

C^* -SIMPLICITY AND BOUNDARY ACTIONS OF DISCRETE QUANTUM GROUPS

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ABSTRACT. We introduce and investigate several quantum group dynamical notions for the purpose of studying C^* -simplicity of discrete quantum groups via the theory of boundary actions. In particular we define a quantum analogue of Powers' Averaging Property (PAP) and a quantum analogue of strongly faithful actions. We show that our quantum PAP implies C^* -simplicity and the uniqueness of σ -KMS states, and that the existence of a strongly C^* -faithful quantum boundary action also implies C^* -simplicity and, in the unimodular case, the quantum PAP. We illustrate these results in the case of the unitary free quantum groups $\mathbb{F}U_F$ by showing that they satisfy the quantum PAP and that they act strongly C^* -faithfully on their quantum Gromov boundary. Moreover we prove that this particular action of $\mathbb{F}U_F$ is a quantum boundary action.

1. INTRODUCTION

A discrete group G is said to be C^* -simple if its reduced C^* -algebra $C_r^*(G)$ is simple, meaning that it has no non-trivial proper closed two-sided ideals. The theory of C^* -simplicity began with the work of Powers in [Pow75] in which he established that the free group on two generators satisfies a certain strong group-theoretic averaging condition now known as Powers' averaging property (PAP), and showed that this condition implies C^* -simplicity. This strong averaging property has been a prominent part of the theory since. Remarkably, it was later shown independently by Haagerup and Kennedy respectively in [Haa17, Ken20] that Powers' averaging property is equivalent to C^* -simplicity.

On the other hand, Kalantar and Kennedy discovered in [KK17] a surprising connection between C^* -simplicity and the theory of boundary actions in topological group dynamics, which had been initiated by Furstenberg in the 1950s. Recall that an action of a group G on a compact space is called a boundary action if it is minimal and strongly proximal. Kalantar and Kennedy established more precisely that a group G is C^* -simple if and only if it acts freely on its Furstenberg boundary $\partial_F G$, if and only if it admits some topologically free boundary action. A key observation for these results is the fact, [Ham78, Remark 4], that the algebra of continuous functions on the Furstenberg boundary is equivariantly isomorphic to Hamana's injective envelope of \mathbb{C} in the category of G -equivariant operator systems.

A bit later it was proved in [BKKO17] that a group acts faithfully on its Furstenberg boundary if and only if it has the unique trace property, i.e., its reduced C^* -algebra has a unique trace. These results gave a solution to one direction of a long-standing open conjecture, namely that C^* -simplicity implies the unique trace property — a counterexample to the converse was found in [LB17].

C^* -simplicity of a discrete *quantum* group \mathbb{G} is defined exactly as in the classical case, using the reduced C^* -algebra $C_r^*(\mathbb{G})$. The study of C^* -simplicity in this framework begins with Banica's work in [Ban97] where he proved that Wang's free unitary discrete quantum groups $\mathbb{F}U_F$ [Wan95, VDW96] are C^* -simple by adapting Powers' methods to the quantum setting. The free orthogonal quantum groups $\mathbb{F}O_F$ and the quantum groups of quantum automorphisms of finite-dimensional C^* -algebras (equipped with their canonical trace) were later proved to be C^* -simple as well under some restrictions on the parameter matrix F , resp. the dimension of the considered C^* -algebra, see [VV07, Bra13].

The theory of boundary actions has also been extended to the setting of discrete quantum groups in [KKS22], using the connection with Hamana's work mentioned above: the quantum Furstenberg boundary $C(\partial_F \mathbb{G})$ is e.g. defined to be Hamana's \mathbb{G} -injective envelope of \mathbb{C} . Then, in [KKS22, ASK24] it was shown that a unimodular discrete quantum group has the unique trace property if and only if it acts faithfully on its Furstenberg boundary. Furthermore, in [ASK24] it was shown that for

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a discrete quantum group \mathbb{G} that is either unimodular or exact, C^* -simplicity implies faithfulness of the action on the Furstenberg boundary. In particular, in the unimodular case, C^* -simplicity implies the unique trace property.

Despite these results, it seems that the theory of C^* -simplicity for quantum groups remains underdeveloped. For instance, an appropriate analogue of the notion of (topological) freeness for actions of discrete quantum groups on noncommutative C^* -algebras for the purposes of studying C^* -simplicity has yet to appear in the literature. A version of Powers' averaging property has not been developed for quantum groups either. In this paper we aim to address this gap in the theory as follows:

- we propose quantum analogues of the PAP and of topologically free boundary actions,
- we prove that these properties imply C^* -simplicity,
- we prove that they are satisfied in our test case, namely the free unitary quantum groups $\mathbb{F}U_F$.

It is an important feature of our approach that we study these properties specially in relation to the “rigidity” of \mathbb{G} -equivariant ucp maps $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$. A deep result of Kennedy in [Ken20], used in its proof of the equivalence between C^* -simplicity and the PAP, is that C^* -simplicity is characterized by uniqueness of a G -equivariant ucp map $C_r^*(G) \rightarrow C(\partial_F G)$. This result has had various generalizations, including a noncommutative analogue in [Zar19] where it was shown that a certain freeness property for an action of a group G on a unital C^* -algebra A is equivalent to the uniqueness of conditional expectations from the reduced crossed product $A \rtimes_r G$ onto A . Concerning a more general context for such problems, there has been interest on the “rigidity” (e.g. uniqueness) of (pseudo)-conditional expectations $A \rightarrow B$ of a C^* -inclusion $A \subseteq B$, cf. [Pit17, PSZ23, PZ15, Zar19].

Our first observation is in Proposition 2.1, which states that \mathbb{G} is C^* -simple if and only if every \mathbb{G} -equivariant ucp map $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ is faithful. We consider also two additional properties: the uniqueness of \mathbb{G} -equivariant ucp maps $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$, and when all \mathbb{G} -equivariant ucp maps $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ factor the canonical Haar state $h : C_r^*(\mathbb{G}) \rightarrow \mathbb{C}$ in the sense of Definition 3.21. Note that we always have the canonical ucp map $a \mapsto h(a)1$ from $C_r^*(\mathbb{G})$ to $C(\partial_F \mathbb{G})$, which is faithful and factors h , but it is \mathbb{G} -equivariant if and only if \mathbb{G} is unimodular — this is one of the difficulties that are specific to the quantum setting.

We consider two versions of Powers' averaging property that generalize the PAP for groups, namely the PAP and PAP_h (see Definition 3.3), where we note that $\text{PAP}_h \implies \text{PAP}$. Again in the unimodular case we have the simplification $\text{PAP} = \text{PAP}_h$. We observe that the PAP implies that \mathbb{G} is C^* -simple and the PAP_h implies additionally that $C_r^*(\mathbb{G})$ has a unique σ -KMS state (Corollary 3.9). Also, we observe that what Banica really showed in [Ban97] is the PAP_h for the free unitary quantum groups $\mathbb{F}U_F$ (Proposition 3.8).

One of Kennedy's main results of [Ken20] states that a group has Powers' averaging property if and only if the only G -boundary contained in $S(C_r^*(G))$ is trivial. In the quantum setting it doesn't make sense to consider \mathbb{G} -boundaries inside $S(C_r^*(\mathbb{G}))$ as the \mathbb{G} -boundaries may be noncommutative. We, however, are able to obtain the following quantum analogue of Kennedy's result by replacing \mathbb{G} -boundaries with \mathbb{G}_h -boundary envelopes (see Definitions 3.11 and 3.13).

Theorem 3.17. *A discrete quantum group \mathbb{G} has the PAP_h if and only if the only \mathbb{G}_h -boundary envelope in $S(C_r^*(\mathbb{G}))$ is trivial.*

Kennedy observed in [Ken20] that the G -boundaries inside $S(C_r^*(G))$ are in bijection with G -equivariant ucp maps $C_r^*(G) \rightarrow C(\partial_F G)$. It is with this observation that he was able to characterize the Powers' averaging property with the uniqueness of \mathbb{G} -equivariant ucp maps $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$. We make similar observations in our work and, as an application of Theorem 3.17, we obtain the following.

Corollary 3.24.

- (1) *A discrete quantum group \mathbb{G} has the PAP iff every \mathbb{G} -equivariant ucp map $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ factors the Haar state.*
- (2) *If \mathbb{G} is unimodular, then it has the PAP iff there is a unique \mathbb{G} -equivariant ucp map $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$.*

Afterwards, we investigate a notion of *freeness* for quantum group actions which generalizes the notion of a group acting freely on a C^* -algebra in the sense of [Zar19]. In fact, we prove that an action $\mathbb{G} \curvearrowright A$ is free if and only if there exists a unique conditional expectation $A \rtimes_r \mathbb{G} \rightarrow A$ (Theorem 4.3),

generalizing one of the main results of [Zar19]. We note, however, that this result implies that $\mathbb{G} \curvearrowright C(\partial_F \mathbb{G})$ is never free when \mathbb{G} is non-unimodular (see Corollary 4.4). This observation suggests that this might not be the right notion to consider in the theory of C^* -simplicity. On the other hand, we apply the techniques used in the proof of Theorem 4.3 to prove that faithfulness of an action $\mathbb{G} \curvearrowright A$ is equivalent to uniqueness of conditional expectations $A \rtimes_r \mathbb{G} \rightarrow A$ that restrict to states on $C_r^*(\mathbb{G})$ (Theorem 4.8).

In the search of a notion in quantum topological dynamics that would characterize, or at least imply, C^* -simplicity, while allowing for boundary actions of non-unimodular quantum groups, we introduce *strong C^* -faithfulness* for actions of discrete quantum groups on C^* -algebras (Definition 4.10). In the classical case of discrete groups acting on locally compact spaces, it recovers strong faithfulness as considered e.g. in [dlH85, Lemma 4] and [FLMMS22, Section 2.1]. Note that in this classical setting, a minimal group action on a compact space is strongly faithful if and only if it is topologically free, and recall that boundary actions are in particular compact and minimal, so that the main result of [KK17] can be rephrased by saying that a discrete group G is C^* -simple if and only if it admits a strongly faithful boundary action. In the quantum setting we are able to prove one direction of that result:

Theorem 4.20 and Corollary 4.21. *If \mathbb{G} admits a \mathbb{G} -boundary with strongly C^* -faithful action, then \mathbb{G} has the PAP. In particular, \mathbb{G} is C^* -simple.*

We finally come back to our test example $\mathbb{F}U_F$ with the aim of showing that it satisfies the hypothesis of the above theorem. We show that the action on its quantum Gromov boundary $C(\partial_G \mathbb{F}U_F)$, introduced in [VVV10], is strongly C^* -faithful by using a combinatorial trick uncovered by Banica in [Ban97]. On the other hand, the fact that $C(\partial_G \mathbb{F}U_F)$ is a \mathbb{G} -boundary has been established in [HHN22] but only with respect to the natural action of the Drinfeld double $\mathbb{G} = D(\mathbb{F}U_F)$. We upgrade this result by showing that $C(\partial_G \mathbb{F}U_F)$ is in fact already an $\mathbb{F}U_F$ -boundary — this is much more intricate and relies on the unique stationarity method used in [KKS22] in the case of $\mathbb{F}O_F$. We prove more precisely the following.

Theorem 5.5. *Let $F \in GL_N(\mathbb{C})$ with $N \geq 3$. Then $C(\partial_G \mathbb{F}U_F)$ has a unique stationary state with respect to the “nearest neighbor” quantum random walk on $\mathbb{F}U_F$.*

As an intermediate step for the non-unimodular case we establish a non-atomicity result for the restriction of stationary states to the *classical* Gromov boundary of $\mathbb{F}U_F$. Taking into account the results of [KKS22] and [VVV10], the above theorem implies that $C(\partial_G \mathbb{F}U_F)$ is indeed an $\mathbb{F}U_F$ -boundary, and the strong C^* -faithfulness property together with our general results yield a new “dynamical proof” of C^* -simplicity of $\mathbb{F}U_F$ when $N \geq 3$.

We complete the introduction by presenting the layout of this paper. In Section 2 we discuss the preliminaries on discrete quantum groups, boundary actions, free unitary discrete quantum groups, and C^* -simplicity. In Section 3 we develop notions of Powers’ averaging property for discrete quantum groups. Here, we prove a characterization in terms of “boundary envelopes” in the state space of $C_r^*(\mathbb{G})$ and, in the unimodular case, a characterization in terms of \mathbb{G} -equivariant ucp maps $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ as discussed above. In Section 4 we develop a notion of a free action and a notion of a strong C^* -faithful action, and investigate the connections with C^* -simplicity and the PAP. Finally, in Section 5 we prove that the Gromov boundary of $\mathbb{F}U_F$ is a $\mathbb{F}U_F$ -boundary when $N \geq 3$.

2. PRELIMINARIES

2.1. Discrete quantum groups and their boundaries. Discrete quantum groups were introduced as duals of compact quantum groups in [PW90]; an operator-theoretic characterization is given in [BS93] at the level of the associated multiplicative unitary, and algebraic characterizations are given in [ER94] and [VD96] at the level of the associated algebras. We shall mostly follow the notation and conventions established in [KV00] for general locally compact quantum groups.

So let \mathbb{G} be locally compact quantum group, given by the reduced Hopf- C^* -algebra $c_0(\mathbb{G})$. We choose a GNS space (H, Λ) for the left Haar weight h_L of \mathbb{G} and construct the left multiplicative unitary $W \in B(H \otimes H)$ of \mathbb{G} , given by the formula $W^*(\Lambda(f) \otimes \Lambda(g)) = (\Lambda \otimes \Lambda)(\Delta(g)(f \otimes 1))$ for $f, g \in c_c(\mathbb{G})$. One can recover $c_0(\mathbb{G})$ as the norm closure of $\{(\text{id} \otimes \omega)(W) \mid \omega \in B(H)_*\}$, equipped with the coproduct $\Delta(f) = W^*(1 \otimes f)W$. As the notation suggests, we assume \mathbb{G} and W to be of

discrete type, i.e. there exists a unit vector $\eta \in H$ such that $W(\eta \otimes \zeta) = \eta \otimes \zeta$ for all $\zeta \in H$. The C^* -algebra $c_0(\mathbb{G})$ then contains a distinguished dense multiplier Hopf $*$ -algebra [VD94] that we denote $c_c(\mathbb{G})$.

In this article, we are interested in the structure of the reduced group C^* -algebra $C_r^*(\mathbb{G})$ of \mathbb{G} , defined as the norm closure of $\{(\omega \otimes \text{id})(W) \mid \omega \in B(H)_*\}$ in $B(H)$. We have then $W \in M(c_0(\mathbb{G}) \otimes C_r^*(\mathbb{G}))$ and we endow $C_r^*(\mathbb{G})$ with the adjoint action of \mathbb{G} given by the coaction $\text{ad} : C_r^*(\mathbb{G}) \rightarrow M(c_0(\mathbb{G}) \otimes C_r^*(\mathbb{G})), x \mapsto W^*(1 \otimes x)W$. Sometimes it will be more convenient to use the adjoint action of the opposite discrete quantum group \mathbb{G}^{op} given by the $*$ -homomorphism $\text{ad}^{\text{op}} : C_r^*(\mathbb{G}) \rightarrow M(c_0(\mathbb{G}) \otimes C_r^*(\mathbb{G})), x \mapsto W(1 \otimes x)W^*$, which is indeed a coaction for the opposite coproduct $\Delta^{\text{op}} = \sigma \circ \Delta$ on $c_0(\mathbb{G}^{\text{op}}) = c_0(\mathbb{G})$. Furthermore, the adjoint actions of \mathbb{G} and \mathbb{G}^{op} are related by the equation $\text{ad}^{\text{op}}(x) = (R \otimes \hat{R}) \text{ad}(\hat{R}(x))$, where R and \hat{R} are the unitary antipodes of $c_0(\mathbb{G})$ and $C_r^*(\mathbb{G})$ respectively, satisfying the identity $(R \otimes \hat{R})(W) = W$.

We denote $\text{Corep}(\mathbb{G})$ the category of non-degenerate finite-dimensional $*$ -representations of $c_0(\mathbb{G})$, and $\text{Irr}(\mathbb{G}) = I$ the set of irreducible objects up to equivalence. Note that such representations $\pi : c_0(\mathbb{G}) \rightarrow B(H_\pi)$ correspond to unitary elements $w = (\pi \otimes \text{id})(W) \in B(H_w) \otimes C_r^*(\mathbb{G})$, where $H_w = H_\pi$, and we will in fact rather use this second picture. The category $\text{Corep}(\mathbb{G})$ has a natural monoidal structure given by $\pi \otimes \rho := (\pi \otimes \rho) \circ \Delta$ and $v \otimes w := v_{13}w_{23}$. Since \mathbb{G} is discrete, we have an isomorphism $c_0(\mathbb{G}) \simeq \bigoplus_{w \in I}^{\text{co}} B(H_w)$ such that $W = \bigoplus_{w \in I} w$, and we denote $\ell^\infty(\mathbb{G}) = M(c_0(\mathbb{G}))$ the corresponding ℓ^∞ -direct sum. We also denote $p_w \in c_0(\mathbb{G})$ the minimal central projection corresponding to $\text{id} \in B(H_w)$ in this isomorphism.

The monoidal category $\text{Corep}(\mathbb{G})$ is rigid, in particular we can find for each $v \in \text{Corep}(\mathbb{G})$ another corepresentation \bar{v} , unique up to isomorphism, and morphisms $t_v : \mathbb{C} \rightarrow H_v \otimes H_{\bar{v}}$, $s_v : \mathbb{C} \rightarrow H_{\bar{v}} \rightarrow H_v$ satisfying the conjugate equations and normalized so that $t_v^* t_v = s_v^* s_v =: \dim_q(v)$. If $v = w$ is irreducible, these morphisms are unique up to a phase and we consider the associated left and right quantum traces, $\text{qTr}_w(a) = t_w^*(a \otimes 1)t_w$ and $\text{qTr}'_w(a) = s_w^*(1 \otimes a)s_w$, for $a \in B(H_w)$, as well as the corresponding states $\text{qtr}_w = (\dim_q w)^{-1} \text{qTr}_w$, $\text{qtr}'_w = (\dim_q w)^{-1} \text{qTr}'_w$. We can also write $\text{qTr}_w(a) = \text{Tr}(Q_w a)$, $\text{qTr}'_w(a) = \text{Tr}(Q_w^{-1} a)$ for a unique positive matrix $Q_w \in B(H_w)$. Compare [VV07, Notation 1.11], where the vectors t_w, s_w are however normalized differently. Note that the conjugate equations yield $(\text{id} \otimes \text{qTr}_w)(t_w t_w^*) = \text{id}_w$, whereas we have $(\text{qTr}_w \otimes \text{id})(t_w t_w^*) = Q_w^{-2}$.

Using this data we can write down explicitly the left- and right-invariant Haar weights on $c_0(\mathbb{G})$:

$$h_L(a) = \sum \dim_q(w)^2 \text{qtr}_w(a_w) \quad \text{and} \quad h_R(a) = \sum \dim_q(w)^2 \text{qtr}'_w(a_w),$$

for $a = (a_w)_w \in c_c(\mathbb{G})$. Compare [VV07, Proposition 1.12]. Note that since we have $h_L(ab) = h_L(bS^2(a))$ in any discrete quantum group algebra, the above formula implies $S^2(a) = Q_w a Q_w^{-1}$ for $a \in B(H_w) \subset c_c(\mathbb{G})$, and we have $S(Q_w) = Q_w^{-1} = S^{-1}(Q_w)$. Recall also that if we switch from \mathbb{G} to \mathbb{G}^{op} we have to exchange h_L with h_R , and to replace S with S^{-1} (but the unitary antipode $R = R^{-1}$ is unchanged).

There are two ways to endow $C_r^*(\mathbb{G})$ with the structure of a Woronowicz C^* -algebra, we choose the coproduct $\Delta : C_r^*(\mathbb{G}) \rightarrow C_r^*(\mathbb{G}) \otimes C_r^*(\mathbb{G})$ such that $(\text{id} \otimes \Delta)(W) = W_{12}W_{13}$, as in [VV07]. We denote $\hat{\mathbb{G}}$ the compact quantum group given by $C^r(\hat{\mathbb{G}}) = C_r^*(\mathbb{G})$ and Δ . Then the unitaries W and $w \in I$ above are corepresentations of $(C_r^*(\mathbb{G}), \Delta)$ in the sense of [Wor87] and $\text{Corep}(\mathbb{G})$ identifies with the category of finite-dimensional unitary representations of $\hat{\mathbb{G}}$. We denote $\mathbb{C}[\mathbb{G}] = \mathcal{O}(\hat{\mathbb{G}})$ the canonical dense Hopf- $*$ -subalgebra spanned by coefficients of finite-dimensional corepresentations.

We denote $h \in C_r^*(\mathbb{G})^*$ the Haar state, and we can write down the Woronowicz-Peter-Weyl orthogonality relations using the matricial states qtr, qtr' introduced above:

$$(\text{id} \otimes h)(w(a \otimes 1)w^*) = \text{qtr}_w(a), \quad (\text{id} \otimes h)(w^*(a \otimes 1)w) = \text{qtr}'_w(a),$$

for $a \in B(H_w)$, $w \in I$. See [VV07, Notation 1.11]. We recognize the modular matrices $F_w = Q_w$ from [Wor87] for $(C_r^*(\mathbb{G}), \Delta)$, which are connected to Woronowicz' characters $f_z \in \mathbb{C}[\mathbb{G}]^*$ by the formula $Q_w^z = (\text{id} \otimes f_z)(w)$.

