Topological quantum groups

(a survey)

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Outline

1. Various frameworks
   - Compact quantum groups
   - Representation categories
   - Discrete quantum groups
   - Locally compact quantum groups
2. Many examples
   - $SU_q(2)$ and q-deformations
   - Orthogonal free quantum groups
   - More examples
3. An analytical property: C^*-simplicity
   - Properties of interest
   - The classical case
   - Quantum boundary actions
   - Quantum Gromov boundaries

Compact quantum groups

Definition

A *Woronowicz C^*-algebra* is a unital C^*-algebra A equipped with a unital $*$-homomorphism $\Delta : A \to A \otimes A$ such that

- $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
- $\overline{\mathrm{Span}}\, \Delta(A)(1 \otimes A) = A \otimes A = \overline{\mathrm{Span}}\, \Delta(A)(A \otimes 1)$.

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C^***-algebra:** Complete, normed $*$-algebra A with $\|a^*a\| = \|a\|^2$.
Always $A \simeq B \subset B(H)$ closed $*$-subalgebra, H Hilbert space.
Commutative case: $A \simeq C_0(X)$, X locally compact.
Positive elements: a^*a, $a \in A$.
Tensor product: $A \otimes B = \overline{A \odot B} \subset B(H \otimes K)$ if $A \subset B(H)$, $B \subset B(K)$.

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Theorem (Woronowicz)

Any Woronowicz C^*-algebra A has a unique Haar state, i.e. a unital positive linear functional $h : A \to \mathbb{C}$ such that $(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = 1h$.

Definition

A is called *reduced* if $h(a^*a) = 0 \Rightarrow a = 0$. A *compact quantum group* $\mathbb{G}$ is given by a reduced Woronowicz C^*-algebra $C^r(\mathbb{G})$.

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There is a reduction procedure $A \twoheadrightarrow A_r$ for Woronowicz C^*-algebras. So a compact quantum group $\mathbb{G}$ can in fact have many associated Woronowicz C^*-algebras $C(\mathbb{G}) \twoheadrightarrow C^r(\mathbb{G})$ — and this is interesting!

Compact quantum groups

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Theorem (Woronowicz)

Any Woronowicz C^*-algebra $C(\mathbb{G})$ contains a unique dense $*$-subalgebra $\mathscr{O}(\mathbb{G}) \subset C(\mathbb{G})$ which is a Hopf $*$-algebra for the restriction of Δ :

$$\Delta(\mathscr{O}(\mathbb{G})) \subset \mathscr{O}(\mathbb{G}) \odot \mathscr{O}(\mathbb{G}).$$

A Hopf $*$-algebra $\mathscr{A}$ is of the form $\mathscr{O}(\mathbb{G})$ **iff** it is generated by coefficients of *unitary* comodules ➔ can define CQG's at that level, too.
$\mathscr{O}(\mathbb{G})$ is the same for all Woronowicz C^*-algebras $C(\mathbb{G})$ associated with $\mathbb{G}$.

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Classical examples.

- G compact group ➔ $C^r(\mathbb{G}) = C(G)$, $\Delta(f) = ((r,s) \mapsto f(rs))$.
  Density condition : $rs = r's$ or $sr = sr' \Rightarrow r = r'$.
  $\mathscr{O}(G) = \{\omega \circ \pi \mid \pi : G \to L(H) \text{ fd rep}, \ \omega \in L(H)^*\}$.
- Γ discrete group ➔ $\mathscr{O}(\mathbb{G}) \simeq \mathbb{C}[\Gamma]$, $\Delta(g) = g \otimes g$.
  $C^r(\mathbb{G}) = C^*_r(\Gamma)$: completion of $\mathbb{C}[\Gamma]$ for $\|x\|_r = \|\lambda(x)\|_{B(\ell^2\Gamma)}$.
  Other completion $C^u(\mathbb{G})$: $\|x\|_u = \sup\{\|\pi(x)\| \mid \pi : \Gamma \to B(H)\}$.

