Gamma-Elements for Free Quantum Groups

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Motivations

– K-amenability [Cuntz]
– groups acting on trees [Julg-Valette]
– amalgamated free products of amenable discrete quantum groups

Compact Quantum Groups
[Woronowicz]

main examples
– discrete groups
– free quantum groups [Wang-Banica]

C^*-**algebras**
– S unital C^*-algebra
– coproduct $\delta : S \to S \otimes S$
There exists a Haar state $h \in S^*$.

corepresentations
$v \in L(H_v) \otimes S$ st $(\mathrm{id} \otimes \delta)(v) = v_{12}v_{13}$
➜ category $\mathcal{C}$ with $\oplus$, $\otimes$, ...
➜ Irr $\mathcal{C}$: irreducible corepresentations

hilbertian objects [Baaj-Skandalis]
– $H = L^2(S, h)$, $\lambda : S \to L(H)$, $S_{\mathrm{red}} = \lambda(S)$
– decomposition $H = \bigoplus_{r \in \mathrm{Irr}\,\mathcal{C}} p_r H$
– Kac system (H, V, U)

Quantum Caley Graphs

data

– quant. discrete group : S, $\mathcal{C}$, (H, V, U)

– $\mathcal{D} \subset \mathrm{Irr}\ \mathcal{C}$ finite st $\bar{\mathcal{D}} = \mathcal{D}$, $1_{\mathcal{C}} \notin \mathcal{D}$

– $p_1 = \sum_{r \in \mathcal{D}} p_r$

classical graph associated to $(\mathcal{C}, \mathcal{D})$

– $\mathfrak{v} = \mathrm{Irr}\ \mathcal{C}$, $\mathfrak{e} = \{(r, r') \in \mathfrak{v}^2 \mid \exists\, s \in \mathcal{D}\ r' \subset r \otimes s\}$

– the reversing map θ is well defined :

$$r' \subset r \otimes s \iff r \subset r' \otimes \bar{s}$$

– geometrical edges, orientation

quantum graph associated to $(\mathcal{C}, \mathcal{D})$

– ℓ^2-space of vertices : H

– ℓ^2-space of edges : $K = H \otimes p_1 H$

– $\Theta = \Sigma(1 \otimes U)V(U \otimes U)\Sigma$, $K_g = \mathrm{Ker}(\Theta - \mathrm{id})$

– $V : K \to H \otimes H$ « endpoints » operator

– $S = (\mathrm{id} \otimes \epsilon)V$ and $T = (\epsilon \otimes \mathrm{id})V$

nb : $\Theta^2 \neq \mathrm{id}$

Ascending Edges

hypothesis
The classical graph of $(\mathcal{C},\mathcal{D})$ is a strict tree, choose the origin $1_{\mathcal{C}}$ ➔ ascending orientation

proposition *In this case the discrete quantum group associated to* $\mathcal{C}$ *is a free product of* $A_o(Q_i)$*'s and* $A_u(R_j)$*'s (with* $Q_i\bar{Q}_i \in \mathbb{C}\mathrm{id}$*).*

definition *(quantum ascending orientation)*
– $p_{\star+} = \sum \{V^*(p_r \otimes p_{r'})V \mid (r,r') \in \mathfrak{e}_+\}$
– $p_{+\star} := \Theta^*(1 - p_{\star+})\Theta \neq p_{\star+}$ **!**
– $p_{\star-} = 1 - p_{\star+}$, $p_{-\star} = 1 - p_{+\star}$
– $p_{++} = p_{+\star}p_{\star+}$, $p_{+-} = \cdots$, $K_{++} = p_{++}K$

definition *(quantum Julg-Valette operator)*

$$F_g^* \ : \ K_g \xrightarrow{p_{++}} K_{++} \xrightarrow{T} H$$

Space of Edges at Infinity

We consider the quantum Cayley tree of $A_o(Q)$, with $Q\bar{Q} \in \mathbb{C}\mathrm{id}$ and Tr $Q^*Q > 2$.

theorem

– $T_{|K_{++}}$ *is injective and its image equals* $(1 - p_0)H$.

– $p_{++|K_g}$ *is injective and its image equals* $\{\zeta \in K_{++} \mid p_{+-}\Theta p_{++}\,\zeta \in \mathrm{Im}\,(\mathrm{id} - p_{+-}\Theta p_{+-})\}$.

definition

$p_{+-}\Theta p_{+-}$ *is a « right shift », we put*

$$H_\infty = \varinjlim \; ((p_k \otimes \mathrm{id})K_{+-}, p_{+-}\Theta p_{+-})$$

theorem

$p_{++}K_g$ *is closed and its orthogonal is naturally isomorphic to* H_∞.