^{-1}\}$.

For $x \in W$ and $i, j \in \mathbb{Z}$, $x_{ij} = a_1^i x a_1^j$.

Made with Goodnotes

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Step 2. Denote $V_m = \mathrm{Span}\{x_{ij}, x \in W, |i|, |j| \geq m\}$. We prove:
For $y \in A^\perp \cap \mathbb{C}[\mathbb{F}O_N]$ and $\zeta_m \in V_m$, $\|\zeta_m\| = 1$, we have $(\zeta_m y \mid y\zeta_m) \to 0$.
Case of A_0: $V_m y \perp y V_m$ if y is supported on elements g with $|g| < m$.

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Case of A_0: $V_m y \perp y V_m$ if y is supported on elements g with $|g| < m$.

Step 3. Denote F_m the projection onto $\mathrm{Span}\{x_{ij} \mid x \in Z, |i| < m, j\}$.

Then for any $(u_i) \subset \mathscr{U}(A)$ converging weakly to 0, $\|F_m u_i F_m\| \to_i 0$.

Case of A_0: ${}_A H^\circ \simeq {}_A L^2(A) \otimes K$, $F_m \simeq f_m \otimes \mathrm{id}$ with $\mathrm{rank}(f_m) < \infty$.

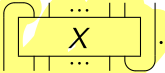
The bimodule basis

We compute using coefficients of $v^{\otimes k} = \bullet\bullet\cdots\bullet$:

$$X \in B(H_\bullet^{\otimes k}) \to v^{\otimes k}(X) \in \mathscr{L}(\mathbb{F}O_N) \subset H.$$

Consider the subspace $B_k \subset B(H_\bullet^{\otimes k})$ of elements X such that

B_k is stable under the rotation map $\rho : B(H_\bullet^{\otimes k}) \to B(H_\bullet^{\otimes k})$:

$\rho(X) =$ 

Let $\mathcal{W}_k \subset B_k$ be an orthonormal basis of eigenvectors of ρ and

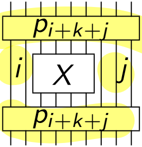
$$W = \{v^{\otimes k}(X) \mid k \in \mathbb{N}^*, X \in \mathcal{W}_k\}. \subset H^\circ$$

Proposition

We have $H^\circ = \overline{\mathrm{Span}}(AWA)$ and $AxA \perp AyA$ for $x \neq y \in W$.

The linear basis

From $X \in B(H_\bullet^{\otimes k})$ one defines $X_{ij} \in B(H_\bullet^{\otimes i+k+j})$ using the Jones-Wenzl projections:

$X_{ij} =$ 

For $x = v^{\otimes k}(X) \in W$, put $x_{ij} = v^{\otimes i+k+k}(X_{ij})$.

$\overline{A x A} \simeq \ell^2(N \times N)$

Theorem

If N is large enough, $\{x_{ij}\}$ is a Riesz basis of $\overline{AxA}$, uniformly over $x \in W$.

Case of the classical generator MASA (Popa): $\{x_{ij}\}$ always orthogonal.
Case of the classical radial MASA (Radulescu): $\{x_{ij}\}$ orthogonal if $k \neq 1$.
Case of the quantum radial MASA : $\{x_{ij}\}$ never orthogonal.
In fact "rapid off-diagonal decay" for the Gramm matrix and its inverse...

Open Problem: show that $N_0 = 3$ works!