Representation categories

Definition

Denote $\mathrm{Rep}(\mathbb{G})$ the category of f.-d. Hilbert $\mathscr{O}(\mathbb{G})$-comodules, equivalently, of *corepresentations* $v \in B(H_v) \otimes C(\mathbb{G}) : (\mathrm{id} \otimes \Delta)(v) = v_{12}v_{13}$.

It is a tensor C^*-category, which is *rigid* (existence of duals), and is equipped with the canonical forgetful functor $\mathrm{Rep}(\mathbb{G}) \to \mathrm{Hilb}$, $v \mapsto H_v$.

If $C(\mathbb{G}) = C(G)$, we have $\mathrm{Rep}(\mathbb{G}) = \mathrm{Rep}(G)$.
If $C(\mathbb{G}) = C^*(\Gamma)$, we have $\mathrm{Rep}(\mathbb{G}) =$ f.d. Γ-graded Hilbert spaces.

Tannaka-Krein duality: from every rigid tensor C^*-category $\mathscr{C}$ and unitary tensor functor $\mathscr{C} \to \mathrm{Hilb}$ one can reconstruct a compact quantum group $\mathbb{G}$ such that $\mathscr{C} \simeq \mathrm{Rep}(\mathbb{G})$ (and the functors agree). [Woronowicz]

Discrete quantum groups

Discrete groups can have interesting actions on *topological spaces*, and their reduced C^*-algebras $C^*_r(\Gamma)$ have interesting *analytical properties*...

Inspired by the second class of examples, we also denote $C(\mathbb{G}) = C^*_r(\mathbb{\Gamma})$, $\mathscr{O}(\mathbb{G}) = \mathbb{C}[\mathbb{\Gamma}]$. "$\mathbb{\Gamma}$ is the discrete dual of $\mathbb{G}$."

Definition

We denote $c_c(\mathbb{\Gamma}) = \{h(a\,\cdot\,) \mid a \in \mathbb{C}[\mathbb{\Gamma}]\} \subset \mathbb{C}[\mathbb{\Gamma}]^*$,

$$c_0(\mathbb{\Gamma}) = \{(\mathrm{id} \otimes \omega)(W_{\mathbb{G}}) \mid \omega \in B(\ell^2\Gamma)_*\}^- \subset B(\ell^2(\mathbb{\Gamma})).$$

These are *non-unital* (C^*-) algebras equipped with coproducts

$$\Delta : c_c(\mathbb{\Gamma}) \to \mathscr{M}(c_c(\mathbb{\Gamma}) \odot c_c(\mathbb{\Gamma})), \Delta : c_0(\mathbb{\Gamma}) \to M(c_0(\mathbb{\Gamma}) \otimes c_0(\mathbb{\Gamma})).$$

As an algebra $c_c(\mathbb{\Gamma}) \simeq \bigoplus_{\alpha \in I} L(H_\alpha)$ over $I = \mathrm{Irr}\,\mathrm{Rep}\,\mathbb{G}$.
Then the *multiplicative unitary* $W_{\mathbb{G}}$ identifies with $\bigoplus_{\alpha \in I} v_\alpha$.

Locally compact quantum groups

Definition (Kustermans, Vaes)

A reduced C^*-algebraic quantum group is given by a C^*-algebra $A = C^r(\mathbb{G})$ and a non-deg. $*$-hom $\Delta : A \to M(A \otimes A)$ such that

- $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
- $\overline{\mathrm{Span}}(\mathrm{id} \otimes A^*)\Delta(A) = A = \overline{\mathrm{Span}}(A^* \otimes \mathrm{id})\Delta(A)$,
- there exist faithful KMS weights φ, ψ on A s.t. $\forall \omega \in A^*_+, a \in \mathscr{M}^+_{\varphi|\psi}$ $\varphi((\mathrm{id} \otimes \omega)\Delta(a)) = \omega(1)\varphi(a)$ and $\psi((\omega \otimes \mathrm{id})\Delta(a)) = \omega(1)\psi(a)$.