Recall that the modular group of h is implemented by these characters, more precisely we have $h(xy) = h(y\sigma_{-i}(x))$ if we put $\sigma_z(x) = f_{iz} * x * f_{iz}$ for $x, y \in \mathbb{C}[\mathbb{G}]$. This yields $(\text{id} \otimes h)((1 \otimes y)w) = (\text{id} \otimes h)((Q_w^{-1} \otimes 1)w(Q_w^{-1} \otimes 1)(1 \otimes y))$ for $w \in I$. Since on the other hand $\text{qtr}_w(ba) = \text{qtr}_w(Q_w^{-1} a Q_w b)$,

we obtain

$$(\text{qtr}_w \otimes h)(w^*(1 \otimes y)w) = (\text{qtr}_w \otimes h)((Q_w^{-2} \otimes 1)ww^*(1 \otimes y)) = \text{qtr}'_w(1)h(y) = h(y).$$

In terms of the adjoint action ad this can be written $(\text{qtr}_w \otimes h) \text{ad} = h$. Similarly one can check that $(\text{qtr}'_w \otimes h) \text{ad}^{\text{op}} = h$.

An action of a discrete quantum group \mathbb{G} on a C^* -algebra A is given by a non-degenerate $*$ -homomorphism $\alpha : A \rightarrow M(c_0(\mathbb{G}) \otimes A)$ such that $(\text{id} \otimes \alpha)\alpha = (\Delta \otimes \text{id})\alpha$ and $\alpha(A)(c_0(\mathbb{G}) \otimes 1)$ is a dense subspace of $c_0(\mathbb{G}) \otimes A$. We will also say that A is a \mathbb{G} - C^* -algebra. For $w \in I$ we denote $\alpha_w = (p_w \otimes 1)\alpha : A \rightarrow M(B(H_w) \otimes A)$. A completely positive (cp) map $\Phi : A \rightarrow B$ between \mathbb{G} - C^* -algebras $(A, \alpha), (B, \beta)$ is called \mathbb{G} -equivariant if $(\text{id} \otimes \Phi)\alpha = \beta\Phi$. For $f \in \ell^1(\mathbb{G}) = c_0(\mathbb{G})^*$ and $a \in A$ we denote $f * a = (f \otimes \text{id})\alpha(a) \in M(A)$. For $\nu \in A^*$ we consider the Poisson transform $\mathcal{P}_\nu = (\text{id} \otimes \nu)\alpha : A \rightarrow M(c_0(\mathbb{G}))$. Throughout we will assume that A is unital unless otherwise specified.

The reduced crossed product $A \rtimes_r \mathbb{G}$ is the closed subspace of $M(K(H) \otimes A)$ generated by the elements $\alpha(a)(x \otimes 1)$ with $a \in A, x \in C_r^*(\mathbb{G})$. It is equipped with an action of the dual of \mathbb{G} and, more importantly in this paper, by the action of \mathbb{G} associated to the map $\text{ad}_A : A \rtimes_r \mathbb{G} \rightarrow M(c_0(\mathbb{G}) \otimes (A \rtimes_r \mathbb{G}))$, $X \mapsto (W^* \otimes 1)(1 \otimes X)(W \otimes 1)$. The non-degenerate injective $*$ -homomorphisms $\alpha : A \rightarrow A \rtimes_r \mathbb{G}$ and $x \in C_r^*(\mathbb{G}) \rightarrow x \otimes 1 \in A \rtimes_r \mathbb{G}$ are then equivariant with respect to ad_A, α and ad . If there is no risk of confusion we will denote $\text{ad}_A = \text{ad}$.

We say that a unital \mathbb{G} - C^* -algebra A is a \mathbb{G} -boundary if for every state $\nu \in S(A)$ the Poisson map \mathcal{P}_ν is completely isometric (ci). Equivalently, every unital cp (ucp) equivariant map $\Phi : A \rightarrow B$ to any unital \mathbb{G} - C^* -algebra B is completely isometric, cf [KKS22, Section 4]. There exists a \mathbb{G} -boundary in which every \mathbb{G} -boundary embeds equivariantly and completely isometrically, and it is unique up to (unique) equivariant $*$ -isomorphism. It is called the Furstenberg boundary of \mathbb{G} , denoted $C(\partial_F \mathbb{G})$ [KKS22, Theorem 4.16]. Moreover, this \mathbb{G} -boundary is \mathbb{G} -injective, meaning that for any equivariant uci map $\Phi : A \rightarrow B$, every equivariant ucp map $\psi : A \rightarrow C(\partial_F \mathbb{G})$ extends to $\Psi : B \rightarrow C(\partial_F \mathbb{G})$.

As in the classical case, a discrete quantum group \mathbb{G} is called C^* -**simple** if $C_r^*(\mathbb{G})$ is a simple C^* -algebra. In the classical case it is known that there are enough ucp maps $\Psi : C_r^*(\mathbb{G}) \rightarrow B$ to detect C^* -simplicity — see, for example, [KS22, Proposition 3.1] and for a generalization to (noncommutative) crossed products (of groups) see [KS19, Theorem 6.6]. This fact holds in the setting of DQGs as well.

More precisely, for a \mathbb{G} -equivariant ucp map $\Psi : A \rightarrow B$ we denote $I_\Psi = \{a \in A : \Psi(a^*a) = 0\}$, which is automatically a closed left ideal by the Schwarz inequality for ucp maps. Recall that Ψ is called faithful if $I_\Psi = \{0\}$.

Proposition 2.1. *Let \mathbb{G} be a DQG and (A, α) a \mathbb{G} - C^* -algebra. For every \mathbb{G} -equivariant ucp map $\Psi : C_r^*(\mathbb{G}) \rightarrow A$, I_Ψ is a two-sided ideal. Moreover, for every closed two-sided ideal $I \subsetneq C_r^*(\mathbb{G})$ there exists a \mathbb{G} -equivariant ucp map $\Phi : C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ such that $I \subseteq I_\Phi$. In particular, \mathbb{G} is C^* -simple if and only if every \mathbb{G} -equivariant ucp map $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ is faithful.*

Proof. Take $u \in \text{Irr}(\mathbb{G})$ and unit vectors $\zeta, \xi \in H_u$. For $x \in I_\Psi$ we can write

$$\begin{aligned} 0 &= (\omega_\zeta \otimes \text{id})\alpha(\Psi(x^*x)) = (\omega_\zeta \otimes \Psi)(\text{ad}(x^*x)) = (\omega_\zeta \otimes \Psi)(u^*(1 \otimes x^*x)u) \\ &\geq (\omega_\zeta \otimes \Psi)(u^*(\xi\xi^* \otimes x^*x)u) = \Psi(u_{\zeta,\xi}^* x^* x u_{\zeta,\xi}). \end{aligned}$$

Hence $xu_{\zeta,\xi} \in I_\Psi$ and since the coefficients $u_{\zeta,\xi} = (\xi^* \otimes 1)u(\zeta \otimes 1)$ span a dense subspace of $C_r^*(\mathbb{G})$ we conclude that I_Ψ is a right ideal.

If $I \subset C_r^*(\mathbb{G})$ is a bilateral ideal, we have $\text{ad}(I)(c_0(\mathbb{G}) \otimes C_r^*(\mathbb{G})) = W^*(1 \otimes I)W(c_0(\mathbb{G}) \otimes C_r^*(\mathbb{G})) \subset c_0(\mathbb{G}) \otimes I$, since $W \in M(c_0(\mathbb{G}) \otimes C_r^*(\mathbb{G}))$. Denoting $q : C_r^*(\mathbb{G}) \rightarrow A = C_r^*(\mathbb{G})/I$ the quotient map, this shows that $(\text{id} \otimes q) \text{ad}$ factors to a coaction α on A such that q is equivariant. Applying \mathbb{G} -injectivity of $C(\partial_F \mathbb{G})$ to the unital inclusion $\mathbb{C} \subset A$ and the canonical unital map $\mathbb{C} \rightarrow C(\partial_F \mathbb{G})$ we obtain a \mathbb{G} -equivariant ucp map $\theta : A \rightarrow C(\partial_F \mathbb{G})$. Then $\Phi = \theta \circ q$ is a \mathbb{G} -equivariant ucp map such that $I \subseteq I_\Phi$. \square

2.2. The free unitary quantum groups and their boundary. Most examples of discrete quantum groups are in fact defined as duals of compact matrix quantum groups. For instance, fix an integer $N \geq 1$ and $F \in \text{GL}_N(\mathbb{C})$. The universal unitary compact quantum group U_F^\pm is the compact quantum group given by the universal unital C^* -algebra $C^u(U_F^\pm)$ generated by the entries of unitary matrix $u \in$

$M_N(C^u(U_F^+))$ and the relations making $F\bar{u}F^{-1}$ a unitary matrix as well, endowed with the coproduct such that $\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}$ [Wan95, VDW96]. Identifying $M_N(C^u(U_F^+))$ with $B(\mathbb{C}^N) \otimes C^u(U_F^+)$, the matrix u becomes a representation of U_F^+ .

The Woronowicz C^* -algebra $C^u(U_F^+)$ has a reduced version $C^r(U_F^+)$, and up to isomorphism there is a unique discrete quantum group $\mathbb{F}U_F$, given by a Hopf- C^* -algebra $(c_0(\mathbb{F}U_F), \Delta)$, such that following the constructions explained previously we have $(C_r^*(\mathbb{F}U_F), \Delta) \simeq (C^r(U_F^+), \Delta)$. We will accordingly denote $C^*(\mathbb{F}U_F) = C^u(U_F^+)$, so that we can write $u \in M_N(\mathbb{C}) \otimes C^*(\mathbb{F}U_F)$. Note that $\mathbb{F}U_F$ is unimodular if and only if F is a scalar multiple of a unitary matrix.

The category of corepresentations of $\mathbb{F}U_F$ has been computed by Banica [Ban97]. The elements of $I = \text{Irr}(\mathbb{G})$ can be labelled by words x on the letters u, \bar{u} in such a way that the empty word 1 corresponds to the trivial corepresentation, the one letter words u and \bar{u} correspond to the fundamental corepresentation (u_{ij}) and its unitary dual $F(u_{ij}^*)F^{-1}$, and we have the recursive fusion rules

$$(1) \quad \begin{aligned} xu \otimes uy &\simeq xuyy, & xu \otimes \bar{u}y &\simeq xu\bar{u}y \oplus x \otimes y, \\ x\bar{u} \otimes \bar{u}y &\simeq x\bar{u}\bar{u}y, & x\bar{u} \otimes uy &\simeq x\bar{u}uy \oplus x \otimes y. \end{aligned}$$

In particular w is a subobject of $x \otimes y$ **iff** we can write $x = x'v, y = \bar{v}y'$ and $w = x'y'$, and for $x, y \in I \setminus \{1\}$ we have $xy \simeq x \otimes y$ **iff** the last letter of x equals the first of y . Note that we identify a word in u, \bar{u} and a representant for the corresponding equivalence class of irreducible corepresentations. We denote $p_x \in \ell^\infty(\mathbb{G})$ the minimal central projection corresponding to x , and this yields an identification $Z(\ell^\infty(\mathbb{G})) = \ell^\infty(I)$. The length of $x \in I$ as a word on u, \bar{u} is denoted $|x|$ and we put $I_n = \{x \in I \mid |x| = n\}$, $p_n = \sum_{|x|=n} p_x \in \ell^\infty(\mathbb{G})$. We denote $z \geq x$ if $z = xy$ for some $y \in I$, and we put $I(x) = \{z \in I \mid z \geq x\}$.

It follows from the fusion rules that inclusions of irreducibles $x \subset y \otimes z$ are always multiplicity free and we choose corresponding isometric intertwiners $V(x, y \otimes z)$, which are unique up to a phase. We also denote $P(x, y \otimes z) = V(x, y \otimes z)V(x, y \otimes z)^* \in B(H_y \otimes H_z)$ the corresponding range projections, and we put $P(x, y \otimes z) = 0$ if $x \not\subset y \otimes z$. These intertwiners can be used to compute the coproduct of $c_0(\mathbb{G})$: for $a \in p_x c_0(\mathbb{G}) \simeq B(H_x)$ and $y, z \in I$ we have

$$(p_y \otimes p_z)\Delta(a) = V(x, y \otimes z)aV(x, y \otimes z)^*.$$

Let q be the unique number in $]0, 1]$ such that $q + q^{-1} = \dim_q(u) = \dim_q(\bar{u})$. For a letter $\alpha = u$ or \bar{u} , denote $\alpha^{(k)} = \alpha\bar{\alpha}\alpha \dots$ (k terms) and $\alpha^{(\infty)}$ the infinite alternating word starting with α . We have $\bar{\alpha} \otimes \alpha^{(k)} \simeq \bar{\alpha}^{(k+1)} \oplus \bar{\alpha}^{(k-1)}$ for $k \geq 1$ and it follows easily that $\dim_q(\alpha^{(k)}) = [k+1]_q$, using the q -numbers $[n]_q = (q^{-n} - q^n)/(q^{-1} - q)$. Then, decomposing $x \in I$ as $x = x_1 \otimes \dots \otimes x_p$ where each x_i is of the form $\alpha^{(k)}$, we obtain $\dim_q(x) = [|x_1|+1]_q \dots [|x_p|+1]_q$.

We will need the following lemma about Woronowicz' modular matrices Q_x .

Lemma 2.2. *Denote $\rho = \max(\|Q_u\|, \|Q_{\bar{u}}\|)$. For all $x \in I$ we have $\|Q_x\|/\dim_q(x) \leq (q\rho)^{|x|}$ and $\|Q_x^{-1}\|/\dim_q(x) \leq (q\rho)^{|x|}$. Moreover if $N \geq 3$ we have $q\rho < 1$.*

Proof. If $x = \alpha_1 \dots \alpha_n$, with $\alpha_i \in \{u, \bar{u}\}$, $n = |x|$, we have in particular $x \subset v := \alpha_1 \otimes \dots \otimes \alpha_n$, hence Q_x appears as a diagonal block of $Q_{\alpha_1} \otimes \dots \otimes Q_{\alpha_n}$ via the decomposition of v into irreducibles. As a result $\|Q_x\| \leq \prod_1^n \|Q_{\alpha_i}\| \leq \rho^n$. The same reasoning holds for Q_x^{-1} with the $Q_{\alpha_i}^{-1}$'s, and since $Q_{\bar{u}} = \bar{Q}_u^{-1}$ we get the same upper bound. On the other hand we have $[k+1]_q = q^{-k}(1 - q^{2k+2})/(1 - q^2) \geq q^{-k}$ for all k , hence $\dim_q(x) \geq q^{-|x|}$ for all $x \in I$. In particular $\|Q_x\|/\dim_q(x) \leq (q\rho)^{|x|}$.

Since $\text{Tr}(Q_u) = \text{Tr}(Q_u^{-1})$ we must have $\|Q_u\|, \|Q_u^{-1}\| \geq 1$. Assume $N \geq 3$. If $\|Q_u\| \geq \|Q_u^{-1}\|$ we have $q + q^{-1} = \text{Tr}(Q_u) > \|Q_u\| + \|Q_u^{-1}\|^{-1} \geq \|Q_u\| + \|Q_u\|^{-1}$, since Q_u has at least a third eigenvalue beyond $\|Q_u\|$ and $\|Q_u^{-1}\|^{-1}$. The same holds if $\|Q_u\| \leq \|Q_u^{-1}\|$ by writing instead $q + q^{-1} = \text{Tr}(Q_u^{-1}) > \|Q_u^{-1}\| + \|Q_u\|^{-1} \geq \|Q_u\| + \|Q_u\|^{-1}$. Since $t \mapsto t + t^{-1}$ is strictly increasing on $[1, +\infty[$ this implies $q^{-1} > \|Q_u\|$. Since we also have $q + q^{-1} = \dim_q(\bar{u}) = \text{Tr}(Q_{\bar{u}}) = \text{Tr}(Q_{\bar{u}}^{-1})$, we have similarly $q^{-1} > \|Q_{\bar{u}}\|$, hence $q^{-1} > \rho$. \square

The construction of the Gromov compactification $\beta_G \mathbb{F}U_F$ of $\mathbb{F}U_F$ relies on the the following ucp maps, defined for $x, y \in \text{Irr}(\mathbb{G})$ and $m \in \mathbb{N}$:

$$\begin{aligned} \psi_{x,xy} &: B(H_x) \rightarrow B(H_{xy}), \quad a \mapsto V(xy, x \otimes y)^*(a \otimes \text{id}_y)V(xy, x \otimes y), \\ \psi_{l,m} &= \sum_{\substack{|x|=l \\ |y|=m-l}} \psi_{x,xy} : p_l \ell^\infty(\mathbb{G}) \rightarrow p_m \ell^\infty(\mathbb{G}) \quad \text{for } m \geq l, \\ \psi_{x,\infty} &= \sum_{y \in I} \psi_{x,xy} : B(H_x) \rightarrow \ell^\infty(\mathbb{G}), \quad \psi_{m,\infty} = \sum_{|x|=m} \psi_{x,\infty} : p_m \ell^\infty(\mathbb{G}) \rightarrow \ell^\infty(\mathbb{G}). \end{aligned}$$

These maps define an inductive system, in the sense that $\psi_{xy,xyz} \circ \psi_{x,xy} = \psi_{x,xyz}$. The quantum Gromov compactification $\beta_G \mathbb{F}U_F$ is then given by the unital C^* -subalgebra $B = C(\beta_G \mathbb{F}U_F) = \bar{B}_0 \subset \ell^\infty(\mathbb{G})$, which is the closure of

$$B_0 = \{\psi_{m,\infty}(a); m \in \mathbb{N}, a \in p_m \ell^\infty(\mathbb{G})\}.$$

Finally the quantum Gromov boundary of $\mathbb{F}U_F$ is given by $C(\partial_G \mathbb{F}U_F) = B_\infty = B/c_0(\mathbb{G})$, see [VV07, VVV10].

Note that as a free monoid the set I has a natural Cayley tree structure (with respect to the generating set $\{u, \bar{u}\}$), and recall that $Z(\ell^\infty(\mathbb{F}U_F)) \simeq \ell^\infty(I)$. The restriction of $\psi_{x,xy}$ to the center is the canonical map $\text{id}_x \mapsto \text{id}_{xy}$ and it then follows that $\ell^\infty(I) \cap C(\beta_G \mathbb{F}U_F) = C(\beta I)$, where βI is the usual compactification of the tree I and $C(\beta I)$ is identified to a subalgebra of $\ell^\infty(I)$ via the restriction map. The boundary $\partial I = \beta I \setminus I$ is canonically identified with the set of infinite words in u, \bar{u} . Then $C(\partial_G \mathbb{F}U_F)$ is a continuous field of unital C^* -algebras over ∂I .

We will make important use of the central projections $\pi_x = \psi_{x,\infty}(p_x) \in C(\beta_G \mathbb{F}U_F)$, for $x \in I$. More precisely we have $\pi_x = \sum_{y \in I} p_{xy}$, since $V(xy, x \otimes y)^* V(xy, x \otimes y) = \text{id}_{xy}$. This is a projection in $\ell^\infty(I) \cap C(\beta_G \mathbb{F}U_F)$, whose image in $C(\partial I)$ corresponds to the subset $\partial I(x) \subset \partial I$ of infinite words starting with x . Any state ν on $C(\partial_G \mathbb{F}U_F)$ induces by restriction a probability measure on ∂I that we denote ν_I . We have then $\nu_I(\partial I(x)) = \nu(\pi_x)$ by definition.

The action $\beta : B \rightarrow M(c_0(\mathbb{G}) \otimes B)$ of \mathbb{G} on B is just given by the restriction of the comultiplication. We have in particular $(p_z \otimes p_t)\beta(\pi_x) = \sum_{y \in I} P(xy, z \otimes t)$, which is non-zero iff $z \otimes t$ contains an irreducible corepresentation starting with x . We will need the following description of the boundary action, which is implicit in [VVV10]:

Lemma 2.3. *Fix $k, n \in \mathbb{N}$ and $\epsilon > 0$. Denote $p_{\geq r} = \sum_{l \geq r} p_l$. Then there exists $r \geq k + n$ such that for all $a_k \in p_k \ell^\infty(\mathbb{F}U_F)$, $\|a_k\| \leq 1$ we have*

$$\|(p_n \otimes p_{\geq r})\Delta(\psi_{k,\infty}(a_k)) - (p_n \otimes \psi_{r,\infty})(b)\| \leq \epsilon,$$

where $b = \sum_{l=0}^n b_{n+r-2l}$, $b_s = (p_n \otimes p_r)\Delta(\psi_{k,s}(a_k))$.

