KK-Theory for Quantum Groups : Functorial and Geometrical Methods

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Quantum Groups

Locally compact quantum groups

- "dual" Hopf C^*-algebras S, $\widehat{S}$
- coproduct $\delta : S \to \tilde{M}(S \otimes S)$ (coassociative)
- regular representations $\lambda : S \twoheadrightarrow S_{\text{red}} \subset L(H)$ and $\widehat{S} \to \widehat{S}_{\text{red}} \subset L(H)$
- trivial representations $\varepsilon : S \to \mathbb{C}$, $\widehat{\varepsilon} : \widehat{S} \to \mathbb{C}$

Ex. : $S = S_{\text{red}} = C_0(G)$, $\delta(f)(r,s) = f(rs)$,
$\widehat{S} = C^*(G)$, $\widehat{S}_{\text{red}} = C^*_{\text{red}}(G)$, $H = L^2(G)$

Discrete quantum groups

- S unital (Woronowicz C^*-algebra)
- the category of corepr. $\mathcal{C}$ induces $\widehat{S} = \widehat{S}_{\text{red}} = \bigoplus_{r \in \text{Irr}\, \mathcal{C}} p_r \widehat{S}$ with $p_r \widehat{S} \simeq L(H_r)$

Ex. : $S = C^*(\Gamma)$, $\widehat{S} = c_0(\Gamma)$, $\text{Irr}\, \mathcal{C} = \Gamma$,
$p_r \widehat{S} = \mathbb{C} 1\!\!1_{\{r\}} \in \ell^2(\Gamma)$

Green-Julg Theorem

- S unital (compact case)
- coactions δ_A, δ_B of S_{red} on A, B
- δ_A trivial, B S_{red}-algebra

Theorem *There is an isomorphism* $KK_{S_{\mathsf{red}}}(A,B) \simeq KK(A, B \rtimes_{\mathsf{red}} \widehat{S})$ *given by*

$$KK_{S_{\mathsf{red}}}(A,B) \xrightarrow{\;j_r\;} KK(A \rtimes_{\mathsf{red}} \widehat{S}, B \rtimes_{\mathsf{red}} \widehat{S}) \xrightarrow{\;\phi^*\;} KK(A, B \rtimes_{\mathsf{red}} \widehat{S}) \xrightarrow{\;\psi\;} KK_{S_{\mathsf{red}}}(A_1, (B \rtimes_{\mathsf{red}} \widehat{S})_1) \xrightarrow{\;\cdot \otimes \beta\;} KK_{S_{\mathsf{red}}}(A,B)$$

K-amenability

- S, S_{red}, $\widehat{S}$, $\widehat{S}_{\mathrm{red}}$: C^*-algebras of a locally compact quantum group
- $\lambda : S \to S_{\mathrm{red}}$: regular representation
- $\varepsilon : S \to \mathbb{C}$: trivial representation

Theorem *We have i $\Rightarrow$ ii $\Rightarrow$ iii $\Rightarrow$ iv, and iv $\Rightarrow$ i when S_{red} is unital (discrete case) :*

i. $\mathbb{1} \in KK_{\widehat{S}_{\mathrm{red}}}(\mathbb{C}, \mathbb{C})$ *is represented by* $(E, 1, F)$ *with* $\delta_E \prec \delta_{\widehat{S}_{\mathrm{red}}}$ *(*K-amenability*)*

ii. $\forall\ A\ \ [\lambda_A] \in KK(A \rtimes S, A \rtimes_{\mathrm{red}} S)$ *is invertible*

iii. $[\lambda] \in KK(S, S_{\mathrm{red}})$ *is invertible*

iv. $\exists\ \alpha \in KK(S_{\mathrm{red}}, \mathbb{C})\ \ \lambda^*(\alpha) = [\varepsilon] \in KK(S, \mathbb{C})$.

Amalgamated Free Products

- ➔ $T \subset S_1$, S_2 : amenable Wor. C^*-algebras
- ➔ $S = S_1 *_T S_2$: amalgamated free product
- ➔ $P : S \twoheadrightarrow T$, $R_i : S \twoheadrightarrow S_i$ cond. expect.
- ➔ E, F_i associated GNS constructions
- ➔ $\varepsilon : T$, $S_i \to \mathbb{C}$ trivial representation

Ex. : $S_i = C^*(\Gamma_i)$, $T = C^*(\Delta)$, $\Gamma = \Gamma_1 *_\Delta \Gamma_2$

Definition *Quantum Bass-Serre Tree*

- ➔ $H = F_1 \otimes_\varepsilon \mathbb{C} \oplus F_2 \otimes_\varepsilon \mathbb{C}$, $K_0 = E \otimes_\varepsilon \mathbb{C}$
- ➔ *GNS representations of S*

