Roland VERGNIOUX
Habilitation à Diriger des Recherches

Research Account

Presented publications

[1] R. Vergnioux : K-amenability for amalgamated free products of amenable discrete quantum groups. *Journal of Functional Analysis*, **212**(1) 206–221, 2004.

[2] R. Vergnioux : Orientation of quantum Cayley trees and applications. *Journal für die reine und angewandte Mathematik*, **580** 101–138, 2005.

[3] R. Vergnioux : The property of rapid decay for discrete quantum groups. *Journal of Operator Theory*, **57**(2) 303–324, 2007.

[4] S. Vaes and R. Vergnioux : The boundary of universal discrete quantum groups, exactness, and factoriality. *Duke Mathematical Journal*, **140**(1) 35–84, 2007.

[5] T. Banica and R. Vergnioux : Fusion rules for quantum reflection groups. *Journal of Noncommutative Geometry*, **3**(3) 327–359, 2009.

[6] T. Banica and R. Vergnioux : Invariants of the half-liberated orthogonal group. *Annales de l'Institut Fourier (Grenoble)*, **60**(6) 2137–2164, 2010.

[7] R. Vergnioux : Paths in quantum Cayley trees and L^2-cohomology. *Advances in Mathematics*, **229**(5) 2686–2711, 2012.

[8] R. Vergnioux and C. Voigt : The K-theory of free quantum groups. Preprint arXiv:1112.3291, 2011.

The following is a short presentation of the content of the articles listed above. More details about the mathematical context, including definitions, references and ideas of proofs, are presented in the french *Mémoire*.

Introduction

My research is devoted to the study of *discrete quantum groups*, from an analytic, algebraic and geometric point of view.

Initiated by G.I. Kac in the 1960's, the theory of quantum groups enjoyed a second take-off in the 1980's with the works of Drinfel'd in mathematical physics and Woronowicz in operator algebras, who introduced the notion of compact (or dually, discrete) quantum group. In the locally compact case, the axiomatic framework has been worked out by Kac-Vainerman and Enock-Schwartz in the 1970's, and then by Kustermans-Vaes ath the end of the 1990's. Since then the theory has been animated by new developments and new applications in various areas such as operator algebras, representation theory, free probability and mathematical physics.

A discrete quantum group $\mathbb{F}$ and its compact dual $\mathbb{G}$ are given by the associated reduced Woronowicz C^*-algebra, $A_r = C^*_r(\mathbb{F}) = C_r(\mathbb{G})$, which is a unital C^*-algebra endowed with a coproduct $\Delta : A_r \to A_r \otimes A_r$ satisfying certain axioms. There is also a full version $A = C^*_f(\mathbb{F})$ of this C^*-algebra, and we say that $\mathbb{F}$ is amenable when the natural map from $C^*_f(\mathbb{F})$ to $C^*_r(\mathbb{F})$ is an isomorphism. In my research I mainly focus on non-amenable discrete quantum groups which satisfy weaker conditions that amenability, e.g. K-amenability, Haagerup's Property, Akemann-Ostrand Property, or exactness. Moreover many results are motivated by the analogy with the case of usual discrete groups, $\mathbb{F} = \Gamma$: then $C^*_r(\Gamma)$ is the norm closed sub-$*$-algebra of $B(\ell^2(\Gamma))$ generated by the operators of left translation by elements of Γ.

Another family of "classical" examples is supplied by usual compact groups G: in this case $C^*_r(\mathbb{F})$ is the algebra of continuous functions $C(G)$, and one says that $\mathbb{F} = \hat{G}$ is the dual of G. The most famous class of "non-classical" quantum groups is probably the one of q-deformations of compact Lie groups, introduced in the framework of envelopping algebras by Jimbo and Drinfel'd: one can e.g. define compact quantum groups $\mathbb{G}_q = SU_q(N)$, $Spin_q(N)$, $Sp_q(N)$ for $q \in]0, 1]$, which coincide with $SU(N)$, $Spin(N)$, $Sp(N)$ when $q = 1$. However the associated discrete quantum groups are amenable.