Proof. The element a_k can be decomposed into a sum of 2^k elements $a_x \in B(H_x)$ with $x \in I_k$. Up to replacing ϵ by $2^{-k}\epsilon$ and using the triangle inequality we can assume $a_k = a_x$ for some $x \in I_k$. Denote $\psi_{x,\infty}(a_x) = a$, so that $a_t = \psi_{x,t}(a_x)$ if $t \geq x$ and $a_t = 0$ else — in particular $a_x = p_x a$ and $a_k = p_k a$ as expected. Take $r \geq k + n$. We first consider $(p_y \otimes p_z)\Delta(a)$ with $|y| = n$, $z = z_1 z_2$, $|z_1| = r$. Since $r \geq n$, subobjects of $y \otimes z$ must be of the form tz_2 with $t \subset y \otimes z_1$, hence we have

$$(p_y \otimes p_z)\Delta(a) = \sum_{tz_2 \subset y \otimes z_1 z_2} V(tz_2, y \otimes z_1 z_2) a_{tz_2} V(tz_2, y \otimes z_1 z_2)^*.$$

Since $|t| \geq |z_1| - |y| \geq |x|$ we have $a_{tz_2} = \psi_{t,tz_2}(a_t)$ — both terms vanish unless $t \geq x$. Recall from [VV08, Lemma 7.8.3, equation (8.47)] that we have $V(tz_2, y \otimes z_1 z_2)V(tz_2, t \otimes z_2)^* \simeq \mu(\text{id}_y \otimes V(z_1 z_2, z_1 \otimes z_2)^*) (V(t, y \otimes z_1) \otimes \text{id}_{z_2})$ up to $Cq^{(|t|+|z_1|-|y|)/2} \leq Cq^{(r-n)/2}$ in operator norm, for some $\mu \in \mathbb{C}$ such that $|\mu| = 1$. This yields

$$\begin{aligned} V(tz_2, y \otimes z_1 z_2) a_{tz_2} V(tz_2, y \otimes z_1 z_2)^* &= \\ &= V(tz_2, y \otimes z_1 z_2) V(tz_2, t \otimes z_2)^* (a_t \otimes \text{id}_{z_2}) V(tz_2, t \otimes z_2) V(tz_2, y \otimes z_1 z_2)^* \\ &\simeq (\text{id}_y \otimes \psi_{z_1, z_1 z_2})(V(t, y \otimes z_1) a_t V(t, y \otimes z_1)^*) = (p_y \otimes p_z \psi_{r,|z|})\Delta(a_t) \end{aligned}$$

up to $2Cq^{(r-n)/2}$. Putting $s = |t|$, t is the only subobject of $y \otimes z_1$ of length s , so that $(p_y \otimes p_{z_1})\Delta(a_t) = (p_y \otimes p_{z_1})\Delta(p_s \psi_{s,\infty}(a_x)) = (p_y \otimes p_{z_1})b_s$. Summing over t we obtain

$$\|(p_y \otimes p_z)\Delta(a) - (p_y \otimes p_z \psi_{r,|z|})(b)\| \leq 2C(n+1)q^{(r-n)/2},$$

since there are at most $n + 1$ possible values of s . For r large enough this yields the result, by taking the supremum over y and z . \square

3. THE POWERS AVERAGING PROPERTY

3.1. Definition and Basic Results. Powers' averaging property is a combinatorial technique originally used by Powers in [Pow75] to prove that free groups are C^* -simple with the unique trace property. Later, it was shown independently by Kennedy [Ken20] and Haagerup [Haa17] that the converse holds, namely that every C^* -simple group has Powers' averaging property. We introduce below an analogue of Powers' averaging property for discrete quantum groups and show that it still implies C^* -simplicity.

Denote $\text{Prob}(\mathbb{G})$ the convex space of normal states on $\ell^\infty(\mathbb{G})$ and $\text{Prob}^c(\mathbb{G})$ the subspace of states with finite support, i.e. states f for which there exists a projection $p \in c_c(\mathbb{G})$ with $f(p) = 1$. We denote $h \in S(C_r^*(\mathbb{G}))$ the Haar state and we consider the following convolution operations, for $x \in C_r^*(\mathbb{G})$, $f \in \text{Prob}(\mathbb{G})$, $\mu \in S(C_r^*(\mathbb{G}))$: $x * f = (f \otimes \text{id}) \text{ad}(x)$, $f * \mu = (f \otimes \mu) \circ \text{ad}$, $\mu * x = (\text{id} \otimes \mu) \Delta(x)$, $x * \mu = (\mu \otimes \text{id}) \Delta(x)$. The state μ on $C_r^*(\mathbb{G})$ is called \mathbb{G} -invariant if we have $f * \mu = \mu$ for all $f \in \text{Prob}(\mathbb{G})$.

Definition 3.1. We define $\text{Prob}_h(\mathbb{G}) = \{f \in \text{Prob}(\mathbb{G}) : f * h = h\}$ and $\text{Prob}_h^c(\mathbb{G}) = \text{Prob}_h(\mathbb{G}) \cap \text{Prob}^c(\mathbb{G})$.

From the preliminaries we have important examples of states in $\text{Prob}_h^c(\mathbb{G})$, namely the quantum traces qtr_w . Before defining the PAP let us note the following stronger stabilization property for elements of $\text{Prob}_h^c(\mathbb{G})$.

Lemma 3.2. *Let \mathbb{G} be a DQG. If $f \in \text{Prob}_h^c(\mathbb{G})$ then $f * \tau = \tau$ for every σ -KMS state $\tau \in S(C_r^*(\mathbb{G}))$.*

Proof. Note that we have $f \in \text{Prob}_h(\mathbb{G})$ **iff** $h(x) = (f \otimes h)(W^*(1 \otimes x)W)$ for all $x \in \mathbb{C}[\mathbb{G}]$. Assume that $f \in \text{Prob}^c(\mathbb{G})$. Recall that the KMS group of h is implemented by the Woronowicz characters $f_z \in \mathbb{C}[\mathbb{G}]^*$ so that $h(xy) = h((f_{-1} * y * f_{-1})x)$ for all $x \in C_r^*(\mathbb{G})$, $y \in \mathbb{C}[\mathbb{G}]$. Denote $Q = (\text{id} \otimes f_1)(W) = (Q_w)_w$, which is a positive unbounded multiplier of $c_0(\mathbb{G})$. We will invoke Sweedler notation and write $W = W_{(1)} \otimes W_{(2)}$. Then we have, since $(S \otimes \text{id})(W) = W^*$ and $(S \otimes \text{id})(W^*) = (Q \otimes 1)W(Q^{-1} \otimes 1)$:

$$\begin{aligned} (f \otimes h)(W^*(1 \otimes x)W) &= (f \otimes h)((1 \otimes W_{(2)})W^*(Q^{-1}W_{(1)}Q^{-1} \otimes x)) \\ &= (fS \otimes h)(W^*(Q^2 \otimes 1)W(1 \otimes x)) \end{aligned}$$

for all $x \in C_r^*(\mathbb{G})$. Hence $f \in \text{Prob}^c(\mathbb{G})$ belongs to $\text{Prob}_h^c(\mathbb{G})$ if and only if

$$(2) \quad (fS \otimes \text{id})(W^*(Q^2 \otimes 1)W) = 1.$$

If this holds, we can roll back the computation with any σ -KMS state τ instead of h , which yields the result. \square

From (2) we also see that we have $\text{Prob}_h^c(\mathbb{G}) = \text{Prob}^c(\mathbb{G})$ **iff** $W^*(Q \otimes 1)W = 1$ **iff** $Q = 1$, i.e. \mathbb{G} is unimodular — this has been known at least since [Izu02]. Note that $\text{Prob}^c(\mathbb{G})$ is norm dense in $\text{Prob}(\mathbb{G})$, but the corresponding result for $\text{Prob}_h(\mathbb{G})$ is not clear.

Definition 3.3. We say that \mathbb{G} has the **Powers averaging property** PAP, resp. PAP_h , if for every $x = x^* \in C_r^*(\mathbb{G})$ we have

$$\inf_{f \in P} \|x * f - h(x)1\| = 0,$$

where $P = \text{Prob}^c(\mathbb{G})$, resp. $\text{Prob}_h^c(\mathbb{G})$. Note that $\text{PAP}_h \implies \text{PAP}$.

Remark 3.4. Let $\mathbb{G} = G$ be a discrete group. The classical PAP, as formulated by Powers in [Pow75], and the formulation used in [Ken20, Haa17], is stated as follows: for every $x = x^* \in C_r^*(G)$,

$$h(x)1 \in \overline{\text{conv}}\{\lambda(s)x\lambda(s)^* : s \in G\}.$$

We can see that this formulation of the PAP is equivalent to the one given in Definition 3.1. Indeed, for $s \in G$ and $x \in C_r^*(G)$, $x * \delta_s = \lambda(s)x\lambda(s)^*$ and so

$$\overline{\text{conv}}\{\lambda(s)x\lambda(s)^* : s \in G\} = \overline{x * \text{Prob}(G)}.$$

In Definitions 3.1 and 3.3 we could have used the coaction ad^{op} of $c_0(\mathbb{G}^{\text{op}})$, instead of ad . Let us show that the resulting notions would have been the same. For this we denote $*^{\text{op}}$ the convolution products associated with ad^{op} , we denote $\text{Prob}_h^c(\mathbb{G}^{\text{op}}) = \{f \in \text{Prob}^c(\mathbb{G}) : f *^{\text{op}} h = h\}$ and we say e.g. that \mathbb{G}^{op} has the PAP_h if $\inf\{\|x *^{\text{op}} f - h(x)1\| : f \in \text{Prob}_h^c(\mathbb{G}^{\text{op}})\} = 0$.

Lemma 3.5. *A discrete quantum group \mathbb{G} has the PAP, resp. PAP_h iff \mathbb{G}^{op} has the PAP, resp. PAP_h .*

Proof. According to the formula $\text{ad}^{\text{op}} = (R \otimes \hat{R}) \circ \text{ad} \circ \hat{R}$ we have

$$x *^{\text{op}} f := (f \otimes \text{id}) \text{ad}^{\text{op}}(x) = \hat{R}(\hat{R}(x) * fR).$$

Since \hat{R} is a $*$ -antiautomorphism such that $h \circ \hat{R} = h$, it follows that $\|x *^{\text{op}} f - h(x)1\| = \|x' * fR - h(x')\|$ for $x' = \hat{R}(x)$. It clearly follows that the PAP is the same for \mathbb{G} and \mathbb{G}^{op} . For the PAP_h it remains to check that $\{f \circ R : f \in \text{Prob}_h^c(\mathbb{G}^{\text{op}})\} = \text{Prob}_h^c(\mathbb{G})$, which again follows from the connection between ad and ad^{op} and the property $h \circ \hat{R} = h$. \square

Lemma 3.6. *A DQG \mathbb{G} has the PAP, resp. the PAP_h , if and only if for every $x = x^* \in \mathbb{C}[\mathbb{G}] \cap \ker(h)$ we have*

$$\inf_{f \in P} \|x * f\| = 0,$$

where $P = \text{Prob}(\mathbb{G})$, resp. $\text{Prob}_h^c(\mathbb{G})$.

Proof. The direct implication is obvious. For the reverse one, we first restrict from $C_r^*(\mathbb{G})$ to $\mathbb{C}[\mathbb{G}]$ by density and because $\|x * f\| \leq \|x\|$ for any $x \in C_r^*(\mathbb{G})$, $f \in \text{Prob}(\mathbb{G})$. Then we apply the assumption to $x - h(x)1$ and observe that $(x - h(x)1) * f = x * f - h(x)1$. \square

In the case of the PAP_h we can remove the limitation to self-adjoint elements.

Lemma 3.7. *A DQG \mathbb{G} has the PAP_h if and only if for every $x \in C_r^*(\mathbb{G})$,*

$$\inf_{f \in P} \|x * f - h(x)1\| = 0,$$

where $P = \text{Prob}_h^c(\mathbb{G})$.

Proof. Assume that \mathbb{G} has the PAP_h . Let $x = x_1 + \mathbf{i}x_2 \in C_r^*(\mathbb{G})$ with $x_1 = x_1^*$, $x_2 = x_2^*$ and fix $\epsilon > 0$. Find $f_1 \in P$ such that $\|x_1 * f_1 - h(x_1)1\| < \epsilon$. Next, put $x'_2 = x_2 * f_1$ and find $f_2 \in \text{Prob}_h(\mathbb{G})$ such that $\|x'_2 * f_2 - h(x'_2)1\| < \epsilon$. Let $f = f_1 * f_2$, which is still in P . Since $h(x'_2) = (f_1 * h)(x_2) = h(x_2)$ and $x'_2 * f_2 = x_2 * f$, we have $\|x_2 * f - h(x_2)1\| < \epsilon$. On the other hand, since $(f_2 \otimes \text{id}) \text{ad}$ is ucp, hence contractive, we have

$$\|x_1 * f - h(x_1)1\| = \|(x_1 * f_1 - h(x_1)1) * f_2\| \leq \|x_1 * f_1 - h(x_1)1\| < \epsilon.$$

As a result $\|(x_1 + \mathbf{i}x_2) * f - h(x_1 + \mathbf{i}x_2)1\| < 2\epsilon$. The converse is obvious. \square

We now verify that our main example $\mathbb{F}U_F$ has the PAP_h . Banica proved in [Ban97] that $\mathbb{F}U_F$ is C^* -simple with a unique σ -KMS state, using an adaptation of Powers' averaging techniques on $\text{Corep}(\mathbb{F}U_F)$. It should be noted that Banica also defined an analogue of Powers' averaging property for DQGs which turns out to not be equivalent to ours. In fact, Banica remarked in [Ban97, Section 8] that U_F^+ does not have the PAP in his sense. However, as we are about to see, what Banica actually proves is that it does have the PAP in our sense.

At heart of the proof of [Ban97, Theorem 3] is indeed the use of maps $\text{AD}(u) : \mathbb{C}[\mathbb{G}] \rightarrow \mathbb{C}[\mathbb{G}]$ from [Ban97, Lemma 9], for $u \in \text{Corep}(\mathbb{G})$, which are given by the formula

$$\text{AD}(u)(z) = (\text{Tr} \otimes \text{id})[(Q_u^b \otimes 1)u(1 \otimes z)u^*(Q_u^{-d} \otimes 1)]$$

where b, d are real parameters. In other words and with our notation, $\text{AD}(u)(z) = (\text{qTr}_u^{b-d} \otimes \text{id}) \text{ad}^{\text{op}}(z) = z *^{\text{op}} \text{qTr}_u^{b-d}$, where $\text{qTr}_u^{b-d}(a) := \text{Tr}_u(Q_u^{b-d}a_u)$ for $a = (a_u)_u \in c_0(\mathbb{F}U_F)$. Note in particular that $\text{qTr}_u^1 = \text{qTr}_u$ and $\text{qTr}_u^{-1} = \text{qTr}'_u$.

Proposition 3.8 ([Ban97]). *The discrete quantum groups $\mathbb{F}U_F$ have the PAP_h .*

Proof. We claim that in the proof of [Ban97, Theorem 3] what Banica really showed is that $\mathbb{F}U_F^{\text{op}}$ has the PAP_h (in our sense) and hence $\mathbb{G} = \mathbb{F}U_F$ has the PAP_h . Fix $z = z^* \in \mathbb{C}[\mathbb{G}] \cap \ker(h)$ and $\epsilon > 0$. In the proof of [Ban97, Theorem 3], a unital linear map $F = V \circ W$, with $V, W : C_r^*(\mathbb{G}) \rightarrow C_r^*(\mathbb{G})$, is constructed. Let us first describe the maps V and W .

The unital map W is constructed at [Ban97, Corollary 4] and given by $W = M^{-1} \text{AD}(r)$, for a specific corepresentation $r \in I$, the parameters $d = -b = \frac{1}{2}$ and $M > 0$. The map V is obtained from [Ban97, Corollary 3]. Here, V is of the form $V = T^m$, for some integer $m \geq 0$ and a unital map $T : C_r^*(\mathbb{G}) \rightarrow C_r^*(\mathbb{G})$ obtained from [Ban97, Proposition 8]. Similarly to W , T is constructed from

[Ban97, Lemma 9] and given by the formula $T = M_1^{-1} \text{AD}(r_1) + M_2^{-1} \text{AD}(r_2) + M_3^{-1} \text{AD}(r_3)$ for some well-chosen corepresentations $r_1, r_2, r_3 \in I$, the parameters $d = -b = 1/2$ and with $M_1, M_2, M_3 > 0$.

With our notation we have $T(z) = z *^{\text{op}} \varphi_1$ with $\varphi_1 = M_1^{-1} \text{qTr}'_{r_1} + M_2^{-1} \text{qTr}'_{r_2} + M_3^{-1} \text{qTr}'_{r_3}$ and $W(z) = z *^{\text{op}} \varphi_2$ with $\varphi_2 = M^{-1} \text{qTr}'_r$. As noted in the Preliminaries, the maps qtr'_w belong to $\text{Prob}_h^c(\mathbb{G}^{\text{op}})$. Altogether, the unital map $F : C_r^*(\mathbb{G}) \rightarrow C_r^*(\mathbb{G})$ is of the form $F(z) = z *^{\text{op}} \varphi$ for some $\varphi \in \text{Prob}_h^c(\mathbb{G}^{\text{op}})$. Moreover, r, r_1, r_2, r_3 and m are chosen so that, for our fixed $z \in \mathbb{C}[\mathbb{G}]$, one has $\|F(z)\| \leq \epsilon \|z\|$. The result thus follows from Lemma 3.6. \square

Such as in the classical case, the PAP easily implies C^* -simplicity and, in the unimodular case, the unique trace property. Note that by [KKSv22, AS24] a state on $C_r^*(\mathbb{G})$ is \mathbb{G} -invariant **iff** it is KMS with respect to the scaling automorphism group τ **iff** it is tracial.

Proposition 3.9. *Let \mathbb{G} be a DQG. If \mathbb{G} has the PAP then \mathbb{G} is C^* -simple and*

- (1) *if \mathbb{G} is unimodular then h is the only \mathbb{G} -invariant state on $C_r^*(\mathbb{G})$;*
- (2) *if \mathbb{G} is non-unimodular then $C_r^*(\mathbb{G})$ has no \mathbb{G} -invariant states.*

If moreover \mathbb{G} has the PAP_h , then h is the unique σ -KMS state.

Proof. Assume \mathbb{G} has the PAP and let $I \subseteq C_r^*(\mathbb{G})$ be a closed two-sided ideal. As already noted in the proof of Proposition 2.1, ad stabilizes I , in particular $x * f \in I$ for all $x \in I, f \in \text{Prob}(\mathbb{G})$. If $I \neq 0$ we can find $x \in I_+$ such that $h(x) = 1$, since h is faithful on $C_r^*(\mathbb{G})$. By the PAP there exists a net of states f_α in $\text{Prob}(\mathbb{G})$ such that $x * f_\alpha \rightarrow 1$, hence $1 \in I$ and $I = C_r^*(\mathbb{G})$.

Let $\tau \in S(C_r^*(\mathbb{G}))$ be a \mathbb{G} -invariant state, i.e. we have $f * \tau = \tau$ for all f in $\text{Prob}(\mathbb{G})$ or, equivalently, in $\text{Prob}^c(\mathbb{G})$. Take $x = x^* \in C_r^*(\mathbb{G})$ and a corresponding net of states f_α given by PAP. Then we have $\tau(x) = (f_\alpha * \tau)(x) = \tau(x * f_\alpha) \rightarrow h(x)\tau(1) = h(x)$. Since the self-adjoint elements span $C_r^*(\mathbb{G})$, this shows that $\tau = h$. If we assume PAP_h , by Lemma 3.2 this computation works also for all σ -KMS states τ .

If \mathbb{G} is unimodular, tracial states on $C_r^*(\mathbb{G})$ are \mathbb{G} -invariant, hence h is the only trace. If \mathbb{G} is not unimodular, h is not \mathbb{G} -invariant (i.e. $\text{Prob}_h^c(\mathbb{G}) \neq \text{Prob}^c(\mathbb{G})$) hence the previous computation shows that $C_r^*(\mathbb{G})$ has no \mathbb{G} -invariant states, and if \mathbb{G} has the PAP_h , it shows that h is the only σ -KMS state. \square

Remark 3.10. The Dixmier property of a C^* -algebra A holds whenever for every $a \in A, \overline{\text{conv}}\{uau^* : u \in \mathcal{U}(A)\} \cap Z(A) \neq \{0\}$. In the case of a classical group C^* -algebra $A = C_r^*(G)$, the PAP clearly implies the Dixmier property, using unitaries $u = \lambda(s), s \in G$. This is not so clear anymore in the quantum case. However, it was established in [HZ84] that if A is simple, then A has the Dixmier property if and only if A has at most one tracial state. Hence Proposition 3.9 together with [AS24] shows that if \mathbb{G} has the PAP then $C_r^*(\mathbb{G})$ has the Dixmier property.

3.2. The PAP and the state space of $C_r^*(\mathbb{G})$. Using Hahn-Banach one can rephrase the PAP_h in terms of functionals on $C_r^*(\mathbb{G})$. This generalizes [Ken20, Proposition 6.1] and part of [Haa17, Theorem 4.5].

Definition 3.11. For $\mu \in C_r^*(\mathbb{G})^*$ we denote $K(\mu)$, resp. $K_h(\mu)$, the space $\overline{P * \mu}^{w*}$, where $P = \text{Prob}^c(\mathbb{G})$, resp. $\text{Prob}_h^c(\mathbb{G})$.

Lemma 3.12. *Let \mathbb{G} be a DQG. The following are equivalent:*

- (1) *\mathbb{G} has the PAP_h ;*
- (2) *for every $\mu \in S(C_r^*(\mathbb{G}))$, $h \in K_h(\mu)$;*
- (3) *for every $\mu \in C_r^*(\mathbb{G})^*$, $\mu(1)h \in K_h(\mu)$.*

The equivalence between 1. and 2. also holds for the PAP using $K(\mu)$ instead of $K_h(\mu)$.

Proof. Write $P = \text{Prob}^c(\mathbb{G})$ resp. $P = \text{Prob}_h^c(\mathbb{G})$.

(1 \implies 2). Take $\mu \in S(C_r^*(\mathbb{G}))$. Using the Hahn-Banach separation theorem, if $h \notin \overline{P * \mu}^{w*}$ then there exists $x \in C_r^*(\mathbb{G})$ such that

$$\inf_{f \in P} \Re((f * \mu)(x) - h(x)) > 0.$$

Since $\mu = \bar{\mu}, \bar{h} = h$ and $\bar{P} = P$, the same holds for x^* , hence we can assume that $x^* = x$ and the PAP resp. the PAP_h fails because $\Re((f * \mu)(x) - h(x)) \leq \|x * f - h(x)\|$.

(2 \implies 1). For a contradiction, assume that the PAP, resp. the PAP_h fails. Let $x \in C_r^*(\mathbb{G})$, $x = x^*$, be such that $\inf_{f \in P} \|x * f - h(x)1\| > 0$. By the Hahn-Banach separation theorem, there exists $\epsilon > 0$ and $\mu \in C_r^*(\mathbb{G})^*$ such that $\Re((f * \mu)(x)) \geq \Re(\mu(1)h(x)) + \epsilon$ for all $f \in P$. Again we can assume that $\mu = \bar{\mu}$, hence $\mu \in S(C_r^*(\mathbb{G}))$ after renormalizing, and this contradicts 2.