The classical « Serre » tree (V, E) associated to « Irr $\mathcal{C}_1 *_\mathcal{D}$ Irr $\mathcal{C}_2$ » induces a J-V operator :

- ➔ F : $E(r,i)^\circ \otimes_\varepsilon \mathbb{C} \xrightarrow{\sim} F_i^\circ \otimes_\varepsilon \mathbb{C}$, $\eta\mathbb{C} \to \eta_2 \otimes_\varepsilon 1_\mathbb{C}$

Theorem

1. $(H \oplus K_0, \pi_{\mathrm{GNS}}, F)$ *defines* $\gamma \in KK(S_{\mathrm{red}}, \mathbb{C})$
2. $(\widehat{S}, \widehat{\delta})$ *is K-amenable*

Quantum Cayley Graphs

- S Wor. C^*-algebra, $p_1 = \sum_{r\in\mathcal{D}} p_r$ with
- $\mathcal{D} \subset \text{Irr } \mathcal{C}$ finite, $1_{\mathcal{C}} \notin \mathcal{D}$, $\bar{\mathcal{D}} = \mathcal{D}$

Ex. : $S = C^*(\Gamma)$, $\hat{S} = C_0(\Gamma)$, $\mathcal{D} \subset \Gamma$

Definition *Quantum Cayley Graph*

- H *: space of the regular repr. of* S, $\hat{S}$
- $K = H \otimes p_1 H$, $S = \text{id} \otimes \epsilon : K \to H$
- $\Theta = \Sigma(1 \otimes U)V(U \otimes U)\Sigma$, $K_g = \text{Ker}(\Theta + \text{id})$
- *regular repr. on* H, *trivial repr. on* $p_1 H$

Definition *Classical Cayley Graph*

- $V = \text{Irr } \mathcal{C}$, $\theta(r, r') = (r', r)$
- $E = \{(r, r', s, i) \in V^2 \mid r' \subset_i r \otimes s,\ s \in \mathcal{D}\}$
- $C_0(V) \to L(H)$, $\mathbb{1}_{\{r\}} \mapsto p_r$

Free Quantum Groups

(E, V) is a tree **iff** S is a free product of free quantum groups $A_o(Q)$, $A_u(Q)$. Then (E, V) induces a projection $p_{\star+} \in L(K)$ « on ascending edges ».

Problems ➜ $\Theta^2 \neq 1$

➜ $p_{+\star} := 1 - \Theta p_{\star+} \Theta^* \neq p_{\star+}$ ➜ $p_{++} = p_{\star+} p_{+\star}$

➜ $[p_{\star+}, u_{ij}]$ *is compact but not of finite rank*

➜ $F = T p_{++} : K_g \to H$ *is not Fredholm*

Theorem *There exists a natural representation* $\pi_\infty : A_o(Q) \to L(H_\infty)$ *such that*

$$\begin{array}{ccccc} K_g & \searrow^{p_{++}} & & & \\ & & K_{++} & \xrightarrow{B} & H \\ H_\infty & \nearrow_{R^*} & & & \end{array}$$

defines $\gamma \in KK_{\widehat{S}}(\mathbb{C}, \mathbb{C})$, *when* Tr $Q^* Q > 2$. *In the classical case* $H_\infty = 0$.

Negative Type Function on $A_o(Q)$

- dense sub-*-Hopf algebra $\mathcal{S} \subset S$
- co-unity $\varepsilon : \mathcal{S} \to \mathbb{C}$, antipode $\kappa : \mathcal{S} \to \mathcal{S}$

Ex. : $\mathcal{S} = \mathbb{C}\Gamma \subset C^*(\Gamma)$, $\varepsilon(g) = 1$, $\kappa(g) = g^{-1}$

Proposition

- $\pi : S \to L(H)$ **-representation.*
- *real π-cocycle : a linear $c : \mathcal{S} \to H$ st :*
- $c(1) = 0$, $c(xy) = \pi(x)c(y) + \varepsilon(y)c(x)$ *and*
- $(c(\kappa(x^*))|c(\kappa(x)^*)) \in \mathbb{R}$.

Put $\varphi = (\,\cdot\,|\,\cdot\,) \circ (c \otimes c) \circ (*\kappa \otimes \mathrm{id}) \circ \delta$ on $\mathcal{S}$. Then $\varphi(x^*x) \leq 0$ for all $x \in \mathrm{Ker}\ \varepsilon$.

Theorem *($A_o(Q)$, $Q\bar{Q} \in \mathbb{C}\mathrm{id}$, Tr $Q^*Q > 2$)*
Then $\mathrm{Ker}(T - S) = K_g^{\perp}$ and $(T - S)(K_g)$ contains $c_0(x) = (\lambda(x) - \varepsilon(x))\,\Lambda_H(1)$ for all $x \in \mathcal{S}$.

- Path and distance to the origin
  in the quantum Cayley graph