- Commutative case: locally compact groups
- Pontrjagin duality $\mathbb{G} \to \hat{\mathbb{G}} \to \mathbb{G}$
- Includes compact and discrete quantum groups
- But also double crossed products $\mathbb{G} \bowtie \mathbb{H}$, including the Drinfeld double $D(\mathbb{G}) = \mathbb{G} \bowtie \hat{\mathbb{G}}$ of a compact quantum group

Outline

q-deformations of compact Lie groups

Let G be a connected, simply connected compact Lie group.
→ $\mathfrak{G} = \mathrm{Lie}(G)_{\mathbb{C}}$, $\mathscr{U}(\mathfrak{G})$ its enveloping algebra,
→ Drinfel'd–Jimbo's $\mathscr{U}_q(\mathfrak{G})$: deformation of Serre's presentation of $\mathscr{U}(\mathfrak{G})$.

The compact real form/$*$-structure is deformed as well.
$\mathscr{U}_q(\mathfrak{G})$ is a Hopf $*$-algebra, but not of the kind described earlier.

Associated compact and discrete quantum groups, for $q \in]0,1[$:

$\mathscr{U}_q(\mathfrak{G})$ – – "dual" – → $\mathscr{O}(G_q)$, $C^r(G_q)$

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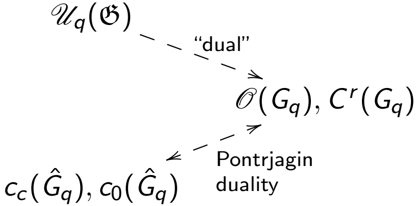

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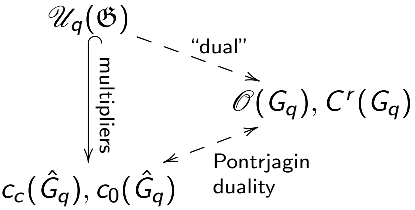

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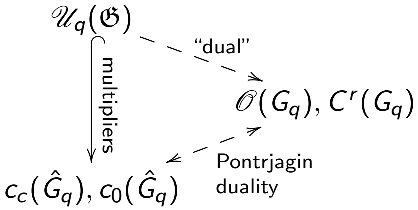

- also "complex semi-simple quantum groups" $D(G_q) = G_q \bowtie \hat{G}_q$.

The case of $SU_q(2)$

- **Drinfel'd–Jimbo:** $\mathscr{U}_q(\mathfrak{sl}(2))$ is the universal algebra generated by elements E, F, K, K^{-1} and the relations $KK^{-1} = K^{-1}K = 1$ and

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

Coproduct: $\Delta(E) = E \otimes K + 1 \otimes E$, $\Delta(F) = F \otimes 1 + K^{-1} \otimes F$.
$\mathscr{U}_q(\mathfrak{su}(2)) = \mathscr{U}_q(\mathfrak{sl}(2))$ with $E^* = FK$, $F^* = K^{-1}E$, $K^* = K$.

- **Woronowicz:** $C(SU_q(2))$ = universal C^*-algebra generated by α, γ and

$$\begin{gathered} \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma \\ \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1. \end{gathered} \tag{1}$$

Coproduct Δ: such that $u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha \end{pmatrix}$ is a corepresentation.

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In fact (1) holds **iff** u is unitary and $u \otimes u$ fixes

$$\xi_q = e_1 \otimes e_2 - q e_2 \otimes e_1.$$

- In other words, $\mathrm{Rep}(SU_q(2))$ is the universal tensor C^*-category
  - generated by one object $\bullet$ and one morphism $t = \cap : 1 \to \bullet \otimes \bullet$
  - subject to $t^* t = \bigcirc = q + q^{-1}$ and the duality equations.

This is the **Temperley-Lieb** category TL_q^-.
The fiber functor is given by $F_q(\bullet) = \mathbb{C}^2$, $F_q(\cap) = \xi_q / \sqrt{|q|}$.