(3 \implies 2) is obvious.

(2 \implies 3). Let $\mu : C_r^*(\mathbb{G}) \rightarrow \mathbb{C}$ be a bounded functional. Decompose $\mu = \mu_1 - \mu_2 + \mathbf{i}(\mu_3 - \mu_4)$ with $\mu_i \in C_r^*(\mathbb{G})_+^*$. By assumption, there exists a net $(f_\alpha) \subseteq P$ such that $f_\alpha * \mu_1 \xrightarrow{w*} \mu_1(1)h$. Using w^* -compactness of $\overline{P * \mu_i}^{w*}$ for each i , we find $\nu_i \in C_r^*(\mathbb{G})_+^*$ such that after passing to a convergent subnet we have $\lim_\alpha^{w*} f_\alpha * \mu_i = \nu_i$ (in particular $\nu_i(1) = \mu_i(1)$). This shows that $\mu_1(1)h - \nu_2 + \mathbf{i}(\nu_3 - \nu_4) \in \overline{P * \mu}^{w*}$. Using P -invariance of h , and by repeating the same argument, we deduce that $\overline{P * \mu}^{w*}$ contains $\mu_1(1)h - \nu_2(1)h + \mathbf{i}(\nu_3(1)h - \nu_4(1)h) = \mu(1)h$. \square

The preceding Lemma motivates the following definition and leads to Theorem 3.17.

Definition 3.13. We say that $X \subseteq S(C_r^*(\mathbb{G}))$ is a \mathbb{G} , resp. \mathbb{G}_h -**boundary envelope** if $X \neq \emptyset$ and $X = K(\mu)$, resp. $K_h(\mu)$, for every $\mu \in X$. We say that such a boundary envelope X is **non-trivial** if $X \neq \{h\}$.

Remark 3.14. Since $\text{Prob}^c(\mathbb{G})$ and $\text{Prob}_h^c(\mathbb{G})$ are convex, boundary envelopes are automatically convex and w^* -closed. In the classical case G -boundary envelopes correspond to G -boundaries inside the space $S(C_r^*(G))$, as shown in the next proposition. In the quantum case, it does not make sense to consider \mathbb{G} -boundaries inside $S(C_r^*(\mathbb{G}))$, since \mathbb{G} -boundaries are in general noncommutative. However *boundary envelopes*, as introduced in the previous definition, will prove useful also in the quantum case.

Proposition 3.15. Let $\mathbb{G} = G$ be a discrete group. The following are equivalent for $X \subseteq S(C_r^*(G))$:

- (1) X is a G -boundary envelope;
- (2) $X = \overline{\text{conv}}^{w*}(Y)$ where $Y \subseteq S(C_r^*(G))$ is a G -boundary;
- (3) $X = \Psi^*(S(C(\partial_F G)))$ for some G -equivariant ucp map $\Psi : C_r^*(G) \rightarrow C(\partial_F G)$.

Proof. The proof comes from a careful inspection of the proof of [Ken20, Proposition 3.1].

(1 \implies 2). It follows from the definitions that X is a minimal affine G -space. Let $\text{ext}(X)$ be the set of extreme points of X . By [Gla76, Theorem III.2.3], $\overline{\text{ext}(X)}^{w*}$ is a G -boundary.

(2 \implies 3). This is exactly the converse of [Ken20, Proposition 3.1].

(3 \implies 1). Because $C(\partial_F G)$ is a G -boundary, for every $\mu \in S(C(\partial_F G))$, $\overline{\text{Prob}(G) * \mu}^{w*} = S(C(\partial_F G))$. It is immediate from here that X is a G -boundary envelope. \square

Lemma 3.16. For any $\mu \in S(C_r^*(\mathbb{G}))$, there exists a \mathbb{G} -boundary envelope $X \subseteq K(\mu)$, and a \mathbb{G}_h -boundary envelope $X \subseteq K_h(\mu)$.

Proof. Let $(X_j)_{j \in J}$ be a descending net of non-empty w^* -closed $\text{Prob}(\mathbb{G})$ -invariant subspaces of $K(\mu)$. By w^* -compactness of $K(\mu)$, the finite intersection property implies that $\bigcap_{j \in J} X_j$ is a non-empty w^* -closed $\text{Prob}(\mathbb{G})$ -invariant subspace of $K(\mu)$. By Zorn's lemma, minimal such subspaces exist, and they are clearly \mathbb{G} -boundary envelopes. The argument is the same for $K_h(\mu)$. \square

We can then prove a quantum version of [Ken20, Theorem 3.6], where G -boundaries are replaced with \mathbb{G}_h -boundary envelopes.

Theorem 3.17. A discrete quantum group \mathbb{G} has the PAP_h if and only if the only \mathbb{G}_h -boundary envelope in $S(C_r^*(\mathbb{G}))$ is trivial.

Proof. Assume that \mathbb{G} has the PAP_h and let X be a \mathbb{G}_h -boundary envelope. Take $\mu \in X$, then by Lemma 3.12 we have $h \in K_h(\mu) = X$. But then we have as well $X = K_h(h) = \{h\}$. Conversely, fix $\mu \in S(C_r^*(\mathbb{G}))$. Using Lemma 3.16 we can find a \mathbb{G}_h -boundary envelope $X \subseteq K_h(\mu)$, and by assumption we must have $X = \{h\}$. In particular $h \in K_h(\mu)$ and the PAP_h follows from Lemma 3.12. \square

3.3. The PAP and ucp maps on $C_r^*(\mathbb{G})$. As in the classical case we establish the following correspondence between \mathbb{G} -boundary envelopes and \mathbb{G} -equivariant ucp maps $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$. Note that by Lemma 3.16 every \mathbb{G} -boundary envelope contains a \mathbb{G}_h -boundary envelope, so that \mathbb{G}_h -boundary envelopes are also connected to \mathbb{G} -equivariant ucp maps $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$.

Definition 3.18. For a \mathbb{G} -equivariant ucp map $\Psi : C_r^*(\mathbb{G}) \rightarrow B$ we denote $X_\Psi = \Psi^*(S(B)) \subseteq S(C_r^*(\mathbb{G}))$.

Lemma 3.19. *Let \mathbb{G} be a DQG. The map $\Psi \mapsto X_\Psi$ is a bijection between \mathbb{G} -equivariant ucp maps $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ and \mathbb{G} -boundary envelopes in $S(C_r^*(\mathbb{G}))$.*

Proof. Let B be a \mathbb{G} -boundary, and $\Psi : C_r^*(\mathbb{G}) \rightarrow B$ a \mathbb{G} -equivariant ucp map. Then for every $\mu \in S(B)$ we have $K(\mu) = S(B)$, see [KKSv22, Lemma 4.2], and since Ψ is equivariant this entails $K(\nu) = X_\Psi$ for all $\nu \in X_\Psi$, i.e. X_Ψ is a \mathbb{G} -boundary envelope. Note that Ψ^* is (w^*, w^*) -continuous and X_Ψ is w^* -closed, as it is the continuous image of the w^* -compact space $S(B)$.

Conversely, let $X \subset S(C_r^*(\mathbb{G}))$ be a \mathbb{G} -boundary envelope and take $\mu \in X$. Recall that we denote $\mathcal{P}_\mu = (\text{id} \otimes \mu) \text{ad} : C_r^*(\mathbb{G}) \rightarrow \ell^\infty(\mathbb{G})$, so that $f \circ \mathcal{P}_\mu = f * \mu$ for all $f \in \text{Prob}(\mathbb{G})$. Choose a \mathbb{G} -equivariant ucp projection $P : \ell^\infty(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ and set $\Psi = P \circ \mathcal{P}_\mu : C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$. Since $\varphi \circ P \in \overline{\text{Prob}(\mathbb{G})}^{w^*}$ for any $\varphi \in S(C(\partial_F \mathbb{G}))$ we have

$$X_\Psi = \{\varphi \circ \Psi : \varphi \in S(C(\partial_F \mathbb{G}))\} \subseteq \overline{\{f \circ \mathcal{P}_\mu : f \in \text{Prob}(\mathbb{G})\}}^{w^*} = K(\mu) = X.$$

Moreover, taking $\nu \in X_\Psi$, we have $X = K(\nu)$ because X is a \mathbb{G} -boundary envelope, hence $X \subset X_\Psi$ because X_Ψ is $\text{Prob}(\mathbb{G})$ -invariant.

On the other hand, if we started from $X = X_\Phi$ in this construction, we have $\mu = \varphi \circ \Phi$ with $\varphi \in S(C(\partial_F \mathbb{G}))$, hence $\Psi = P \circ \mathcal{P}_\varphi \circ \Phi$ by equivariance of Φ . But by rigidity of $\partial_F \mathbb{G}$, the equivariant ucp map $P \circ \mathcal{P}_\varphi : C(\partial_F \mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ must be the identity, hence $\Psi = \Phi$. \square

Remark 3.20. There is a clear analogue of a \mathbb{G} -boundary envelope in the state space of an arbitrary \mathbb{G} - C^* -algebra. In Lemma 3.19 and its proof, we can replace $C_r^*(\mathbb{G})$ with any \mathbb{G} - C^* -algebra — to be precise, there is a bijection between \mathbb{G} -boundary envelopes in $S(A)$ and \mathbb{G} -equivariant ucp maps $A \rightarrow C(\partial_F \mathbb{G})$ for any \mathbb{G} - C^* -algebra A .

Definition 3.21. Given a C^* -algebra A , we say that a ucp map $\Psi : C_r^*(\mathbb{G}) \rightarrow A$ factors h if $h \in \Psi^*(S(A))$.

Lemma 3.22. *Let \mathbb{G} be a DQG and A a C^* -algebra. A ucp map $\Psi : C_r^*(\mathbb{G}) \rightarrow A$ factors h iff for all $x \in \mathbb{C}[\mathbb{G}]$ we have $h(x) \in \overline{\{\mu(\Psi(x)) : \mu \in S(A)\}}$.*

Proof. Since $S(A)$ is w^* -compact and convex, $\Psi^*(S(A))$ is a w^* -closed convex subset of $S(C_r^*(\mathbb{G}))$. If $h \notin \Psi^*(S(A))$, the Hahn-Banach separation theorem shows that there exists $x \in C_r^*(\mathbb{G})$ and $\epsilon > 0$ such that $\Re(\mu(\Psi(x))) \geq \Re(h(x)) + 3\epsilon$ for all $\mu \in S(A)$. Taking $y \in \mathbb{C}[\mathbb{G}]$ such that $\|x - y\| \leq \epsilon$ we have $|\mu(\Psi(y)) - h(y)| \geq \epsilon$ for all $\mu \in S(A)$. The reverse direction is obvious. Note that $\{\mu(\Psi(x)) : \mu \in S(A)\}$ is in fact closed. \square

Lemma 3.23. *Let A be a \mathbb{G} -boundary and $\Psi : C_r^*(\mathbb{G}) \rightarrow A$ a \mathbb{G} -equivariant ucp map.*

- (1) *Assume \mathbb{G} has the PAP. Then Ψ factors h .*
- (2) *Assume \mathbb{G} is unimodular. Then Ψ factors h iff $\Psi = 1h$.*

Proof. First we note that by (the beginning of the proof of) Lemma 3.19 the set $X_\Psi = \Psi^*(S(A))$ is a \mathbb{G} -boundary envelope.

1. By Lemma 3.12, $h \in K(\mu) = X_\Psi$, where we have chosen some $\mu \in X_\Psi$.

2. If Ψ factors h , X_Ψ contains h , hence $X_\Psi = K(h)$. Since \mathbb{G} is unimodular, $K(h) = \{h\}$. As a result we have $\mu \circ \Psi = h = \mu \circ (1h)$ for all $\mu \in S(A)$, hence $\Psi = 1h$. The converse is trivial. \square

Corollary 3.24. *A DQG \mathbb{G} has the PAP iff every \mathbb{G} -equivariant ucp map $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ factors h . In particular, if \mathbb{G} is unimodular then \mathbb{G} has the PAP iff $1h$ is the unique \mathbb{G} -equivariant ucp map $C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$.*

Proof. The direct implication is a particular case of Lemma 3.23. For the converse, take $\mu \in S(C_r^*(\mathbb{G}))$. By Lemma 3.16 there exists a \mathbb{G} -boundary $X \subset K(\mu)$, which can be written $X = X_\Psi$ for some \mathbb{G} -equivariant ucp map $\Psi : C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ by Lemma 3.19. By hypothesis Ψ factors h , hence we have $h \in X \subset K(\mu)$, and Lemma 3.12 shows that \mathbb{G} has the PAP. The unimodular case follows immediately from Lemma 3.23 and the general case. \square

We end this section by stating explicitly the new results that we obtain for the unitary free quantum groups.

Corollary 3.25. *The only $(\mathbb{F}U_F)_h$ -boundary envelope in $S(C_r^*(\mathbb{F}U_F))$ is trivial. Every $\mathbb{F}U_F$ -equivariant ucp map $C_r^*(\mathbb{F}U_F) \rightarrow C(\partial_F \mathbb{F}U_F)$ factors h , and if F is unitary $1h$ is the unique such map.*

4. FREENESS AND FAITHFULNESS OF BOUNDARY ACTIONS

4.1. Freeness. The following definition was formulated by Masuda and Tomatsu [MT07, Definition 2.7] in the case when A is a von Neumann algebra equipped with a cocycle action of \mathbb{G} . Recall that W^* -coactions on a von Neumann algebra A are also C^* -coactions on A , see the discussion in [KKS22, Section 2.5]. The same notion was also considered in [Zar19] in the case when $\mathbb{G} = G$ is a classical discrete group. See also [Kal69, CKN72] in the case of a single automorphism. One should be aware that in the case of classical actions of groups on locally compact spaces one recovers the notion of *topological* freeness ; we shall nevertheless keep the terminology of [MT07] and [Zar19].

In this section we denote $p_0 \in c_c(\mathbb{G})$ the support projection of the counit, which coincides with the minimal central projection $p_u \in c_c(\mathbb{G})$ corresponding to the trivial corepresentation $1 \in I$. Recall that for a coaction $\alpha : A \rightarrow M(c_0(\mathbb{G}) \otimes A)$, $a \in A$ and $u \in I$ we denote $\alpha_u(a) = (p_u \otimes 1)\alpha(a) \in B(H_u) \otimes A$.

Definition 4.1. We say that $\mathbb{G} \curvearrowright A$ is **free** if for any $X \in M(c_0(\mathbb{G}) \otimes A)$, $X(1 \otimes a) = \alpha(a)X$ for all $a \in A$ implies $X \in p_0 \otimes Z(A)$. Equivalently, for every $1 \neq u \in I$ and every $X \in B(H_u) \otimes A$, $\forall a \in A \quad X(1 \otimes a) = \alpha_u(a)X \implies X = 0$.

Proposition 4.2. *Let $\mathbb{G} = G$ be a discrete group acting on a G - C^* -algebra A . We have that $G \curvearrowright A$ is free (in our sense) if and only if G acts freely on A (in the sense of [Zar19]), i.e. $\alpha_g(a)b = ba$ for all $a \in A$ and some $g \in G \setminus \{e\}$ implies $b = 0$. In particular, if $A = C(X)$ is commutative then $G \curvearrowright C(X)$ is free if and only if $G \curvearrowright X$ is topologically free.*

Proof. Since $B(H_g) \simeq \mathbb{C}$ for every $g \in I = G$, the equation $X(1 \otimes a) = \alpha(a)X$, for $X \in B(H_g) \otimes A$, is equivalent to $Xa = \alpha_g(a)X$ for every $g \in G$. This establishes the claim that $G \curvearrowright A$ is free if and only if G acts freely on A .

Now, assume $G \curvearrowright C(X)$ is free. Take $g \in G$ and an open subset $U \subseteq \text{Fix}(g)$. Take any $b \in C(X)$ whose support is contained in U . Then for any $a \in C(X)$ we have $\alpha_g(a)b = ab$, hence $b = 0$. This shows that $\text{Fix}(g)$ has empty interior as desired. Conversely, suppose that $G \curvearrowright C(X)$ is not free. Then there exists $b \in C(X)$, $b \neq 0$ and $g \in G$, $g \neq e$ such that $b\alpha_g(a) = ab$ for every $a \in C(X)$. Let U be an non empty open subset contained in the support of b . Then $U \subseteq \text{Fix}(g)$. \square

Compare the following with [MT07, Lemma 2.8], [MT07, Theorem 2.14], and [KS19, Section 7.4]. Our theorem is a quantum analogue of [Zar19, Theorem 3.2] and the statement 2. \iff 3. of our theorem is a C^* -algebraic analogue of [MT07, Theorem 2.14].

Recall that $A \rtimes_r \mathbb{G}$ is the closed subspace of $M(K(\ell^2(\mathbb{G})) \otimes A)$ generated by elements $(x \otimes 1)\alpha(a)$ where $x \in C_r^*(\mathbb{G})$ and $a \in A$. We can and will identify $C_r^*(\mathbb{G})$ and A with subalgebras of $M(A \rtimes_r \mathbb{G})$ — moreover in our setting $A \rtimes_r \mathbb{G}$ will be unital. We denote $E_0 = \omega_{\xi_0} \otimes \text{id} : A \rtimes_r \mathbb{G} \rightarrow A$ the canonical conditional expectation, where $\xi_0 \in \ell^2(\mathbb{G})$ is the canonical $C_r^*(\mathbb{G})$ -cyclic vector.

Theorem 4.3. *Let \mathbb{G} be a DQG and A be a \mathbb{G} - C^* -algebra. The following are equivalent:*

- (1) *there exists a unique conditional expectation $A \rtimes_r \mathbb{G} \rightarrow A$;*
- (2) $\alpha(A)' \cap (A \rtimes_r \mathbb{G}) = Z(\alpha(A))$;
- (3) $\mathbb{G} \curvearrowright^\alpha A$ is free.

Proof. 1 \implies 2. This follows from [Zar19, Proposition 3.1] while using the fact that the canonical conditional expectation $E_0 : A \rtimes_r \mathbb{G} \rightarrow A$ is faithful.

2 \implies 3. Take $1 \neq u \in I$ and $X \in B(H_u) \otimes A$ such that $X(1 \otimes a) = \alpha_u(a)X$ for all $a \in A$. Denote $X^\alpha = (\text{id} \otimes \alpha)X \in B(H_u) \otimes M(c_0(\mathbb{G}) \otimes A)$. Then

$$\begin{aligned} X^\alpha(1 \otimes \alpha(a)) &= ((\text{id} \otimes \alpha)\alpha_u(a))X^\alpha = (p_u \otimes 1 \otimes 1)W_{12}^* \alpha(a)_{23} W_{12} X^\alpha \\ &= (u^* \otimes 1)(1 \otimes \alpha(a))(u \otimes 1)X^\alpha, \end{aligned}$$

where we identify $p_u c_0(\mathbb{G})$ with $B(H_u)$. By assumption we obtain

$$(u \otimes 1)X^\alpha \in (1 \otimes \alpha(A))' \cap (B(H_u) \otimes (A \rtimes_r \mathbb{G})) = B(H_u) \otimes Z(\alpha(A)),$$

hence $X^\alpha \in (u \otimes 1)(B(H_u) \otimes \alpha(A))$. Applying the canonical conditional expectation E_0 on the second leg of $B(H_u) \otimes (A \rtimes_r \mathbb{G})$ we obtain $X = 0$, since for $u \neq 1$ we have $(\text{id} \otimes E_0)(u) = 0$.

3 \implies 1. Let $E : A \rtimes \mathbb{G} \rightarrow A$ be a conditional expectation; we have in particular $E((x \otimes 1)\alpha(a)) = E(x)a$ and $E(\alpha(a)(x \otimes 1)) = aE(x)$ for $x \in C_r^*(\mathbb{G})$, $a \in A$. Denote $X = (\text{id} \otimes E)(W_{12}^*) \in M(c_0(\mathbb{G}) \otimes A)$, where $W_{12}^* \in M(c_0(\mathbb{G}) \otimes A \rtimes_r \mathbb{G})$. For $a \in A$ we have $W_{12}^*(1 \otimes \alpha(a)) = (\Delta \otimes \text{id})\alpha(a)W_{12}^* = (\text{id} \otimes$

$\alpha(a)W_{12}^*$. Applying $\text{id} \otimes E$ yields $X(1 \otimes a) = \alpha(a)X$. By assumption this implies $X = p_0 \otimes a_0$ with $a_0 \in Z(A)$; in particular we have $(\text{id} \otimes E)(w) = 0$ for any $1 \neq w \in I$, thus $E = E_0$. \square

In the special case of $A = C(\partial_F \mathbb{G})$, freeness implies the PAP, and in particular, C^* -simplicity. However, it also implies unimodularity and so freeness is probably not the appropriate notion to consider for boundary actions of non-unimodular discrete quantum groups. Moreover it is not clear that freeness passes from G -boundaries to the Furstenberg boundary as does topological freeness in the classical case.

Corollary 4.4. *Let $\mathbb{G} \curvearrowright A$ be a free action.*

- (1) *If A is \mathbb{G} -injective then \mathbb{G} is unimodular.*
- (2) *If $A = C(\partial_F \mathbb{G})$ is the Furstenberg boundary then \mathbb{G} has the PAP.*

Proof. 1. Assume \mathbb{G} is not unimodular. In particular h is not \mathbb{G} -equivariant on $C_r^*(\mathbb{G})$, see Section 3.1, hence the canonical expectation $E_0 : A \rtimes_r \mathbb{G} \rightarrow A$ is not \mathbb{G} -equivariant either. On the other hand, since A is \mathbb{G} -injective, the identity map $A \rightarrow A$ extends to a \mathbb{G} -equivariant ucp projection $A \rtimes_r \mathbb{G} \rightarrow A$. Then Theorem 4.3 implies that $\mathbb{G} \curvearrowright A$ cannot be free.