In 1995 Wang introduced a new class of examples: the one of universal compact quantum groups, unitary or orthogonal, given by full Woronowicz C^*-algebras denoted $A_u(Q)$, $A_o(Q)$, where the parameter $Q \in M_N(\mathbb{C})$ is an invertible matrix. Their discrete duals are also called unitary or orthogonal *free quantum groups* and denoted $\mathbb{F}U(Q)$, $\mathbb{F}O(Q)$. They are non amenable as soon as $N \geq 3$ and can be considered in some respects as quantum analogues of the usual free groups F_N. Several results of this rapport deal with the operator algebras associated to these quantum groups and are motivated by the parallel with free groups.

To each discrete quantum groups $\mathbb{\Gamma}$ is associated its category of finite-dimensional corepresentations, $\operatorname{Corep} \mathbb{\Gamma}$. When $\mathbb{\Gamma}$ is the dual of a usual compact group, $\mathbb{\Gamma} = \hat{G}$, its corepresentations correspond to representations of G and the classical Peter-Weyl theory shows that $\operatorname{Corep} \mathbb{\Gamma}$ is semisimple. Woronowicz shows that semisimplicity still holds for discrete quantum groups, and we denote $\operatorname{Irr} \mathbb{\Gamma}$ the set of classes of irreducible corepresentations of $\mathbb{\Gamma}$. The tensor product of two irreducible corepresentations can be decomposed in $\operatorname{Irr} \mathbb{\Gamma}$: the multiplicities that appear in this way form the *fusion rules* of $\mathbb{\Gamma}$. In the case when $\mathbb{\Gamma} = \Gamma$ is a usual discrete group, $\operatorname{Irr} \mathbb{\Gamma}$ identifies with Γ and the fusion rules are given by the product of the group.

Fusion rules of liberated quantum groups

In the article [5], we determine the fusion rules of the quantum reflection groups $\mathbb{G} = H_N^{s+}$, which were introduced by Banica, Belinschi, Capitaine and Collins in 2011. In the $s = 1$ case one recovers an important family of quantum groups: the one of quantum permutation groups S_N^+ introduced by Wang in 1998, whose fusion rules were computed by Banica in 1999. We show in [5] that H_N^{s+} is isomorphic to the *free wreath product* $(\mathbb{Z}/s\mathbb{Z}) \wr_* S_N^+$, and can be interpreted as the group of quantum symmetries of an explicit finite graph, namely the disjoint union of N oriented cycles of length s.

The notion of free wreath product $\mathbb{G} \wr_* S_N^+$ has been introduced by Bichon in 2004 for any compact quantum group $\mathbb{G}$. There is no general result allowing to describe the fusion rules of the wreath product in terms of the ones of $\mathbb{G}$. The following result can be seen as a first step in this direction, dealing with the case $\mathbb{G} = \mathbb{Z}/s\mathbb{Z}$:

Theorem 1 [5, Thm. 7.3] *For $N \geq 4$ the irreducible representations r_x of H_N^{s+} can be indexed by words x on $\mathbb{Z}/s\mathbb{Z}$ in such a way that we have the following recursive fusion rules, where x, y are words and i, j are letters:*

$$r_{xi} \otimes r_{jy} = \begin{cases} r_{xijy} \oplus r_{x(i+j)y} & \text{if } i + j \neq 0 \mod s, \\ r_{xijy} \oplus r_{x(i+j)y} \oplus (r_x \otimes r_y) & \text{else.} \end{cases}$$

In the article [6], we describe the fusion rules of the half-liberated orthogonal groups $\mathbb{G} = O_N^*$ introduced by Banica and Speicher in 2009. The results we obtain, and the methods used, are inspired by the classical theory of *weights* for representations of usual compact Lie groups.