Orthogonal free quantum groups

The Temperley-Lieb categories $TL_q^{\pm}$ have higher-dimensional fiber functors.

Fix $N \in \mathbb{N}$, $N \geq 2$ and $Q \in GL_N(\mathbb{C})$ such that $Q\bar{Q} = \pm I_n$.
Write $\mathrm{Tr}(Q^*Q) = q + q^{-1}$ with $q \in]0, 1]$.
Then there is a unique fiber functor $F_Q : TL_q^{\pm} \to \mathrm{Hilb}$ given by

$$F_Q(\bullet) = \mathbb{C}^N, \quad F_Q(\cap) = \sum_i e_i \otimes Qe_i.$$

Moreover all fiber functors on $TL_q^{\pm}$ are of this form, up to isomorphism.

Definition

The compact quantum group associated with F_Q is denoted O_Q^+, and its discrete dual is denoted $\mathbb{F}O_Q$. For $Q = I_N$ they are denoted O_N^+, $\mathbb{F}O_N$.

Orthogonal free quantum groups

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Proposition (Wang, Van Daele, Banica)

The algebras $A = C^u(O_Q^+)$, $\mathscr{O}(O_Q^+)$ are presented by the entries of $u = (u_{ij}) \in M_N(A)$ with the relations:

$$uu^* = 1 = u^*u \quad \text{and} \quad Q\bar{u}Q^{-1} = u, \quad \text{where } \bar{u} = (u_{ij}^*).$$

These are exactly the CQG having the same fusion ring as $SU(2)$.
For $N = 2$: $\{O_Q^+ \mid N = 2\} = \{SU_{\mp q}(2) \mid 0 < q \leq 1\}$.

We call O_Q^+ the universal orthogonal quantum groups,
$\mathbb{F}O_Q$ the orthogonal free quantum groups.

We have $C^u(O_N^+)/\langle [x, y] \rangle \simeq C(O_N)$,
$C^*(\mathbb{F}O_N)/\langle u_{ij}, i \neq j \rangle \simeq C_u^*(FO_N)$, $FO_N = (\mathbb{Z}/2)^{*N}$.

Open question: does $\mathrm{Rep}(SU_q(3))$ have higher-dim. fiber functors?

More examples

Universal unitary quantum groups

For $N \geq 2$, $Q \in GL_N(\mathbb{C})$, define a C^*-algebra by generators and relations:

$$C^u(U_Q^+) = C_u^*(\mathbb{F}U_Q) = \langle u_{ij} \mid u = (u_{ij}) \text{ and } Q\bar{u}Q^{-1} \text{ unitary}\rangle.$$

It is a Woronowicz C^*-algebra for the coproduct s.t. $(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}$. The fusion ring is non-commutative ($v \otimes w \not\simeq w \otimes v$), isomorphic to the ring of the free monoid on two letters. [Banica]

Partition/easy quantum groups

A tensor C^*-category of partitions $\mathscr{P}$ has objects in $\mathbb{N}$ and $\mathrm{Hom}(m, n)$ spanned by partitions of $m + n$ points, with the "usual operations".
If $\bigcirc = N \in \mathbb{N}^*$, there is a functor $T : \mathscr{P} \to \mathrm{Hilb}$ with $T(\bullet) = \mathbb{C}^N$ (not always faithful) and an associated CQG $\mathbb{G}_N(\mathscr{P})$. For instance :

- $\mathscr{P} = \{\text{non crossing pair partitions}\} \to \mathbb{G}_N(\mathscr{P}) = O_N^+$.
- $\mathscr{P} = \{\text{all partitions}\} \to \mathbb{G}_N(\mathscr{P}) = S_N$,
- $\mathscr{P} = \{\text{non crossing partitions}\} \to \mathbb{G}_N(\mathscr{P}) = S_N^+$.

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What kind of properties?

Fix $A = C^r(\mathbb{G}) = C_r^*(\mathbb{\Gamma})$.