2. Let $\Psi : C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ be a \mathbb{G} -equivariant ucp map. Using the \mathbb{G} -equivariant inclusion $C_r^*(\mathbb{G}) \subseteq C(\partial_F \mathbb{G}) \rtimes_r \mathbb{G}$ and \mathbb{G} -injectivity, we obtain a \mathbb{G} -equivariant ucp extension $\tilde{\Psi} : C(\partial_F \mathbb{G}) \rtimes_r \mathbb{G} \rightarrow C(\partial_F \mathbb{G})$. By \mathbb{G} -rigidity $\tilde{\Psi}$ restricts to the identity map on $C(\partial_F \mathbb{G})$, hence it is a conditional expectation. Theorem 4.3 implies $\tilde{\Psi} = E_0$, hence $\Psi = 1h$ and Theorem 3.24 gives us the PAP, since \mathbb{G} is unimodular by the first point. \square

If $\mathbb{G} = G$ is a discrete group, it is a well-known and easy fact that the action $G \curvearrowright^\Delta \ell^\infty(G)$ is free. We have the following converse:

Corollary 4.5. *The action $\mathbb{G} \curvearrowright \ell^\infty(\mathbb{G})$ is free if and only if $\ell^\infty(\mathbb{G})$ is commutative.*

Proof. We apply the von Neumann version of Theorem 4.3, namely [MT07, Theorem 2.14], which says that $\mathbb{G} \curvearrowright \ell^\infty(\mathbb{G})$ is free if and only if $\ell^\infty(\mathbb{G})' \cap (\ell^\infty(\mathbb{G}) \rtimes \mathbb{G})'' = Z(\ell^\infty(\mathbb{G}))$. Since $(\ell^\infty(\mathbb{G}) \rtimes \mathbb{G})'' = B(\ell^2(\mathbb{G}))$ this is equivalent to commutativity of $\ell^\infty(\mathbb{G})'$, hence of $\ell^\infty(\mathbb{G})$ which is standardly represented on $\ell^2(\mathbb{G})$. \square

4.2. Faithfulness. In this subsection we will prove an analogue of Theorem 4.3 for faithfulness of actions. The notion of faithfulness was introduced in [KKSv22] in connection with boundary actions. We give below four new characterizations of faithfulness, which allow in particular to see that freeness implies faithfulness, by comparing e.g. point 3. of Proposition 4.7 and the definition of freeness, or Theorem 4.8 and Theorem 4.3.

Definition 4.6 ([KKSv22]). The action $\mathbb{G} \curvearrowright^\alpha A$ is **faithful** if $N_\alpha := \{\mathcal{P}_\mu(a) : \mu \in A^*, a \in A\}'' = \ell^\infty(\mathbb{G})$.

Recall that the *cockernel* N_α of α is a Baaĳ-Vaes subalgebra of $\ell^\infty(\mathbb{G})$ [KKSv22, Proposition 2.9], in particular it can be realized as the group von Neumann algebra $\ell^\infty(\mathbb{H})$ of a closed quantum subgroup $\mathbb{H} \subset \hat{\mathbb{G}}$ of the dual of \mathbb{G} [BV05, Proposition 10.5], and there is a *group-like projection* $P_\alpha \in \ell^\infty(\mathbb{G})$ such that $N_\alpha = \{f \in \ell^\infty(\mathbb{G}) \mid (1 \otimes P_\alpha)\Delta(f) = f \otimes P_\alpha\}$ [FK18, Theorem 3.1].

Proposition 4.7. *Let \mathbb{G} be a DQG and A be a \mathbb{G} - C^* -algebra. TFAE:*

- (1) $\mathbb{G} \curvearrowright^\alpha A$ is faithful;
- (2) if $\varphi \in \ell^1(\mathbb{G})$ satisfies $\varphi * \mu = \mu$ for every $\mu \in A^*$, then $\varphi = \epsilon$;
- (3) if $f \in \ell^\infty(\mathbb{G})$ satisfies $(f \otimes 1)\alpha(a) = f \otimes a$ for every $a \in A$, then $f \in \mathbb{C}p_0$;

Proof. 1 \Rightarrow 2. Assume there exists $\epsilon \neq \varphi \in \ell^1(\mathbb{G})$ such that $\varphi * \mu = \mu$ for every $\mu \in A^*$. Then, $\psi = \varphi - \epsilon$ is a non-zero element of $\ell^1(\mathbb{G})$ such that $\psi * \mu = 0$ for all $\mu \in A^*$. In particular, $\psi(\mathcal{P}_\mu(a)) = 0$ for all $\mu \in A^*$ and $a \in A$. This shows that $\psi|_{N_\alpha} = 0$ and hence $N_\alpha \neq \ell^\infty(\mathbb{G})$.

2 \Rightarrow 3. Assume that there exists $f \in \ell^\infty(\mathbb{G})$, $f \notin \mathbb{C}p_0$, such that $(f \otimes 1)\alpha(a) = f \otimes a$ for every $a \in A$. We can choose $\varphi \in \ell^1(\mathbb{G})$ such that $\varphi(f(1 - p_0)) \neq 0$ and $\varphi(f) = 1$. Then, for $\mu \in A^*$ and $a \in A$, an application of $\varphi \otimes \mu$ to the above equation implies $(\varphi f) * \mu(a) = \mu(a)$. In other words, $(\varphi f) * \mu = \mu$ for every $\mu \in A^*$, but we have $\varphi f \neq \epsilon$.

3 \Rightarrow 1. Let $P_\alpha \in N_\alpha$ be the group-like projection corresponding to N_α . For any $f \in N_\alpha$ we have $(1 \otimes P_\alpha)\Delta(f) = f \otimes P_\alpha$, hence $P_\alpha f = \epsilon(f)P_\alpha$. Applying this to $f = \mathcal{P}_\mu(a)$, for $a \in A$, $\mu \in A^*$, we get

$(P_\alpha \otimes \mu)\alpha(a) = \mu(a)P_\alpha$. It follows that $(P_\alpha \otimes 1)\alpha(a) = P_\alpha \otimes a$ for all $a \in A$, which by assumption implies $P_\alpha \in \mathbb{C}p_0$. Since p_0 is the support of the co-unit, the condition $(1 \otimes P_\alpha)\Delta(f) = f \otimes P_\alpha$ becomes void and we have $N_\alpha = \ell^\infty(\mathbb{G})$. \square

Note that every \mathbb{G} -invariant state $\tau : C_r^*(\mathbb{G}) \rightarrow \mathbb{C}$ extends by \mathbb{G} -injectivity to a \mathbb{G} -equivariant conditional expectation $E : C(\partial_F \mathbb{G}) \rtimes_r \mathbb{G} \rightarrow C(\partial_F \mathbb{G})$ such that $E(C_r^*(\mathbb{G})) = \mathbb{C}1$. This fact was exploited in [KKS22] to show that faithfulness of $\mathbb{G} \curvearrowright C(\partial_F \mathbb{G})$ implies that h is the only possible \mathbb{G} -invariant state on $C_r^*(\mathbb{G})$, see also [AS24]. In the same spirit, our next result establishes a characterization of faithfulness of $\mathbb{G} \curvearrowright A$ in terms of the uniqueness of conditional expectations $A \rtimes_r \mathbb{G} \rightarrow A$ that restrict to states on $C_r^*(\mathbb{G})$.

Theorem 4.8. *Let \mathbb{G} be a DQG and A be a \mathbb{G} -C*-algebra. TFAE:*

- (1) *there exists a unique conditional expectation $E : A \rtimes_r \mathbb{G} \rightarrow A$ such that $E(C_r^*(\mathbb{G})) = \mathbb{C}1$;*
- (2) *$\alpha(A)' \cap (C_r^*(\mathbb{G}) \otimes 1) = \mathbb{C}1$;*
- (3) *$\mathbb{G} \curvearrowright^\alpha A$ is faithful.*

Proof. The proofs of $(2 \Rightarrow 3)$ and $(3 \Rightarrow 1)$ follow from essentially the same reasoning as in the proof of Theorem 4.3. The proof of $(1 \Rightarrow 2)$ follows from the same argument used in the proof of [Zar19, Proposition 3.1] with only a minor adjustment.

$1 \Rightarrow 2$. The canonical expectation $E_0 : A \rtimes_r \mathbb{G} \rightarrow A$ is the unique conditional expectation satisfying $E_0(C_r^*(\mathbb{G})) = h(C_r^*(\mathbb{G})) = \mathbb{C}$. Let $x \in \alpha(A)' \cap (C_r^*(\mathbb{G}) \otimes 1)$ be a self-adjoint element with $\|x\| < 1$, so that $1 - x \in \alpha(A)' \cap (C_r^*(\mathbb{G}) \otimes 1)$ is positive and invertible. Then $1 - E_0(x) \in \mathbb{C}1$ is positive and invertible. Now, define a ucp map $\theta : A \rtimes_r \mathbb{G} \rightarrow A$ by setting

$$\theta(z) = E_0((1 - x)^{1/2} z (1 - x)^{1/2}) (1 - E_0(x))^{-1}.$$

It is clear that $\theta(a) = a$ for $a \in A$ and $\theta(C_r^*(\mathbb{G})) = \mathbb{C}$, where the latter follows because $(1 - x)^{1/2} \in C_r^*(\mathbb{G})$. Therefore $\theta = E_0$ by assumption. So,

$$E_0(x)(1 - E_0(x)) = E_0((1 - x)^{1/2} x (1 - x)^{1/2}) = E_0(x - x^2)$$

which implies that $E_0(x^2) = E_0(x)^2$ and x is in the multiplicative domain of E_0 . We have thus shown that E_0 restricts to a *-character on $\alpha(A)' \cap (C_r^*(\mathbb{G}) \otimes 1)$. Since E_0 is faithful, this restriction is injective, hence we must have $\alpha(A)' \cap (C_r^*(\mathbb{G}) \otimes 1) = \mathbb{C}1$.

$2 \Rightarrow 3$. Take $1 \neq u \in I$ and $x \in B(H_u)$ such that $\alpha(a)(x \otimes 1) = x \otimes a$ for all $a \in A$. An application of $\text{id} \otimes \alpha$ shows that

$$\begin{aligned} (x \otimes 1 \otimes 1)\alpha(a)_{23} &= (\text{id} \otimes \alpha)\alpha(a)(x \otimes 1 \otimes 1) = W_{12}^* \alpha(a)_{23} W_{12} (x \otimes 1 \otimes 1) \\ &= u_{12}^* \alpha(a)_{23} u_{12} (x \otimes 1 \otimes 1). \end{aligned}$$

Therefore, $(u \otimes 1)(x \otimes 1 \otimes 1) \in (1 \otimes \alpha(A))'$. Then for any $\varphi \in B(H_u)^*$ the element $((x\varphi) \otimes \text{id})(u) \otimes 1 \in \alpha(A)' \cap (C_r^*(\mathbb{G}) \otimes 1)$ is scalar by assumption, and since $u \neq 1$ this is only possible if it vanishes, so that necessarily $x = 0$. This shows that the third characterization of Proposition 4.7, is satisfied, since we must have $x := p_u f = 0$ for all $u \neq 1$.

$3 \Rightarrow 1$. As in the proof of Theorem 4.3, if $E : A \rtimes_r \mathbb{G} \rightarrow A$ is a conditional expectation and $X = (\text{id} \otimes E)(W_{12}^*) \in M(c_0(\mathbb{G}) \otimes A)$, the coaction equation implies $X(1 \otimes a) = \alpha(a)X$ for all $a \in A$. If moreover $E(C_r^*(\mathbb{G})) = \mathbb{C}1$ we have $X = f \otimes 1$ with $f \in \ell^\infty(\mathbb{G})$ and Proposition 4.7 implies $f \in \mathbb{C}p_0$. This shows that $E = E_0$. \square

Example 4.9. Clearly, the action of \mathbb{G} on $\ell^\infty(\mathbb{G})$ is always faithful.

For examples with \mathbb{G} -boundaries, recall Theorem 6.9 from [ASK24] which proves that for \mathbb{G} exact, C*-simplicity implies the faithfulness of the action on $C(\partial_F \mathbb{G})$. This shows that the action on $C(\partial_F \mathbb{G})$ is faithful in the following cases: $\mathbb{G} = \mathbb{F}U_F$, $\mathbb{G} = \mathbb{F}O_F$ with $F\bar{F} = \pm I_N$ and $\|F\|^8 \leq \frac{3}{8} \text{Tr}(FF^*)$, and the dual \mathbb{G} of $\text{Aut}(B, \psi)$ where ψ is a δ -trace and $\dim(B) \geq 8$.

Note on the other hand that faithfulness of the action of $\mathbb{F}O_F$ on its quantum Gromov boundary $C(\partial_G \mathbb{F}O_F)$ is proved in [KKS22] without restriction on F (subject to $F\bar{F} = \pm I_N$). The faithfulness of the action on $C(\partial_F \mathbb{F}O_F)$ follows because we have a \mathbb{G} -equivariant uci map $C(\partial_G \mathbb{F}O_F) \rightarrow C(\partial_F \mathbb{F}O_F)$.

4.3. Strong C^* -Faithfulness.

Definition 4.10. Let A be a \mathbb{G} - C^* -algebra. We say $\mathbb{G} \curvearrowright A$ is C^* -**faithful** if for every $\eta > 0$ and every minimal central projection $p \in c_c(\mathbb{G})$ with $\epsilon(p) = 0$ there exists $k \in \mathbb{N}^*$ and $b \in (A \otimes M_k(\mathbb{C}))_+$ such that $\|b\| = 1$ and $\|(p \otimes b)(\alpha \otimes \text{id})(b)\| \leq \eta$. We say that $\mathbb{G} \curvearrowright A$ is **strongly C^* -faithful** if the same holds for every $\eta > 0$ and for every *finite rank* central projection $p \in c_c(\mathbb{G})$.

Remark 4.11. One could consider a strengthened version of (strong) C^* -faithfulness by allowing $\eta = 0$ — and this is indeed what we will achieve in the case of the Gromov boundary of $\mathbb{F}U_F$ below. But we think that it would be too strong in general ; and moreover we need the restriction to $\eta > 0$ to show that (strong) C^* -faithfulness passes to the universal Furstenberg boundary. We could also strengthen the definition by imposing $k = 1$, but this time our verification for the Gromov boundary of $\mathbb{F}U_F$ would not work.

We prove below that the notions of (strong) C^* -faithfulness reduce to the classical notions for actions of classical groups on classical spaces, and we make the connection with the classical notion of topological freeness. Recall that an action $G \curvearrowright X$ is called strongly faithful if for any finite subset $F \subset G \setminus \{e\}$ there exists $x \in X$ such that $gx \neq x$ for all $g \in F$, see e.g. [dlH85, Lemma 4] and [FLMMS22, Section 2.1].

Proposition 4.12. *Assume $\mathbb{G} = G$ is a classical discrete group acting on a commutative C^* -algebra $A = C_0(X)$. Then $\mathbb{G} \curvearrowright A$ is (resp. strongly) C^* -faithful iff $G \curvearrowright X$ is (resp. strongly) faithful. If the action $G \curvearrowright X$ is topologically free then it is strongly faithful, and the converse is true if X is compact and the action of G is minimal.*

Proof. Assume that $\mathbb{G} \curvearrowright A$ is strongly C^* -faithful and take $F \subset G \setminus \{e\}$ finite. Considering the characteristic function $p \in c_c(\mathbb{G})$ of F and $\eta = \frac{1}{4}$ we obtain $b \in (A \otimes M_k(\mathbb{C}))_+ \simeq C_0(X, M_k(\mathbb{C}))_+$ such that $\|b\| = 1$ and $\|b \times gb\| \leq \frac{1}{4}$ for all $g \in F$. Pick a point $x \in X$ such that $\|b(x)\| > \frac{1}{2}$. We claim that x cannot be fixed by any $g \in F$: otherwise we would have $\|(b \times gb)(x)\| = \|b(x)b(g^{-1}x)\| = \|b(x)\|^2 > \frac{1}{4}$.

For the reverse implication, take also $F \subset G \setminus \{e\}$ finite and $x \in X$ such that $x \neq gx$ for all $g \in F$. For each $g \in F$ choose open subsets $U_g, V_g \subset X$ such that $x \in U_g, g^{-1}x \in V_g$ and $U_g \cap V_g = \emptyset$. Consider $U = \bigcap_{g \in F} U_g \cap gV_g$, which is still an open subset containing x . By construction we have $U \cap g^{-1}U = \emptyset$ for all $g \in F$. Now it suffices to take $a \in C_0(X)_+$ that vanishes on $X \setminus U$ and such that $\|a\| = 1$: then for any $y \in X$ and $g \in F$ at least one of $a(y)$ and $(ga)(y)$ vanishes, so that $a \times ga = 0$. This proves strong C^* -faithfulness with $\eta = 0$ and $k = 1$.

Assume now that $G \curvearrowright X$ is topologically free and take $F \subset G \setminus \{e\}$ finite. By assumption $\text{Fix}(g)$ has empty interior for any $g \in G$. Since F is finite, $Y = \bigcup_{g \in F} \text{Fix}(g)$ still has empty interior and we can find $x \in X \setminus Y$, thus proving strong faithfulness.

Assume finally that X is compact and that $G \curvearrowright X$ is minimal but not topologically free. Then there exists $g \in G, g \neq e$ such that $\text{Fix}(g)$ contains a non-empty open subset U . By minimality, $(hU)_{h \in G}$ is an open cover of X and by compactness we can find $h_1, \dots, h_n \in G$ such that $\bigcup_i h_i U = X$. We have then $\bigcup_i \text{Fix}(h_i g h_i^{-1}) = X$, and this shows that strong faithfulness fails for the finite subset $\{h_i g h_i^{-1}; i = 1, \dots, n\} \subset G \setminus \{e\}$. \square

Proposition 4.13. *Let A be a \mathbb{G} - C^* -algebra. If $\mathbb{G} \curvearrowright^\alpha A$ is C^* -faithful, then it is faithful.*

Proof. It suffices to find, for every $\eta > 0$ and every minimal central projection $p \in c_c(\mathbb{G})$ such that $\epsilon(p) = 0$, an element $f \in N_\alpha$ such that $\|f\| \leq 1, p_0 f = p_0$ and $\|p f\| \leq \eta$. Indeed this implies that $p_0 \in N_\alpha$, which entails $N_\alpha = \ell^\infty(\mathbb{G})$, see [KKS22]. Take the element $b \in (A \otimes M_k(\mathbb{C}))_+$ given by C^* -faithfulness (with respect to η and p). By Hahn-Banach, find a state $\nu \in (A \otimes M_k(\mathbb{C}))^*$ which restrict to evaluation at 1 on $C^*(b) \simeq C(\text{Sp}(b))$, and put $f = (\text{id} \otimes \nu)[(1 \otimes b)(\alpha \otimes \text{id})(b)]$. We have $\|b\| = 1$, hence $\|(\alpha \otimes \text{id})(b)\| = 1$, and since ν is a state, $\|f\| \leq 1$. Since p_0 is the support of the co-unit we have moreover $p_0 f = \nu(b^2)p_0 = 1^2 p_0 = p_0$, and by choice of b we have $\|p f\| \leq \|(p \otimes b)(\alpha \otimes \text{id})(b)\| \leq \eta$. Finally, writing $\nu = \sum \mu_i \otimes \tau_i$ and $b = \sum a_j \otimes m_j$ with $\mu_i \in A^*, \tau_i \in M_k(\mathbb{C})^*, a_j \in A, m_j \in M_k(\mathbb{C})$ we have $f = \sum \tau_i(m_j m_k)(\text{id} \otimes \mu_i b_j)(\alpha(b_k)) \in N_\alpha$. \square

Remark 4.14. Proposition 4.12 shows that for classical groups acting on classical spaces, the implication proved in the previous proposition is an equivalence. This is not the case in general, already for classical groups acting on noncommutative C^* -algebras. Indeed, fix $\theta \in \mathbb{R}$ and consider the automorphism α of $M_2(\mathbb{C})$ given by $\alpha(a) = u_\theta^* a u_\theta$, where $u_\theta = \text{diag}(1, e^{i\theta})$. Then one can check that

for any $b \in M_2(\mathbb{C}) \otimes B(\ell^2\mathbb{N})$ positive such that $\|b\| = 1$, one has $\|b(\alpha \otimes \text{id})(b)\| \geq 1 - |1 - e^{i\theta}|$. As a result, the faithful action of \mathbb{Z} or $\mathbb{Z}/N\mathbb{Z}$ generated by α is not C^* -faithful if $0 < |1 - e^{i\theta}| < 1$. In fact, rather than C^* -faithfulness, the interesting notion for us is strong C^* -faithfulness “in presence of minimality” which we use as a replacement of freeness for quantum boundary actions.

As a first example let us remark that $\mathbb{G} \curvearrowright^\Delta \ell^\infty(\mathbb{G})$ is strongly C^* -faithful. Indeed, the support of the counit $p_0 \in \ell^\infty(\mathbb{G})$ is a norm one positive element and $(p \otimes p_0)\Delta(p_0) = 0$ for every central projection $p \in c_c(\mathbb{G})$ such that $\epsilon(p) = 0$.

We prove now strong C^* -faithfulness for our central example, the quantum Gromov boundary of $\mathbb{F}U_F$. The proof relies on a nice combinatorial trick uncovered by Banica in his proof of Power’s Property for $\mathbb{F}U_F$, see [Ban97, Lemme 13]. It will then follow from Corollary 5.6 and Corollary 4.21 that $\mathbb{F}U_F$ is C^* -simple. As mentioned earlier, it was already established by Banica that $\mathbb{F}U_F$ possesses the PAP and hence is C^* -simple.