The C^*-algebra $C_f(O_N^*)$ is generated by elements u_{ij} which form the fundamental representation $u = (u_{ij})$ of O_N^*. It is easy to see that the quotient of $C_f(O_N^*)$ by the ideal generated by "off-diagonal" generators (u_{ij}, $i \neq j$) identifies with the C^*-algebra $C^*(L)$ of a usual discrete group L, called *diagonal group* of O_N^*. It turns out that L plays the role of a noncommutative weight lattice for O_N^*. More precisely, the image of any representation $v \in \operatorname{Rep} O_N^*$ in $C^*(L)$ decomposes as a direct sum of elements of L. Denoting $P(v)$ the set (with multiplicities) of elements of L appearing in this way, we have the following classification result:

Theorem 2 [6, Thm. 6.2] *Let L be the diagonal group of O_N^*, and v, w two representations of O_N^*. If we have $P(v) = P(w)$ as subsets of L with multiplicities, then v and w are equivalent.*

We compute also the group L: it identifies in a natural way with a subgroup of $\mathbb{Z}^N \rtimes (\mathbb{Z}/2\mathbb{Z})$ isomorphic to $\mathbb{Z}^{N-1} \rtimes (\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts as $-\mathrm{id}$ on $\mathbb{Z}^N$ and $\mathbb{Z}^{N-1}$. Using the natural map from $\mathbb{Z}^N \rtimes (\mathbb{Z}/2\mathbb{Z})$ onto $X = \mathbb{Z}^N$ identified with the weight lattice of the compact group U_N, we manage to describe the fusion rules of O_N^* in terms of the ones of U_N, and to compute the dimensions of its irreducible representations. It turns out that these fusion rules are noncommutative, and that the dual of O_N^* has polynomial growth — in particular it is amenable.

The Property of Rapid Decay

The article [3] deals with the Property of Rapid Decay in the framework of discrete quantum groups. The definitions and equivalent characterizations of the classical case are easy to generalize to the quantum case. Given a generating subset of $\operatorname{Corep} \mathbb{\Gamma}$, there is a notion of length for irreducible objects of $\operatorname{Corep} \mathbb{\Gamma}$, and an associated graduation of the C^*-algebra $C_r^*(\mathbb{\Gamma})$ into subspaces $C_r^*(\mathbb{\Gamma})_k$, $k \in \mathbb{N}$. Denoting $\|\cdot\|_r$ the norm of $C_r^*(\mathbb{\Gamma})$, and $\|\cdot\|_2$

the hilbertian norm induced by the Haar state on $C_r^*(\mathbb{F})$, we have in general $\|x\|_2 \leq \|x\|_r$ for all $x \in C_r^*(\mathbb{F})$. We say that $\mathbb{F}$ as the *Property of Rapid Decay* if there exists a polynomial P such that $\|x\|_r \leq P(k)\|x\|_2$ for all $x \in C_r^*(\mathbb{F})_k$ and all $k \in \mathbb{N}$.

In [3], I show that for amenable discrete quantum groups, the Property of Rapid Decay corresponds to polynomial growth, correctly formulated in the quantum framework. In particular duals of compact Lie groups have the Property of Rapid Decay. I also show that the Property of Rapid Decay implies that the discrete quantum group under consideration is *unimodular* — which is not automatic in the quantum case. In particular duals of q-deformations à la Jimbo-Drinfel'd do not have the Property of Rapid Decay. Finally I show that the classical application of the Property of Rapid Decay to K-theory extends to the quantum case: the subspace $H^\infty(\mathbb{F}) \subset C_r^*(\mathbb{F})$ of "functions with rapid decay" is a dense sub-$*$-algebra which has the same K-theory as $C_r^*(\mathbb{F})$.

However the main result of [3] is the following one:

Theorem 3 [3, Thm. 4.9] *Let $Q \in GL_N(\mathbb{C})$ be a matrix such that $Q\bar{Q} \in \mathbb{C}I_N$ and $QQ^* \in \mathbb{C}I_N$. Then $\mathbb{F}O(Q)$ has the Property of Rapid Decay.*

In particular the Property of Rapid Decay is a common feature of free quantum groups $\mathbb{F}O(Q)$ and classical free groups, for which the proof, of combinatorial nature, goes back to Haagerup's foundational paper in 1979. The proof of the Theorem above is quite different because there is no "free combinatorics" inside $\mathbb{F}O(Q)$. It relies instead on computations in the category $\mathrm{Corep}\,\mathbb{F}$, and in particular on the study of the *geometry* of fusion rules. Let us note that the hypothesis $QQ^* \in \mathbb{C}I_N$ is equivalent to unimodularity of the discrete quantum group $\mathbb{F}O(Q)$.