- Is A the only Woronowicz C^*-algebra associated with $\mathbb{G}$?
  Equivalent to amenability of $\mathbb{\Gamma}$.
  $\mathbb{G} = G_q$: yes. O_Q^+, U_q^+: no if $N \geq 3$. S_N^+: no if $N \geq 5$.
- Weaker approximation properties: Haagerup approximation property, weak amenability, exactness... True for F_N, $\mathbb{F}O_Q$, $\mathbb{F}U_Q$...
  Non-approximation properties: Property (T). True for $D(SU_q(3))$.
- Classification → K-theory → Baum-Connes.
  $K_0(C^r(S_N^+)) \simeq \mathbb{Z}^{N^2-2N+2}$; $K_0(C^r(O_N^+)) \simeq \mathbb{Z} \simeq K_1(C^r(O_N^+))$.
  Open question: $C^r(O_N^+) \simeq C^r(O_M^+)$ for $N \neq M$?
- Structure of $C_r^*(\mathbb{\Gamma})$: simplicity? traces? maximal abelian subalgebras? (Also in the von Neumann context.)

Slogan: $C(O_N^+) = C_r^*(\mathbb{F}O_N)$, $C_r^*(\mathbb{F}U_N)$ are very similar to $C_r^*(F_N)$!

Classical boundary actions

Simplicity of A: no proper *closed* bilateral ideals $I \subset A$.
Note: $\mathbb{C}[F_N]$ is not (alg.) simple, but $C_r^*(F_N)$ is simple [Powers 1975].

Trace on A: positive functional $\varphi \in A_+^*$ such that $\varphi(ab) = \varphi(ba)$.
Note: $C_r^*(\Gamma)$ has a canonical trace $h(\sum x_g g) = x_e$.

Theorem (Breuillard, Kalantar, Kennedy, Ozawa 2017)

- $C_r^*(\Gamma)$ *is simple* **iff** Γ *admits an essentially free boundary action.*
- $C_r^*(\Gamma)$ *has a unique trace* **iff** Γ *admits a faithful boundary action.*

A boundary action is an action $\Gamma \curvearrowright X$ on a compact space X which is:

- minimal: $\forall x, y \in X \quad \exists g_n \in \Gamma \quad \lim g_n \cdot x = y$,
- strongly proximal: $\forall \mu, \nu \in \mathrm{Prob}(X) \quad \exists g_n \in \Gamma \quad \lim g_n \cdot \mu = \lim g_n \cdot \nu$.

Equivalently: $\forall \nu \in \mathrm{Prob}(X) \quad \overline{\Gamma \cdot \nu} \supset X$.
Example: Gromov boundary of F_N.

Quantum boundary actions

Action of a DQG $\mathbb{\Gamma}$ on a C^*-algebra A: given by $\alpha : A \to M(c_0(\mathbb{\Gamma}) \otimes A)$.
A unital map $T : A \to B$ between C^*-algebras is

- completely positive (UCP) if $(T \otimes \mathrm{id})(M_n(A)_+) \subset M_n(B)_+$ for all n,
- completely isometric (UCI) if $T \otimes \mathrm{id}$ is isometric on $M_n(A)$ for all n.

Definition (Kasprzak, Kalantar, Skalski, V.)

A unital $\mathbb{\Gamma}$-C^*-algebra A is a $\mathbb{\Gamma}$-boundary if all UCP $\mathbb{\Gamma}$-equivariant maps $T : A \to B$ are automatically UCI.

In other words, the extension $\mathbb{C} \hookrightarrow A$ is an "essential extension" in the category of unital $\mathbb{\Gamma}$-C^*-algebras with UCP $\mathbb{\Gamma}$-maps as morphisms and UCI $\mathbb{\Gamma}$-maps as embeddings.