Recall the notation $C(\partial_G \mathbb{F}U_F) = B_\infty = B/c_0(\mathbb{F}U_F)$ from Section 2.2, the associated coaction $\beta : B_\infty \rightarrow M(c_0(\mathbb{F}U_F) \otimes B_\infty)$ and the central elements $\pi_x \in B_\infty$, for $x \in I$, given by the formula $\pi_x = \sum_{y \in I} p_{xy}$ in B .

Proposition 4.15. *The action of $\mathbb{F}U_F$ on $C(\partial_G \mathbb{F}U_F)$ is strongly C^* -faithful.*

Proof. Let $p \in c_c(\mathbb{F}U_F)$ be a finite-rank central projection such that $\epsilon(p) = 0$. We have $p = \sum_{x \in F} p_x$ for a finite subset $F \subset I$ not containing 1. According to [Ban97, Lemme 13] we can find $N \in \mathbb{N}$ such that all irreducible subobjects of $U \otimes x \otimes U$, with $x \in F$ and $U = (u\bar{u})^N$, are indexed by words starting with u and ending with \bar{u} . Consider $b = (p_U \otimes 1)\beta(\pi_{\bar{u}}) \in B(H_U) \otimes B \subset B(H_U) \otimes \ell^\infty(\mathbb{G})$. We will show below that $(1 \otimes p \otimes 1)(\text{id} \otimes \beta)(b)$ and $(1 \otimes p \otimes 1)b_{13}$ have “disjoint support” at the right leg in $\ell^\infty(\mathbb{G})$, meaning that for any $s \in I$ we have $(1 \otimes p \otimes p_s)(\text{id} \otimes \beta)(b) = 0$ or $(1 \otimes p \otimes p_s)b_{13} = 0$. In particular $(1 \otimes p \otimes 1)b_{13}(\text{id} \otimes \beta)(b) = 0$. This identity still holds modulo $1 \otimes 1 \otimes c_0(\mathbb{G})$ and this shows that the image of $\sigma(b)$ in $B_\infty \otimes B(H_U)$ satisfies Definition 4.10, with $k = \dim U$ and even for $\eta = 0$.

To prove the above claim, take $s \in I$ such that $(1 \otimes p_s)b \neq 0$. Equivalently, $U \otimes s$ contains a corepresentation y starting with \bar{u} . By Frobenius reciprocity, this is equivalent to $s \subset U \otimes y$ (note that $\bar{U} = U$). Assume now that we have as well $(1 \otimes p \otimes p_s)(\text{id} \otimes \beta)(b) \neq 0$. Equivalently, there exists $x \in F$ such that $U \otimes x \otimes s$ contains a corepresentation z starting with \bar{u} . We have then $z \subset U \otimes x \otimes U \otimes y$. By irreducibility, there exists a subobject $t \subset U \otimes x \otimes U$ such that $z \subset t \otimes y$. But by construction t starts with u and ends with \bar{u} , while y starts with \bar{u} . Hence $t \otimes y = ty$ and $z = ty$. In particular z starts with u , a contradiction. \square

We come back to the general case. The following reformulation of (strong) C^* -faithfulness is crucial for the application to C^* -simplicity.

Proposition 4.16. *The action $\mathbb{G} \curvearrowright A$ is C^* -faithful (resp. strongly C^* -faithful) iff for every $\eta > 0$ and every minimal (resp. finite rank) central projection $p \in c_c(\mathbb{G})$ with $\epsilon(p) = 0$ there exists $k \in \mathbb{N}^*$ and $b \in (A \otimes M_k(\mathbb{C}))_+$ such that $\|b\| = 1$ and $\|p \otimes b + (p \otimes 1)(\alpha \otimes \text{id})(b)\| \leq 1 + \eta$.*

The proposition follows immediately from the following two lemmas, which are probably known to experts, by taking $A = p \otimes b$ and $B = (p \otimes 1)(\alpha \otimes \text{id})(b) \in c_0(\mathbb{G}) \otimes A \otimes M_k(\mathbb{C})$.

Lemma 4.17. *Fix $\epsilon \in]0, \frac{1}{2}]$. Then there exists $n \in \mathbb{N}$ such that, for any positive operators $A, B \in B(H)$ satisfying $\|A\| \leq 1$, $\|B\| \leq 1$ and $\|A + B\| \leq 1 + \epsilon$, we have $\|A^n B^n\| \leq 14\epsilon$.*

Proof. The assumption can also be written $0 \leq A, B \leq 1$, $0 \leq A + B \leq 1 + \epsilon$.

Step 1 (well-known). If $\zeta \in H$ satisfies $\|\zeta\| = 1$ and $\|A\zeta\|^2 \geq 1 - \epsilon^2$, then $\|A\zeta - \zeta\| \leq \epsilon$. Indeed, writing $\zeta = \zeta - A\zeta + A\zeta$ and noting that $A - A^2 \geq 0$ we can write

$$\epsilon^2 \geq \|\zeta\|^2 - \|A\zeta\|^2 = \|\zeta - A\zeta\|^2 + 2\Re(\zeta - A\zeta \mid A\zeta) \geq \|\zeta - A\zeta\|^2.$$

This implies also $(\zeta \mid A\zeta) \geq 1 - \epsilon$.

Step 2. Fix ζ as above and $\xi \in H$ such that $\|\xi\| = 1$, $\|B\xi\|^2 \geq 1 - \epsilon^2$. We have the following inequalities: $(\zeta \mid A\zeta) \geq 1 - \epsilon$, $(\xi \mid B\xi) \geq 1 - \epsilon$, $(\zeta \mid B\zeta) \geq 0$, $(\xi \mid A\xi) \geq 0$. Also,

$$(\zeta \mid A\xi) + (\xi \mid A\zeta) = 2\Re(\xi \mid A\zeta) \geq 2\Re(\xi \mid \zeta) - 2\|\xi\|\|\zeta - A\zeta\| \geq 2\Re(\zeta \mid \xi) - 2\epsilon.$$

Similarly $(\xi \mid B\zeta) + (\zeta \mid B\xi) \geq 2\Re(\zeta \mid \xi) - 2\epsilon$. Summing all these inequalities we get

$$(\zeta + \xi \mid (A + B)(\zeta + \xi)) \geq 2 + 4\Re(\zeta \mid \xi) - 6\epsilon.$$

But on the other hand we have $(\zeta + \xi | (A + B)(\zeta + \xi)) \leq \|\zeta + \xi\|^2 \|A + B\| \leq (2 + 2\Re(\zeta | \xi))(1 + \epsilon)$. Subtracting we obtain $\Re(\zeta | \xi) \leq 4\epsilon/(1 - \epsilon) \leq 8\epsilon$, if $\epsilon \leq \frac{1}{2}$. We can apply this to $-\xi$ and we thus get $|\Re(\zeta | \xi)| \leq 8\epsilon$. We can also apply this to $i\xi$ and we get $|\Im(\zeta | \xi)| \leq 8\epsilon$, so that finally $|(\zeta | \xi)| \leq 8\sqrt{2}\epsilon \leq 12\epsilon$.

This can be reformulated as follows: denote P, Q the spectral projections of A, B for the interval $[\sqrt{1 - \epsilon^2}, 1]$. Then for any $\zeta, \xi \in H$ such that $P\zeta \neq 0, Q\xi \neq 0$ we can apply the previous result to $\zeta' = P\zeta/\|P\zeta\|, \xi' = Q\xi/\|Q\xi\|$ and we get

$$|(P\zeta | Q\xi)| \leq 12\epsilon\|P\zeta\|\|Q\xi\| \leq 12\epsilon\|\zeta\|\|\xi\|.$$

This holds as well if $P\zeta = 0$ or $Q\xi = 0$, so that we have proved $\|PQ\| \leq 12\epsilon$.

Step 3. Take $n \in \mathbb{N}$ such that $(1 - \epsilon^2)^n \leq \epsilon^2$. For any unit vectors $\zeta, \xi \in H$ we have then $\|A^n(1 - P)\zeta\| \leq (1 - \epsilon^2)^{n/2}\|(1 - P)\zeta\| \leq \epsilon$ and similarly $\|B^n(1 - P)\xi\| \leq \epsilon$. We can then write

$$\begin{aligned} |(A^n\zeta | B^n\xi)| &\leq |(A^n(1 - P)\zeta | B^n\xi)| + |(A^n P\zeta | B^n(1 - Q)\xi)| + |(A^n P\zeta | B^n Q\xi)| \\ &\leq \epsilon + \epsilon + |(A^n\zeta | PQB^n\xi)| \leq 14\epsilon. \end{aligned}$$

This proves that $\|A^n B^n\| \leq 14\epsilon$. \square

The second lemma is much easier and certainly well known as well.

Lemma 4.18. *Fix $\epsilon \in [0, +\infty[$. Then for any positive operators $A, B \in B(H)$ such that $\|A\| = \|B\| = 1$ and $\|AB\| \leq \epsilon$, we have $\|A + B\| \leq 1 + 2\epsilon$.*

Proof. Since $A + B \geq A$ the assumption entails $\|A + B\| \geq 1$. Also, $0 \leq A, B \leq 1$ implies $0 \leq A^2 + B^2 \leq A + B$. We can thus write

$$\begin{aligned} \|A + B\|^2 &= \|(A + B)^2\| = \|A^2 + B^2 + AB + BA\| \\ &\leq \|A^2 + B^2\| + 2\epsilon \leq \|A + B\| + 2\epsilon\|A + B\| \end{aligned}$$

and the result follows. \square

The following is a direct but important consequence of the previous reformulation of strong C^* -faithfulness.

Corollary 4.19. *If $\Phi : A \rightarrow B$ is \mathbb{G} -equivariant, unital and completely isometric, then (strong) C^* -faithfulness passes from A to B . In particular, if a \mathbb{G} -boundary A is (strongly) C^* -faithful, then so is the universal Furstenberg boundary $C(\partial_F \mathbb{G})$.*

Theorem 4.20. *Let A be a C^* -algebra with strongly C^* -faithful action of \mathbb{G} , and let $E : A \rtimes_r \mathbb{G} \rightarrow A$ be a conditional expectation. Then the restriction of E to $C_r^*(\mathbb{G})$ factors h .*

Proof. Let A be faithfully represented on a Hilbert space H . Recall that we identify A and $C_r^*(\mathbb{G})$ as C^* -subalgebras of $A \rtimes_r \mathbb{G}$, in such a way that $(1 \otimes a)W = W\alpha(a) \in M(c_0(\mathbb{G}) \otimes A \rtimes_r \mathbb{G})$ for any $a \in A$. Denote $\Psi : C_r^*(\mathbb{G}) \rightarrow B(H)$ the restriction of E , and $X = (\text{id} \otimes \Psi)(W) \in M(c_0(\mathbb{G}) \otimes A)$. Applying E to the previous relation we obtain $(1 \otimes a)X = X\alpha(a)$.

Take $\eta > 0$ and $p \in c_c(\mathbb{G})$ a finite-rank central projection such that $\epsilon(p) = 0$. Let $k \in \mathbb{N}, b \in A \otimes M_k(\mathbb{C})$ be the elements given by Definition 4.10 for η and p . Since $\|b\| = 1$ we can find $\zeta \in H \otimes \mathbb{C}^k$ such that $\|\zeta\| \leq \sqrt{2}$ and $\|b\zeta\| = 1$. Put $\xi = b\zeta$. We have

$$\begin{aligned} (p \otimes \omega_\xi)(X \otimes 1) &= (\text{id} \otimes \omega_\zeta)[(p \otimes b)(X \otimes 1)(p \otimes b)] \\ &= (\text{id} \otimes \omega_\zeta)[(X \otimes 1)(\alpha \otimes \text{id})(b)(p \otimes b)] \end{aligned}$$

hence $\|(p \otimes \omega_\xi)(X \otimes 1)\| \leq 2\eta$.

Denoting $\Psi^k : A \rtimes_r \mathbb{G} \rightarrow B(H \otimes \mathbb{C}^k)$ the amplification $\Psi^k(a) = \Psi(a) \otimes \text{id}$, and $L(\mu) = (\mu \otimes \text{id})(W) \in C_r^*(\mathbb{G})$ for $\mu \in \ell^1(\mathbb{G})$, this implies $|\omega_\xi(\Psi^k(L(\mu)))| \leq 2\eta\|\mu\|_1$ if μ is supported on p . Note that $h(L(\mu)) = \mu(p_0)$, where p_0 is the support of ϵ , and $L(\epsilon) = 1$. Hence if now μ is supported on $p_0 + p$ we have $|(\omega_\xi \circ \Psi^k - h)(L(\mu))| \leq 2\eta\|\mu\|_1$. Writing $\xi = \sum_{i=1}^k \xi_i \otimes e_i$ and denoting $\varphi \in S(A)$ the sum of the restrictions of the forms ω_{ξ_i} to A , we have $\omega_\xi(\Psi^k(L(\mu))) = \varphi(\Psi(L(\mu)))$. Thus by letting $\eta \rightarrow 0$ we see that $h(L(\mu))$ belongs to the closure of $\{\varphi(\Psi(L(\mu))) : \varphi \in S(A)\}$. Finally since all elements $x \in \mathbb{C}[\mathbb{G}]$ can be written $x = L(\mu)$ as above, Lemma 3.22 shows that Ψ factors h . \square

Corollary 4.21. *If \mathbb{G} admits a \mathbb{G} -boundary with a strongly C^* -faithful action, then \mathbb{G} has the PAP. In particular, \mathbb{G} is C^* -simple.*

Proof. By Corollary 4.19, the action of \mathbb{G} on $C(\partial_F \mathbb{G})$ is strongly C^* -faithful. Let $\Psi : C_r^*(\mathbb{G}) \rightarrow C(\partial_F \mathbb{G})$ be a \mathbb{G} -equivariant ucp map. By \mathbb{G} -injectivity, extend it to $E : C(\partial_F \mathbb{G}) \rtimes_r \mathbb{G} \rightarrow C(\partial_F \mathbb{G})$. By \mathbb{G} -rigidity, E is a conditional expectation. We can thus apply Theorem 4.20 which shows that Ψ factors h , and by Corollary 3.24, \mathbb{G} has the PAP. By Proposition 3.9, \mathbb{G} is C^* -simple. \square

5. THE BOUNDARY OF $\mathbb{F}U_F$ IS A \mathbb{G} -BOUNDARY

In this section we prove that $C(\partial_G \mathbb{F}U_F)$ is an $\mathbb{F}U_F$ -boundary, using the unique stationarity method from [KKS22]. We use the notation from Section 2.2. The main object of the proof is the canonical “harmonic state” on $C(\partial \mathbb{F}U_F)$ that we describe now.

Denote $\text{qTr}_n = \sum_{|x|=n} \text{qTr}_x$, $\dim_q(n) = \sum_{|x|=n} \dim_q(x)$ and $\text{qtr}_n = \text{qTr}_n(\cdot) / \dim_q(n)$ — which is different from $\sum_{|x|=n} \text{qtr}_x$. We have the following convolution formula in $\ell^1(\mathbb{F}U_F)$:

$$(3) \quad \text{qTr}_x * \text{qTr}_y = \sum_z m(z, x \otimes y) \text{qTr}_z,$$

where $m(z, x \otimes y) = \dim \text{Hom}(z, x \otimes y)$. According to this formula and to the fusion rules of $\mathbb{F}U_F$ we have $\text{qTr}_1 * \text{qTr}_n = \text{qTr}_{n+1} + 2\text{qTr}_{n-1}$ for $n \geq 1$. Since $\text{qTr}_1(1) = \dim_q(u) + \dim_q(\bar{u}) = 2[2]_q$, this gives in particular $2[2]_q \dim_q(n) = \dim_q(n+1) + 2\dim_q(n-1)$. Denoting κ the unique number in $]0, 1[$ such that $\kappa + \kappa^{-1} = \sqrt{2}[2]_q$, this yields $\dim_q(n) = \sqrt{2}^n [n+1]_\kappa$.

More generally, if μ is a probability measure on I we denote $\psi_\mu = \sum \mu(x) \text{qtr}_x$ the associated state on $c_0(\mathbb{G})$. It is shown in [VV10] that if μ is generating and finitely supported, then the sequence ψ_μ^{*n} converges $*$ -weakly on $B = C(\beta_G \mathbb{F}U_F) \subset \ell^\infty(\mathbb{F}U_F)$ to a KMS state $\omega \in B^*$, which factors to a faithful state still denoted ω on B_∞ . Moreover the “ μ -harmonic state” ω is ψ_μ -stationary, meaning that $\psi_\mu * \omega = \omega$, and the corresponding Poisson map $P_\mu : C(\partial_G \mathbb{F}U_F) \rightarrow H^\infty(\mathbb{F}U_F, \mu)$ is completely isometric.

Now we restrict to the case $\mu = \mu_1 := \frac{1}{2}(\delta_u + \delta_{\bar{u}})$, i.e. $\psi_\mu = \text{qtr}_1$, which is sufficient to prove that $C(\partial_G \mathbb{F}U_F)$ is an $\mathbb{F}U_F$ -boundary. From the remarks above it follows that qtr_1^{*n} is a convex combination of the states qtr_k , $0 \leq k \leq n$, $k+n$ even. We show below that already the sequence $(\text{qtr}_n)_n$ converges $*$ -weakly on B to a state $\omega \in B^*$, which then coincides clearly with the harmonic state of the previous paragraph. We also compute ω explicitly, although we will not need it in the sequel.

Lemma 5.1. *Fix $x \in I$, $x \neq 1$, and $a \in p_x \ell^\infty(\mathbb{F}U_F) \simeq B(H_x)$. Then $\text{qtr}_n(\psi_{x,\infty}(a))$ converges as $n \rightarrow \infty$. Moreover, let $l \in \mathbb{N}^*$ be the unique integer such that $x \simeq x' \otimes \alpha^{(l)}$ with $x' \in I$, $\alpha \in \{u, \bar{u}\}$. Then the limit is:*

$$\lim_{n \rightarrow \infty} \text{qtr}_n(\psi_{x,\infty}(a)) = \omega(\psi_{x,\infty}(a)) = \text{qtr}_x(a) \times \omega_I(\partial I(x)), \quad \text{where}$$

$$\omega_I(\partial I(x)) = \dim_q(x) \left(\frac{\kappa}{\sqrt{2}} \right)^{|x|} \left(1 - \frac{\kappa}{\sqrt{2}} \frac{[l]_q}{[l+1]_q} \right).$$

Proof. If $z \subset x \otimes y$ with $|z| = |x| + |y|$ we have $\text{qTr}_z = \text{qTr}_x \otimes \text{qTr}_y$ on $B(H_z) \subset B(H_x) \otimes B(H_y)$. Observe moreover that $(\text{id} \otimes \text{qTr}_y)(P(z, x \otimes y)) = (\dim_q(z) / \dim_q(x)) \text{id}_x$ since this element of $B(H_x)$ is an intertwiner, hence a multiple of id_x . From this we get

$$\text{qTr}_z(\psi_{x,z}(a)) = (\text{qTr}_x \otimes \text{qTr}_y)[P(z, x \otimes y)(a \otimes \text{id}_y)] = \frac{\dim_q(z)}{\dim_q(x)} \text{qTr}_x(a).$$

Summing over $z \in I_n$, with $n \geq |x|$, we get $\text{qTr}_n(\psi_{x,\infty}(a)) = \dim_q(x, n) \text{qtr}_x(a)$, where we put $\dim_q(x, n) = \sum \{ \dim_q(z) \mid |z| = n, z \supset x \}$. Thus it remains to show that $\dim_q(x, n) / \dim_q(n)$ has the limit given in the statement when $n \rightarrow \infty$. According to the fusion rules the sequence $(\dim_q(x, n))_n$ satisfies the same recursion equation than $(\dim_q(n))_n$, but with a different initialization. This already shows the existence of the limit.

To compute the limit, first note that $\dim_q(x, n) = \dim_q(x') \dim_q(\alpha^{(l)}, n - |x'|)$. Moreover $\dim_q(n - |x'|) / \dim_q(n) \rightarrow (\kappa / \sqrt{2})^{|x'|}$ as $n \rightarrow \infty$, thanks to the formula $\dim_q(n) = \sqrt{2}^n [n+1]_\kappa$. As a result it suffices to consider the case $x = \alpha^{(l)}$. In this case we obtain the result by solving the above recursion relation with the initialization $\dim_q(x, l) = \dim_q(x) = [l+1]_q$ and $\dim_q(x, l+1) = \dim_q(xu) + \dim_q(x\bar{u}) = [2]_q [l+1]_q + [l+2]_q$. \square

The result above completely determines the harmonic state ω on $C(\partial_G \mathbb{F}U_F)$ and in particular the classical probability measure ω_I on βI . We get in particular $\omega_I(\partial I(x)) \leq ([2]_q \kappa / \sqrt{2})^{|x|}$. One can check that $[2]_q \kappa / \sqrt{2} < 1$, and thus ω_I is non-atomic. Note that for arbitrary μ it is already shown in [VV10] that ω_I has no atoms at points of the form $x\alpha^{(\infty)}$, and in [VV08] that ω_I has no atoms at

all. The following Lemma can be seen as a quantitative and uniform version of the fact that ω_I has no atoms at points of the form $x\alpha^{(\infty)}$, in the case $\mu = \mu_1$: indeed by taking the limit $n \rightarrow \infty$ it yields $\omega_I(\partial I(x\alpha^{(k+1)})) \leq 2^{-k}$ for any $x \in I$, $k \in \mathbb{N}$.

Lemma 5.2. *Denote $I_n^{(p,k)} \subset I_n$ the set of words $y \in I_n$ which have a repetition of letters between the p th and $(p+k)$ th letter, i.e. we have $y \simeq y' \otimes y''$ with $p \leq |y'| \leq p+k$. Denote $\text{qtr}_n^{(p,k)} = \dim_q(n)^{-1} \sum \{\text{qTr}_y \mid y \in I_n^{(p,k)}\}$. Then for all $n \geq p+k$ we have*

$$\| \text{qtr}_n - \text{qtr}_n^{(p,k)} \| \leq 2^{-k}.$$

Proof. We have $\dim_q(n) \| \text{qtr}_n - \text{qtr}_n^{(p,k)} \| = \sum \{\text{qTr}_y(1) \mid y \in {}^c I_n^{(p,k)}\}$. Any $y \in {}^c I_n^{(p,k)} = I_n \setminus I_n^{(p,k)}$ can be written $y = y_1 \alpha^{(k+1)} y_2$ with $|y_1| = p-1$, $\alpha = u$ or \bar{u} . Observe moreover that $\dim_q(y_1 \alpha^{(k+1)} y_2) \leq \dim_q(y_1 z y_2)$ for any $z \in I_{k+1}$, hence $\text{qTr}_y(1) = \dim_q(y) \leq 2^{-k-1} \sum_{z \in I_{k+1}} \dim_q(y_1 z y_2)$. Summing over $y_1 \in I_{p-1}$, $\alpha \in \{u, \bar{u}\}$ and $y_2 \in I_{n-p-k}$ this yields $\dim_q(n) \| \text{qtr}_n - \text{qtr}_n^{(p,k)} \| \leq 2^{-k} \sum_{y_1, z, y_2} \dim_q(y_1 z y_2) = 2^{-k} \dim_q(n)$. \square

We prove now that $\partial_G \mathbb{F}U_F$ is indeed a $\mathbb{F}U_F$ -boundary, using the unique stationarity method from [KKSv22]. We still work with the harmonic state ω on $C(\partial_G \mathbb{F}U_F)$ induced by $\psi_\mu = \text{qtr}_1 \in \ell^\infty(\mathbb{F}U_F)_*$. Recall that ω is qtr_1 -stationary, meaning that $\text{qtr}_1 * \omega = \omega$, and the corresponding Poisson map $P_1 = (\text{id} \otimes \omega)\beta : C(\partial_G \mathbb{F}U_F) \rightarrow H^\infty(\mathbb{F}U_F, \text{qtr}_1) \subset \ell^\infty(\mathbb{F}U_F)$ is completely isometric [VVV10]. In this setting it follows that $\partial_G \mathbb{F}U_F$ is an $\mathbb{F}U_F$ -boundary if one can prove that ω is the *unique* qtr_1 -stationary state ν on $C(\partial_G \mathbb{F}U_F)$ [KKSv22].

This unique stationarity result was established in [HHN22] under the additional assumption that ν is invariant with respect to the adjoint action of the dual compact quantum group — in other words ω is the unique stationary state with respect to the natural action of the Drinfeld double $D(\mathbb{F}U_F)$ on $\partial_G \mathbb{F}U_F$. Considering the action of the Drinfeld double makes the situation much more rigid: for instance in the orthogonal case, $\partial_G \mathbb{F}O_F$ is the *only* non-trivial $D(\mathbb{F}O_F)$ -boundary (see Example 3.19 of [HHN22]).

We start with the intermediate goal of proving that a qtr_1 -stationary state ν on $C(\partial_G \mathbb{F}U_F)$ has no atoms at classical points of the form $x\alpha^{(\infty)} \in \partial I$. If $\mathbb{F}U_F$ is unimodular, i.e. F is a multiple of a unitary matrix, this is easy. Indeed in this case, the left and right quantum traces are equal and in particular $\ell^\infty(I) \subset \ell^\infty(\mathbb{F}U_F)$ is stable under left convolution by qtr_1 , which then corresponds to the Markov operator of a classical random walk on I . But Proposition 3.23 of [HHN22] shows that this classical random walk has a unique stationary probability measure, so that necessarily $\nu_I = \omega_I$, which is already known to have no atoms at points $x\alpha^{(\infty)}$, as recalled above.

The general case is more involved but still follows the classical strategy: if ν is a μ -stationary measure on the Gromov boundary $\partial_G F_N$ of a classical free group, the function $h : x \rightarrow \nu(\{x\})$ on $\partial_G F_N$ is summable and μ -harmonic with respect to the action of F_N on its boundary. Since all orbits of the action are infinite, this implies $h = 0$. One can also work with the associated functions $\gamma \mapsto h(\gamma \cdot x_0)$ on F_N , which are easier to manipulate in the quantum case — this is the “classical analogue” for the elements h, \bar{h} introduced below.

Fix a qtr_1 -stationary state ν on $C(\partial_G \mathbb{F}U_F)$; we still denote ν the induced state on $C(\beta_G \mathbb{F}U_F) \subset \ell^\infty(\mathbb{F}U_F)$. Denote $h^{(k)} = P_\nu(\pi_{u^{(k)}}) = (\text{id} \otimes \nu)\Delta(\pi_{u^{(k)}})$ and consider the decreasing limit $h = \lim_{k \rightarrow \infty} h^{(k)} \subset \ell^\infty(\mathbb{F}U_F)$. Similarly, denote $\bar{h}^{(k)} = P_\nu(\pi_{\bar{u}^{(k)}})$ and $\bar{h} = \lim_{k \rightarrow \infty} \bar{h}^{(k)}$. Our goal is to prove that $h = \bar{h} = 0$.

Lemma 5.3. *Consider the element $h = (h_y)_{y \in I} \in \ell^\infty(\mathbb{F}U_F)$ associated to a state $\nu \in C(\partial_G \mathbb{F}U_F)^*$ as above. For $x \in I(\bar{u})$ and $l \in \mathbb{N}$ we have $h_{u^{(2l)}x} = p_{u^{(2l)}} \otimes h_x$ in the identification $B(H_{u^{(2l)}x}) \simeq B(H_{u^{(2l)}}) \otimes B(H_x)$. Similarly for $x \in I(u)$ we have $h_{u^{(2l+1)}x} = p_{u^{(2l+1)}} \otimes \bar{h}_x$.*

Proof. Take $k > 2l$. Start with $x = 1$. We have $(p_{u^{(2l)}} \otimes p_y)\Delta(p_{u^{(k)}z}) \neq 0$ iff $u^{(k)}z \subset u^{(2l)} \otimes y$ iff $y \subset u^{(2l)} \otimes u^{(k)}z$ iff $y = u^{(k+2m)}z$ with $|m| \leq l$. In particular $(p_{u^{(2l)}} \otimes p_y)\Delta(\pi_{u^{(k)}}) \neq 0 \Rightarrow y \in I(u^{(k-2l)})$. On the other hand for $y \in I(u^{(k+2l)})$, all subobjects of $u^{(2l)} \otimes y$ are in $I(u^{(k)})$. As a result we have

$$(4) \quad p_{u^{(2l)}} \otimes \pi_{u^{(k+2l)}} \leq (p_{u^{(2l)}} \otimes 1)\Delta(\pi_{u^{(k)}}) \leq p_{u^{(2l)}} \otimes \pi_{u^{(k-2l)}}.$$

Coming back to a general $x \in I(\bar{u})$ we have as well $(p_{u^{(2l)}x} \otimes p_y)\Delta(p_{u^{(k)}z}) \neq 0$ iff $u^{(k)}z \subset u^{(2l)}x \otimes y$ iff $y \subset \bar{x}u^{(2l)} \otimes u^{(k)}z$ iff $y = \bar{x}u^{(k+2m)}z$ with $|m| \leq l$. Moreover in this case we have $u^{(2l)}x \otimes y = u^{(2l)} \otimes x \otimes \bar{x} \otimes u^{(k+2m)}z = \bigoplus_{t \subset x \otimes \bar{x}} u^{(2l)} \otimes t \otimes u^{(k+2m)}z$, and the objects t appearing in the sum start with \bar{u} , except $t = 1$. As a result the isometric morphism $V(u^{(k)}z, u^{(2l)}x \otimes y)$ has to factor as $(\text{id} \otimes V(u^{(k+2m)}z, x \otimes \bar{x}u^{(k+2m)}z))V(u^{(k)}z, u^{(2l)} \otimes u^{(k+2m)}z)$. Recalling from Section 2.2 that one can use such morphisms to compute the coproduct, we obtain for $y = \bar{x}u^{(k+2m)}z$:

$$\begin{aligned} (p_{u^{(2l)}x} \otimes p_y)\Delta(p_{u^{(k)}z}) &= (p_{u^{(2l)}} \otimes p_x \otimes p_y)(\text{id} \otimes \Delta)[(p_{u^{(2l)}} \otimes p_{u^{(k+2m)}z})\Delta(p_{u^{(k)}z})] \\ &= (p_{u^{(2l)}} \otimes p_x \otimes p_y)(\text{id} \otimes \Delta)[(p_{u^{(2l)}} \otimes 1)\Delta(p_{u^{(k)}z})], \end{aligned}$$

since z and m are uniquely determined by y . Here we are using the identifications $p_{u^{(2l)}x}\ell^\infty(\mathbb{F}U_F) \simeq B(H_{u^{(2l)}x}) \simeq B(H_{u^{(2l)}}) \otimes B(H_x) \simeq p_{u^{(2l)}}\ell^\infty(\mathbb{F}U_F) \otimes p_x\ell^\infty(\mathbb{F}U_F)$. Summing over z (first) and y gives

$$(p_{u^{(2l)}x} \otimes 1)\Delta(\pi_{u^{(k)}}) = (p_{u^{(2l)}} \otimes p_x \otimes 1)(\text{id} \otimes \Delta)[(p_{u^{(2l)}} \otimes 1)\Delta(\pi_{u^{(k)}})].$$

Now we use (4) which yields:

$$p_{u^{(2l)}} \otimes [(p_x \otimes 1)\Delta(\pi_{u^{(k+2l)}})] \leq (p_{u^{(2l)}x} \otimes 1)\Delta(\pi_{u^{(k)}}) \leq p_{u^{(2l)}} \otimes [(p_x \otimes 1)\Delta(\pi_{u^{(k-2l)}})].$$

Finally we apply $(\text{id} \otimes \text{id} \otimes \nu)$ to obtain $p_{u^{(2l)}} \otimes h_x^{(k+2l)} \leq h_{u^{(2l)}x}^{(k)} \leq p_{u^{(2l)}} \otimes h_x^{(k-2l)}$. Letting $k \rightarrow \infty$ yields $h_{u^{(2l)}x} = p_{u^{(2l)}} \otimes h_x$.

The odd case works similarly. We have $(p_{u^{(2l+1)}x} \otimes p_y)\Delta(p_{u^{(k)}z}) \neq 0$ iff $u^{(k)}z \subset u^{(2l+1)}x \otimes y$ iff $y \subset \bar{x}\bar{u}^{(2l+1)} \otimes u^{(k)}z$ iff $y = \bar{x}\bar{u}^{(k+2m+1)}z$ with $-l-1 \leq m \leq l$. When $x = 1$ we obtain

$$p_{u^{(2l+1)}} \otimes \pi_{\bar{u}^{(k+2l+1)}} \leq (p_{u^{(2l+1)}} \otimes 1)\Delta(\pi_{u^{(k)}}) \leq p_{u^{(2l+1)}} \otimes \pi_{\bar{u}^{(k-2l-1)}}.$$

In general if $y = \bar{x}\bar{u}^{(k+2m+1)}z$ we obtain successively:

$$\begin{aligned} (p_{u^{(2l+1)}x} \otimes p_y)\Delta(p_{u^{(k)}z}) &= (p_{u^{(2l+1)}} \otimes p_x \otimes p_y)(\text{id} \otimes \Delta)[(p_{u^{(2l+1)}} \otimes p_{\bar{u}^{(k+2m+1)}z})\Delta(p_{u^{(k)}z})], \\ (p_{u^{(2l+1)}x} \otimes 1)\Delta(\pi_{u^{(k)}}) &= (p_{u^{(2l+1)}} \otimes p_x \otimes 1)(\text{id} \otimes \Delta)[(p_{u^{(2l+1)}} \otimes 1)\Delta(\pi_{u^{(k)}})] \quad \text{and} \\ p_{u^{(2l+1)}} \otimes [(p_x \otimes 1)\Delta(\pi_{\bar{u}^{(k+2l+1)}})] &\leq (p_{u^{(2l+1)}x} \otimes 1)\Delta(\pi_{u^{(k)}}) \leq \\ &\leq p_{u^{(2l+1)}} \otimes [(p_x \otimes 1)\Delta(\pi_{\bar{u}^{(k-2l-1)}})]. \end{aligned}$$

Applying $(\text{id} \otimes \text{id} \otimes \nu)$ this yields $p_{u^{(2l+1)}} \otimes \bar{h}_x^{(k+2l+1)} \leq h_{u^{(2l+1)}x}^{(k)} \leq p_{u^{(2l+1)}} \otimes \bar{h}_x^{(k-2l-1)}$ and the result follows by taking the limit $k \rightarrow \infty$. \square

Proposition 5.4. *Let ν be a qtr_1 -stationary state on $C(\partial_G\mathbb{F}U_F)$. Then ν_I has no atoms at points of the form $x\alpha^{(\infty)}$, i.e. we have $\lim_{k \rightarrow \infty} \nu(\pi_{x\alpha^{(k)}}) = 0$ for all $x \in I$.*

Proof. From the stationarity property $(\text{qtr}_1 \otimes \nu)\Delta = \nu$ one easily obtains harmonicity of the elements h, \bar{h} considered in Lemma 5.3: we have $(\text{id} \otimes \text{qtr}_1)\Delta(h) = h$ and $(\text{id} \otimes \text{qtr}_1)\Delta(\bar{h}) = \bar{h}$. We have $(p_x \otimes \text{qtr}_1)\Delta(p_z) \neq 0$ iff $z \subset x \otimes (u \oplus \bar{u})$, hence the harmonicity equation for h can be written

$$(5) \quad h_x = \frac{1}{2}(p_x \otimes \text{qtr}_\alpha)\Delta(h_{x\alpha}) + \frac{1}{2}(p_x \otimes \text{qtr}_{\bar{\alpha}})\Delta(h_{x\bar{\alpha}}) + \frac{1}{2}(p_x \otimes \text{qtr}_{\bar{\alpha}})\Delta(h_{x'})$$

if $x = x'\alpha$ with $\alpha \in \{u, \bar{u}\}$ (we put $h_{x'} = 0$ if $x = 1$).

Consider the function $\varphi : I \rightarrow \mathbb{R}_+$, $x \mapsto \text{qtr}_x(h_x)$. Since quantum traces are compatible with subobjects we have, for $z \subset x \otimes y$ and $a_z \in B(H_z)$: $(\text{qTr}_x \otimes \text{qTr}_y)\Delta(a_z) = \text{qTr}_z(a_z)$. Thus, applying qtr_x to (5) we obtain

$$\varphi(x) = \frac{1}{2}p(x, x\alpha)\varphi(x\alpha) + \frac{1}{2}p(x, x\bar{\alpha})\varphi(x\bar{\alpha}) + \frac{1}{2}p(x, x')\varphi(x'),$$

where $p(x, y) = \dim_q(y) / \dim_q(x) \dim_q(u)$ if x, y are neighbours in the classical Cayley graph of $\mathbb{F}U_F$, i.e. $y \subset x \otimes (u \oplus \bar{u})$. Note that with our notation $x\alpha = x \otimes \alpha$ so that $p(x, x\alpha) = 1$, and on the other hand $x\bar{\alpha} \oplus x' \simeq x \otimes \bar{\alpha}$ so that $p(x, x\bar{\alpha}) + p(x, x') = 1$. In short, φ is harmonic with respect to the classical random walk on I with transition probabilities $p(x, y)$.

Besides, we claim that $\sum_{x \in I^+(\bar{u})} \varphi(x) \leq 1$, where $I^+(\bar{u}) = I(\bar{u}) \cup \{1\}$. To see this, take a finite subset $F \subset I^+(\bar{u})$ and put $k = 1 + \max_{x \in F} |x|$, so that the subsets $I(\bar{x}u^{(k)}) \subset \partial I$ are pairwise disjoint when $x \in F$. Consider the positive elements

$$\eta_x = (\text{qtr}_x \otimes \text{id})\Delta(\pi_{u^{(k)}}) \in C(\beta_G\mathbb{F}U_F).$$

We have $p_y\eta_x \neq 0$ iff there exists $z \in I$ such that $u^{(k)}z \subset x \otimes y$, equivalently, $y \subset \bar{x} \otimes u^{(k)}z$. If $x \in I^+(\bar{u})$ we have $\bar{x} \otimes u^{(k)}z = \bar{x}u^{(k)}z$, hence we get $\eta_x = \pi_{\bar{x}u^{(k)}}\eta_x$. The projections $\pi_{\bar{x}u^{(k)}}$ are pairwise

orthogonal when $x \in F$, hence we have $\|\sum_{x \in F} \eta_x\| = \max_{x \in F} \|\eta_x\| \leq 1$. Since ν is a state we can then write

$$1 \geq \nu\left(\sum_{x \in F} \eta_x\right) = \sum_{x \in F} \text{qtr}_x(h_x^{(k)}) \geq \sum_{x \in F} \text{qtr}_x(h_x) = \sum_{x \in F} \varphi(x).$$

This holds for any finite $F \subset I^+(\bar{u})$, hence the claim follows. Working with the elements $\bar{\eta}_x = (\text{qtr}_x \otimes \text{id})\Delta(\pi_{\bar{u}^{(k)}}) \in C(\beta_G \mathbb{F}U_F)$ we see similarly that $\sum_{x \in I^+(\bar{u})} \bar{\varphi}(x) \leq 1$, where $\bar{\varphi}(x) = \text{qtr}_x(\bar{h}_x)$.

In particular φ attains a maximum M on $I^+(\bar{u})$ and $\bar{\varphi}$ attains a maximum \bar{M} on $I^+(u)$. For $y \in I \setminus I^+(\bar{u})$ we can write $y = u^{(2l)}x$ with $x \in I(\bar{u})$ or $y = u^{(2l+1)}x$ with $x \in I(u)$. In the first case we apply $\text{qtr}_{u^{(2l)}x} = \text{qtr}_{u^{(2l)}} \otimes \text{qtr}_x$ to the identity of Lemma 5.3, which yields $\varphi(u^{(2l)}x) = \varphi(x) \leq M$. In the second case we have similarly $\varphi(u^{(2l+1)}x) = \bar{\varphi}(x) \leq \bar{M}$. Altogether we see that φ attains the maximum $\max(M, \bar{M})$ on I . Since it is harmonic, it has to be constant. Since it is summable on $I^+(\bar{u})$, which is infinite, we have in fact $\varphi = 0$. Hence $h = 0$. Similarly $\bar{\varphi} = 0$ and $\bar{h} = 0$.

In particular we already have $\epsilon(h) = \lim_{k \rightarrow \infty} \nu(\pi_{u^{(k)}}) = 0$. Consider more generally a point $x\alpha^{(\infty)} \in \partial I$. Without loss of generality, one can assume that the last letter of x is α , i.e. $\bar{x} \in I(\bar{\alpha})$, so that $x\alpha^{(k)z} = x \otimes \alpha^{(k)z}$ for all $k \in \mathbb{N}^*$, $z \in I$. We have then $(p_{\bar{x}} \otimes p_y)\Delta(p_{\alpha^{(k)z}}) \neq 0$ iff $y = x\alpha^{(k)z}$ and in that case one can take $V(\alpha^{(k)z}, \bar{x} \otimes y) = (\dim_q \bar{x})^{-1/2}(t_{\bar{x}} \otimes \text{id}_{\alpha^{(k)z}})$ so that

$$(p_{\bar{x}} \otimes 1)\Delta(p_{\alpha^{(k)z}}) = (\dim_q \bar{x})^{-1}(t_{\bar{x}}^* \otimes p_{\alpha^{(k)z}}),$$

where we identify $H_{x\alpha^{(k)z}} \simeq H_x \otimes H_{\alpha^{(k)z}}$. We apply $\text{qtr}'_{\bar{x}} \otimes \text{id}_x \otimes \text{id}_{\alpha^{(k)z}}$ to this identity, where $\text{qtr}'_{\bar{x}} = t_x^*(\text{id} \otimes \cdot)t_x$ is the *right* quantum trace. According to the conjugate equations, this yields $(\text{qtr}'_{\bar{x}} \otimes \text{id})\Delta(p_{\alpha^{(k)z}}) = (\dim_q \bar{x})^{-1}(p_x \otimes p_{\alpha^{(k)z}}) = (\dim_q \bar{x})^{-1}p_{x\alpha^{(k)z}}$. Summing over z we obtain $(\text{qtr}'_{\bar{x}} \otimes 1)\Delta(\pi_{\alpha^{(k)}}) = (\dim_q \bar{x})^{-1}\pi_{x\alpha^{(k)}}$. Finally we can apply ν and take the limit $k \rightarrow \infty$ to get

$$\lim_{k \rightarrow \infty} \nu(\pi_{x\alpha^{(k)}}) = (\dim_q \bar{x}) \lim_{k \rightarrow \infty} \text{qtr}'_{\bar{x}}(h_{\bar{x}}^{(k)}) = (\dim_q \bar{x}) \text{qtr}'_{\bar{x}}(h_{\bar{x}}) = 0,$$

if $\alpha = u$. If $\alpha = \bar{u}$ the same argument holds with h replaced by \bar{h} . \square

We can now prove our main unique stationarity result. The global strategy is the same as in the orthogonal case, see [KKSv22], however the techniques are very different. The analysis of $\partial \mathbb{F}O_F$ ultimately relies on the geometry of the fusion rules in the Temperley-Lieb category. This geometry also appears in the case of $\mathbb{F}U_F$, but only over the points of the form $x\alpha^{(\infty)} \in \partial I$. Thanks to the non-atomicity result above, we will be able to ignore these points, and thus our analysis will rely only on the combinatorics of the free monoid over u, \bar{u} , which is more elementary than the geometry of the Temperley-Lieb category.