[Habbestad, Hataishi, Neshveyev 2022] constructs for any rigid tensor C^*-category $\mathscr{C}$ the universal $\mathscr{C}$-boundary (which is a $\mathscr{C}$-tensor category) which corresponds to the universal $D(\mathbb{\Gamma})$-boundary of the Drinfeld double.

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The action $\mathbb{\Gamma} \curvearrowright^{\alpha} A$ is *faithful* if $(c_0(\mathbb{\Gamma})^* \otimes \mathrm{id})\alpha(A)$ generates $M(c_0(\mathbb{\Gamma}))$.

Theorem (KKSV 2020)

Assume that $\mathbb{\Gamma}$ acts faithfully on some $\mathbb{\Gamma}$-boundary A. Then:

- *if $\mathbb{\Gamma}$ is unimodular, $C_r^*(\mathbb{\Gamma})$ has a unique trace ;*
- *else $C_r^*(\mathbb{\Gamma})$ has no KMS state wrt the scaling group.*

Theorem (Anderson-Sackaney, Khosravi 2024)

$\mathbb{\Gamma}$ unimodular and $C_r^*(\mathbb{\Gamma})$ unique trace $\Rightarrow$ there exists a faithful $\mathbb{\Gamma}$-boundary.

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Note: for $\Gamma \curvearrowright X$ compact minimal, essentially free $\Leftrightarrow$ strongly faithful:

$$\forall g_1, \dots, g_n \in \Gamma \setminus \{1\} \quad \exists x \in X \quad \forall i \quad g_i \cdot x \neq x.$$

Definition (Anderson-Sackaney, V.)

$\mathbb{\Gamma} \curvearrowright A$ is *strongly C^*-faithful* if for every projection $p \in Z(c_c(\mathbb{\Gamma}))$ with $\epsilon(p) = 0$ and every $\eta > 0$ there exists $k \in \mathbb{N}^*$ and $b \in (A \otimes M_k(\mathbb{C}))_+$ such that $\|b\| = 1$ and $\|(p \otimes b)(\alpha \otimes \mathrm{id})(b)\| \leq \eta$.

Theorem (ASV 2024)

If $\mathbb{\Gamma}$ admits a strongly C^*-faithful boundary action, then $C^*_r(\mathbb{\Gamma})$ is simple.

Quantum Gromov boundaries

Recall that O_Q^+ has the same fusion rules as $SU(2)$.
In particular $c_c(\mathbb{F}O_Q) \simeq \bigoplus_{n\in\mathbb{N}} L(H_n)$ with $H_{n+1} \subset H_n \otimes H_1$.
By analogy with the free group case $c_c(F_N) \simeq \bigoplus_{n\in\mathbb{N}} C(S_n)$ one puts

$$C(\partial\mathbb{F}O_Q) = \varinjlim L(H_n).$$

It has a natural structure of a unital $\mathbb{F}O_Q$-C^*-algebra [Vaes-V. 2007].
There is a similar construction for $\mathbb{F}U_Q$ [Vaes-Vander Vennet].

Theorem (ASV 2024)

For $N \geq 3$, $C(\partial\mathbb{F}U_Q)$ is an $\mathbb{F}U_Q$-boundary and it is strongly C^*-faithful.

[Habbestad, Hataishi, Neshveyev 2022] shows the weaker result that $C(\partial\mathbb{F}U_Q)$ is a $D(\mathbb{F}U_Q)$-boundary.
Simplicity of $C_r^*(\mathbb{F}U_Q)$ is already known [Banica 1997].

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Theorem (KKSV 2020)

Assume $N \geq 3$. Then $C(\partial\mathbb{F}O_Q)$ is an $\mathbb{F}O_Q$-boundary and it is faithful.

$N = 2$: the dual of $SU_q(2)$ is amenable $\Rightarrow$ the only $\mathbb{F}O_Q$-boundary is $\mathbb{C}$.
In the unimodular case, uniqueness of trace was already known.
Simplicity is known only with restrictions on Q [Vaes-V.].
Open question: is $C(\partial\mathbb{F}O_Q)$ strongly C^*-faithful?