Theorem 5.5. *If $N \geq 3$, the harmonic state ω is the only qtr_1 -stationary state on $C(\partial \mathbb{F}U_F)$.*

Proof. Let ν be a qtr_1 -stationary state on B_∞ . As in [KKSv22] we shall prove that $\nu \leq \omega$, which implies $\nu = \omega$ because both maps are states. For this we fix $x_1 \in I$, $a_{x_1} \in B(H_{x_1})_+$ with $\|a_{x_1}\| \leq 1$, and we put $a = \psi_{x_1, \infty}(a_{x_1})$. It suffices then to prove $\nu(a) \leq \omega(a)$. For any $z \in I$ we denote $a_z = p_z a = \psi_{x_1, z}(a_{x_1}) \in B(H_z)$ (which vanishes unless $z \geq x_1$).

By Proposition 5.4 the measure ν_I has no atoms at points of the form $x_1\alpha^{(\infty)}$ with $\alpha = u$ or \bar{u} , hence we have the decreasing limit $\nu(\pi_{x_1\alpha^{(k)}}) \rightarrow_k \nu_I(\{x_1\alpha^{(\infty)}\}) = 0$. Take $\epsilon > 0$ and choose k such that $(|x_1| + k)2^{-k} \leq \epsilon$ and $\nu(\pi_{x_1\alpha^{(k)}}) \leq \epsilon$ for all $\alpha = u, \bar{u}$. The elements $a = \sum_{y \in I} \psi_{x_1, x_1 y}(a_{x_1})$ and $a' := \sum_{x_2 \in I_k} \psi_{x_1 x_2, \infty}(a_{x_1 x_2}) = \sum_{x_2 \in I_k, y \in I} \psi_{x_1, x_1 x_2 y}(a_{x_1})$ are equal from length $|x_1| + k$ on, meaning that $p_l a = p_l a'$ if $l \geq |x_1| + k$. Since ν , seen as a state on B , vanishes on $c_0(\mathbb{G})$, we obtain the decomposition $\nu(a) = \nu(a') = \sum_{x_2 \in I_k} \nu(\psi_{x_1 x_2, \infty}(a_{x_1 x_2}))$. For $x_2 = \alpha^{(k)}$, $\alpha = u, \bar{u}$ and $x = x_1 x_2$ we have $\nu(\psi_{x, \infty}(a_x)) \leq \epsilon$ since $\|\nu \psi_{x, \infty}\| = \nu(\pi_{x_1\alpha^{(k)}}) \leq \epsilon$. Hence $\nu(a) \leq \sum_{x_2 \in I_k'} \nu(\psi_{x_1 x_2, \infty}(a_{x_1 x_2})) + 2\epsilon$, where $I_k' = I_k \setminus \{u^{(k)}, \bar{u}^{(k)}\}$.

Fix now $x = x_1 x_2$ with $x_2 \in I_k'$ and consider $\nu(\psi_{x, \infty}(a_x))$. By iterating the stationarity identity $\text{qtr}_1 * \nu = \nu$ we obtain $\text{qtr}_n * \nu = \nu$ for any n . Applying Lemma 2.3 with k replaced by $|x_1| + k$ and ϵ replaced by $2^{-k}\epsilon$ we get r such that

$$(6) \quad \nu(\psi_{x, \infty}(a_x)) = (\text{qtr}_n \otimes \nu)\Delta(\psi_{x, \infty}(a)) \leq 2^{-k}\epsilon + \sum_{l=0}^n (\text{qtr}_n \otimes \nu \psi_{r, \infty})(b_{n+r-2l}),$$

where $b_s = (p_n \otimes p_r)\Delta(a_s) = \sum (p_y \otimes p_z)\Delta(a_w)$, with the sum running over $y \in I_n$, $z \in I_r$ and $w \in I_s$, $w \geq x$ (so that $\psi_{|x|, s}(a_x) = \sum a_w$).

We start with the terms $l > n - |x|$ in (6). For such an l we shall show that the positive linear form $\varphi : a_x \mapsto (\text{qtr}_n \otimes \nu\psi_{r,\infty})(b_{n+r-2l})$ has a small norm for large n . We first apply Lemma 5.2 with $p = |x|$ and k replaced by $2k$, getting $\|\varphi - \varphi'\| \leq 2^{-2k}$, where $\varphi'(a_x) = (\text{qtr}_n^{(|x|,2k)} \otimes \nu\psi_{r,\infty})(b_{n+r-2l})$. We now prove a non-trivial estimate for $\|\varphi'\| = \varphi'(p_x) = (\text{qtr}_n^{(|x|,2k)} \otimes \nu\psi_{r,\infty})(c_{n+r-2l})$, where $c_s = \sum_{x \leq w \in I_s} (p_n \otimes p_r)\Delta(p_w)$.

Take $y \in I_n^{(|x|,2k)}$ and $z \in I_r$ such that $(p_y \otimes p_z)c_{n+r-2l} \neq 0$: then there exists a unique $w \in I_{n+r-2l}$ such that $w \geq x$ and $w \subset y \otimes z$. We have then $(p_y \otimes p_z)\Delta(p_w) = V(w, y \otimes z)V(w, y \otimes z)^*$. Since $n - l < |x|$ we can write $y \simeq y_1y_2 \otimes y_3$ with $|y_1| = n - l$ and $|x| \leq |y_1y_2| \leq |x| + 2k$. Then we must have $z \simeq \bar{y}_3 \otimes \bar{y}_2z'$, because $y \otimes z$ contains w of length $n+r-2l$, hence the last l letters y_2y_3 of y must simplify with the first l letters of z . Thus we can take $V(w, y \otimes z) = (\text{id} \otimes t_{y_3} \otimes \text{id})V(w, y_1y_2 \otimes \bar{y}_2z')/\|t_{y_3}\|$. We have as well $\text{qTr}_y = \text{qTr}_{y_1y_2} \otimes \text{qTr}_{y_3}$ and we can perform the following computation:

$$\begin{aligned} \|t_{y_3}\|^2 \times (\text{qTr}_y \otimes p_z)[V(w, y \otimes z)V(w, y \otimes z)^*] &= \\ &= (\text{qTr}_{y_1y_2} \otimes \text{qTr}_{y_3} \otimes p_z)[(\text{id}_{y_1y_2} \otimes t_{y_3} \otimes \text{id}_{\bar{y}_2z'})P(w, y_1y_2 \otimes \bar{y}_2z')(\text{id}_{y_1y_2} \otimes t_{y_3}^* \otimes \text{id}_{\bar{y}_2z'})] \\ &= (\text{qTr}_{y_3} \otimes p_z)[(t_{y_3} \otimes \text{id}_{\bar{y}_2z'}) (\text{qTr}_{y_1y_2} \otimes \text{id}) [P(w, y_1y_2 \otimes \bar{y}_2z')] (t_{y_3}^* \otimes \text{id}_{\bar{y}_2z'})] \\ &= Q_{y_3}^{-2} \otimes (\text{qTr}_{y_1y_2} \otimes \text{id}) [P(w, y_1y_2 \otimes \bar{y}_2z')] \leq \|Q_{y_3}^{-2}\| \dim_q(y_1y_2)p_z, \end{aligned}$$

since $\|\text{qTr}_{y_1y_2}\| = \dim_q(y_1y_2)$ and $(\text{qTr}_{y_3} \otimes \text{id}_{\bar{y}_3})(t_{y_3}t_{y_3}^*) = Q_{\bar{y}_3}^{-2}$.

Observe moreover that $\|t_{y_3}\|^2 = \dim_q(y_3) = \dim_q(y)/\dim_q(y_1y_2)$. Summing the previous estimate over $y \in I_n^{(|x|,2k)}$ and $z \in I_r$ we obtain, since $\nu\psi_{r,\infty}$ is a state:

$$\begin{aligned} \|\varphi'\| &= \dim_q(n)^{-1} \sum_{y \in I_n^{(|x|,2k)}} (\text{qTr}_y \otimes \nu\psi_{r,\infty})(c_{n+r-2l}) \\ &\leq \dim_q(n)^{-1} \sum_{y \in I_n^{(|x|,2k)}} \frac{\|Q_{\bar{y}_3}^{-1}\|^2}{\dim_q(y_3)^2} \dim_q(y). \end{aligned}$$

According to Lemma 2.2 we have $\|Q_{\bar{y}_3}^{-1}\|/\dim_q(y_3) \leq (q\rho)^{|y_3|} \leq (q\rho)^{n-|x|-2k}$. For the remaining sum we have by definition $\dim_q(n)^{-1} \sum_y \dim_q(y) \leq 1$, hence altogether $\|\varphi'\| \leq (q\rho)^{2n-2|x|-4k}$ and $\|\varphi\| \leq 2^{-2k} + (q\rho)^{2n-2|x|-4k}$. Recall now that $q\rho < 1$ when $N \geq 3$. Letting $n \rightarrow \infty$ we thus obtain

$$\limsup_{n \rightarrow \infty} \sum_{l=n-|x|+1}^n (\text{qtr}_n \otimes \nu\psi_{r,\infty})(b_{n+r-2l}) \leq |x|2^{-2k}\|a_x\| \leq |x|2^{-2k},$$

since there are $|x|$ terms in the sum.

Then we consider the terms $l \leq n - |x|$ in (6). Take $y, z \in I$ with $|y| = n, |z| = r$. For each $s = n + r - 2l$ there is at most one $w_l = xy'_lz'_l \geq x$ of length s such that $w_l \subset y \otimes z$, corresponding to a unique v_l of length l such that one can write $y = xy'_lv_l, z = v_lz'_l$. Denote p the maximal length of such a word v_l , with $p \leq n - |x|$. We have then $\bigoplus_{l=0}^p w_l \subset y \otimes z$ and

$$\sum_{l=0}^{n-|x|} (\text{qTr}_y \otimes p_z)(b_s) = \sum_{l=0}^p (\text{qTr}_y \otimes p_z)[V(w_l, y \otimes z)(a_x \otimes \text{id}_{y'_lz'_l})V(w_l, y \otimes z)^*].$$

Now recall that we have $x = x_1x_2$, with $x_2 \in I'_k$ which can thus be written $x_2 = x'_2 \otimes x''_2$. We have then $y = x_1x'_2 \otimes x''_2y'_lv_l, w_l = x_1x'_2 \otimes x''_2y'_lz'_l$, so that $\text{qTr}_y = \text{qTr}_{x_1x'_2} \otimes \text{qTr}_{x''_2y'_l}$ and we can take $V(w_l, y \otimes z) = \text{id}_{x_1x'_2} \otimes V(x''_2y'_lz'_l, x''_2y'_lv_l \otimes z)$. On the other hand we have by definition $a_x = \psi_{x_1,x}(a_{x_1}) = \psi_{x_1x'_2,x}(a_{x_1x'_2})$, and since $x = x_1x'_2 \otimes x''_2$, we can take $V(x, x_1x'_2 \otimes x''_2) = \text{id}$ to compute $\psi_{x_1x'_2,x}$, so that finally $a_x = a_{x_1x'_2} \otimes \text{id}_{x''_2}$. We have then

$$\begin{aligned} (\text{qTr}_y \otimes p_z)[V(w_l, y \otimes z)(a_x \otimes \text{id})V(w_l, y \otimes z)^*] &= \\ &= \text{qTr}_{x_1x'_2}(a_{x_1x'_2}) \times (\text{qTr}_{x''_2y'_l} \otimes p_z)(P(x''_2y'_lz'_l, x''_2y'_lv_l \otimes z)) \\ &= \text{qTr}_{x_1x'_2}(a_{x_1x'_2}) \frac{\dim_q(x''_2y'_lz'_l)}{\dim_q(z)} \times p_z. \end{aligned}$$

Since $\bigoplus_{l=0}^p x_2'' y_l' z_l' \subset x_2'' y_l' \bar{v}_l \otimes z$ we get the following upper bound for the sum over l :

$$\begin{aligned} \sum_{l=0}^{n-|x|} (\text{qTr}_y \otimes p_z)(b_s) &\leq \text{qTr}_{x_1 x_2'}(a_{x_1 x_2'}) \frac{\dim_q(x_2'' y_l' \bar{v}_l \otimes z)}{\dim_q(z)} \times p_z \\ &= \text{qtr}_{x_1 x_2'}(a_{x_1 x_2'}) \dim_q(y) p_z = \text{qtr}_x(a_x) \dim_q(y) p_z. \end{aligned}$$

Now we make the sum over y and z . Since we are in the case where less than $n - |x|$ letters are simplified in the tensor product $y \otimes z$, the terms where y does not start with x vanish and we get

$$\sum_{l=0}^{n-|x|} (\text{qTr}_n \otimes p_r)(b_s) \leq \text{qtr}_x(a_x) p_r \times \sum \{\dim_q(y); |y| = n, y \geq x\}.$$

Finally since ν is a state we have $\nu \psi_{r,\infty}(p_r) \leq 1$ and we obtain, using the notation $\dim_q(x, n)$ from the previous lemma:

$$\sum_{l=0}^{n-|x|} (\text{qtr}_n \otimes \nu \psi_{r,\infty})(b_s) \leq \text{qtr}_x(a_x) \frac{\dim_q(x, n)}{\dim_q(n)}.$$

Note that as $n \rightarrow \infty$ the right-hand side tends to $\text{qtr}(a_x) \omega_I(\partial I(x)) = \omega(\psi_{x,\infty}(a_x))$.

Altogether (6) yields $\nu(\psi_{x,\infty}(a_x)) \leq 2^{-k}\epsilon + |x|2^{-2k} + \omega(\psi_{x,\infty}(a_x))$ for any $x = x_1 x_2$, $x_2 \in I'_k$. Summing over x_2' , which takes less than 2^k values, this yields $\nu(a) \leq \epsilon + (|x_1| + k)2^{-k} + \omega(a) + 2\epsilon \leq \omega(a) + 4\epsilon$. This is true for any $\epsilon > 0$, hence $\nu(a) \leq \omega(a)$. This is true for every $a \in \psi_{x_1,\infty}(B(H_{x_1})_+)$, hence $\nu = \omega$ on $\psi_{x_1,\infty}(B(H_{x_1}))$ for every x_1 , and finally $\nu = \omega$. \square

Corollary 5.6. *If $N \geq 3$, the Gromov boundary of $\mathbb{F}U_F$ is an $\mathbb{F}U_F$ -boundary.*

Proof. It is already known that the Poisson map $P_1 = (\text{id} \otimes \omega)\beta : C(\partial_G \mathbb{F}U_F) \rightarrow H^\infty(\mathbb{F}U_F, \text{qtr}_1) \subset \ell^\infty(\mathbb{F}U_F)$ is completely isometric [VVV10]. Taking into account the unique stationarity result of Theorem 5.5, we can thus apply [KKS22, Theorem 4.19] which shows that that $\partial_G \mathbb{F}U_F$ is an $\mathbb{F}U_F$ -boundary. \square

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REFERENCES

- [AS24] Benjamin Anderson-Sackaney. Tracial States and \mathbb{G} -Invariant States of Discrete Quantum Groups. *Studia Mathematica, to appear*, 2024.
- [ASK24] Benjamin Anderson-Sackaney and Fatemeh Khosravi. Topological boundaries of representations and coideals. *Adv. Math.*, 452:Paper No. 109830, 2024.
- [Ban97] Teodor Banica. Le Groupe Quantique Compact Libre $U(n)$. *Comm. Math. Phys.*, 190:143–172, 1997.
- [BKKO17] Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa. C^* -simplicity and the unique trace property for discrete groups. *Publ. Math. Inst. Hautes Études Sci.*, 126:35–71, 2017.
- [Bra13] Michael Brannan. Reduced Operator Algebras of Trace-Preserving Quantum Automorphism Groups. *Doc. Math.*, 18:1349–1402, 2013.
- [BS93] Saad Baaj and Georges Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C^* -algèbres. *Ann. Sci. Éc. Norm. Supér. (4)*, 26(4):425–488, 1993.
- [BV05] Saad Baaj and Stefaan Vaes. Double crossed products of locally compact quantum groups. *J. Inst. Math. Jussieu*, 4(1):135–173, 2005.
- [CKN72] Marie Choda, Isamu Kasahara, and Ritsuo Nakamoto. Dependent elements of an automorphism of a C^* -algebra. *Proc. Japan Acad.*, 48:561–565, 1972.
- [dlH85] Pierre de la Harpe. Reduced C^* -algebras of discrete groups which are simple with a unique trace. In *Operator algebras and their connections with topology and ergodic theory (Buzeni, 1983)*, volume 1132 of *Lecture Notes in Math.*, pages 230–253. Springer, Berlin, 1985.
- [ER94] Edward G. Effros and Zhong-Jin Ruan. Discrete quantum groups. I. The Haar measure. *Internat. J. Math.*, 5(5):681–723, 1994.
- [FK18] Ramin Faal and Paweł Kasprzak. Group-like projections for locally compact quantum groups. *J. Operator Theory*, 80(1):153–166, 2018.

- [FLMMS22] Pierre Fima, François Le Maître, Soyoung Moon, and Yves Stalder. A characterization of high transitivity for groups acting on trees. *Discrete Anal.*, 2022. Paper No. 8, 63 pages.
- [Gla76] Shmuel Glasner. *Proximal flows*, volume Vol. 517 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1976.
- [Haa17] Uffe Haagerup. A new look at C^* -simplicity and the unique trace property of a group. In *Operator algebras and applications—the Abel Symposium 2015*, volume 12 of *Abel Symp.*, pages 167–176. Springer, 2017.
- [Ham78] Masamichi Hamana. Injective envelopes of banach modules. *Tohoku Math. J.*, 30(2):439–453, 1978.
- [HHN22] Erik Habbestad, Lucas Hataishi, and Sergey Neshveyev. Noncommutative Poisson boundaries and Furstenberg-Hamana boundaries of Drinfeld doubles. *J. Math. Pures Appl. (9)*, 159:313–347, 2022.
- [HZ84] Uffe Haagerup and László Zsidó. Sur la propriété de Dixmier pour les C^* -algèbres. *C. R. Acad. Sci. Paris Sér. I Math.*, 298(8):173–176, 1984.
- [Izu02] Masaki Izumi. Non-commutative Poisson Boundaries and Compact Quantum Group Actions. *Adv. Math.*, 169(1), 2002.
- [Kal69] Robert R. Kallman. A generalization of free action. *Duke Math. J.*, 36:781–789, 1969.
- [Ken20] Matthew Kennedy. An intrinsic characterization of C^* -simplicity. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(5):1105–1119, 2020.
- [KK17] Mehrdad Kalantar and Matthew Kennedy. Boundaries of reduced C^* -algebras of discrete groups. *J. Reine Angew. Math.*, 727:247–267, 2017.
- [KKS22] Mehrdad Kalantar, Paweł Kasprzak, Adam Skalski, and Roland Vergnioux. Noncommutative Furstenberg boundary. *Anal. PDE*, 15(3):795–842, 2022.
- [KS19] Matthew Kennedy and Christopher Schafhauser. Non-commutative Boundaries and the Ideal Structure of Reduced Crossed Products. *Duke Math. J.*, 168:3215–3260, 2019.
- [KS22] Mehrdad Kalantar and Eduardo Scarparo. Boundary maps, germs and quasi-regular representations. *Adv. Math.*, 394:Paper No. 108130, 31, 2022.
- [KV00] Johan Kustermans and Stefaan Vaes. Locally Compact Quantum Groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 33(6):837–934, 2000.
- [LB17] Adrien Le Boudec. C^* -simplicity and the amenable radical. *Invent. Math.*, 209(1):159–174, 2017.
- [MT07] Toshihiko Masuda and Reiji Tomatsu. Classification of minimal actions of a compact kac algebra with amenable dual. *Comm. Math. Phys.*, 274:487–551, 2007.
- [Pit17] David R. Pitts. Structure for regular inclusions. I. *J. Operator Theory*, 78(2):357–416, 2017.
- [Pow75] Robert T. Powers. Simplicity of the C^* -algebra associated with the free group on two generators. *Duke Math. J.*, 42:151–156, 1975.
- [PS23] David R. Pitts, Roger R. Smith, and Vrej Zarikian. Norming in discrete crossed products. Preprint, arXiv:2307.15571, 2023.
- [PW90] Piotr Podleś and Stanisław L. Woronowicz. Quantum deformation of Lorentz group. *Comm. Math. Phys.*, 130(2):381–431, 1990.
- [PZ15] David R. Pitts and Vrej Zarikian. Unique pseudo-expectations for C^* -inclusions. *Illinois J. Math.*, 59(2):449–483, 2015.
- [VD94] Alfons Van Daele. Multiplier Hopf algebras. *Trans. Amer. Math. Soc.*, 342(2):917–932, 1994.
- [VD96] Alfons Van Daele. Discrete quantum groups. *J. Algebra*, 180(2):431–444, 1996.
- [VDW96] Alfons Van Daele and Shuzhou Wang. Universal quantum groups. *Internat. J. Math.*, 7(2):255–263, 1996.
- [VV07] Stefaan Vaes and Roland Vergnioux. The Boundary of Universal Discrete Quantum Groups, Exactness, and Factoriality. *Duke Math. J.*, 140(1):35–84, 2007.
- [VV08] Nikolas Vander Venet. *Probabilistic boundaries of discrete quantum groups*. Phd thesis, Katholieke Universiteit Leuven, 2008.
- [VVV10] Stefaan Vaes and Nikolas Vander Venet. Poisson boundary of the discrete quantum group $\widehat{A_u(F)}$. *Compos. Math.*, 146(4):1073–1095, 2010.
- [Wan95] Shuzhou Wang. Free Products of Compact Quantum Groups. *Comm. Math. Phys.*, 167:671–692, 1995.
- [Wor87] Stanisław L. Woronowicz. Compact Matrix Pseudogroups. *Comm. Math. Phys.*, 111, 1987.
- [Zar19] Vrej Zarikian. Unique expectations for discrete crossed products. *Ann. Funct. Anal.*, 10(1):60–71, 2019.

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