Quantum Cayley graphs and applications

In the article [2] I introduce a general notion of quantum Cayley graph for discrete quantum groups and I investigate it in detail in the case of free quantum groups $\mathbb{F}$, that is, in the case of free products of quantum groups $\mathbb{F}O(Q)$ and $\mathbb{F}U(Q)$. A quantum Cayley graph $\mathbb{X}$ is given by C^*-algebras $C_0(\mathbb{X}^{(0)})$, $C_0(\mathbb{X}^{(1)})$ represented on Hilbert spaces $\ell^2(\mathbb{X}^{(0)})$, $\ell^2(\mathbb{X}^{(1)})$, and by source, target and reversing operators S, $T : \ell^2(\mathbb{X}^{(1)}) \to \ell^2(\mathbb{X}^{(0)})$ and $\Theta : \ell^2(\mathbb{X}^{(1)}) \to \ell^2(\mathbb{X}^{(1)})$. Moreover the spaces $\ell^2(\mathbb{X}^{(i)})$ as endowed with representations of $C_r^*(\mathbb{F})$ which are intertwined by these operators.

In the case of a usual tree $\mathbb{X} = X$ endowed with an origin, one can define a Fredholm operator from the space of antisymetric edges, $\ell^2_\wedge(X^{(1)}) = \mathrm{Ker}(\Theta + \mathrm{id})$, to the space of vertices $\ell^2(X^{(0)})$: it is the so-called *Julg-Valette operator*, which maps each edge to its furthest endpoint from the origin. One shows moreover that this operator defines an equivariant KK-theoretic element $\gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$, and using this element Julg and Valette established in 1984 the *K-amenability* of groups acting on trees with amenable stabilizers, i.e. the fact the the natural map $C_f^*(\Gamma) \to C_r^*(\Gamma)$ induces an isomorphism in KK-theory.

Motivated by this result I introduce a notion of ascending orientation for quantum Cayley trees $\mathbb{X}$, given by two "ascending projectors" $p_{+\star}$ and $p_{\star+}$ acting "on the right and on the left" of $\ell^2(\mathbb{X}^{(1)})$, and I use this notion to describe the space of antisymetric edges $\ell^2_\wedge(\mathbb{X}^{(1)}) \subset \ell^2(\mathbb{X}^{(1)})$. It turns out in the quantum case that Θ is not involutive, and that the space $\ell^2_\wedge(\mathbb{X}^{(1)})$ is "to small". However, the interaction between the orientation projectors $p_{+\star}(1 - p_{\star+})$ and the reversing operator Θ yields an inductive system inside $\ell^2(\mathbb{X}^{(1)})$, and I study its inductive limit, denoted $\ell^2_\infty(\mathbb{X}^{(1)})$. This space of "edges at infinity" is equipped with a natural operator $P : p_{++}\ell^2(\mathbb{X}^{(1)}) \to \ell^2_\infty(\mathbb{X}^{(1)})$, where $p_{++} = p_{+\star}p_{\star+}$. Using computations in the category $\mathrm{Corep}\,\mathbb{F}$ and fine analytic arguments one obtains:

Theorem 4 [2, Thm. 6.5] *Let $\mathbb{X}$ be the quantum Cayley graph associated to a free quantum group $\mathbb{F}$. If the quantum dimensions of generators of* $\mathrm{Corep}\,\mathbb{F}$ *are different from* 2 *we have*

$$p_{++}\ell^2(\mathbb{X}^{(1)}) = p_{++}\ell^2_\wedge(\mathbb{X}^{(1)}) \oplus P^*\ell^2_\infty(\mathbb{X}^{(1)}).$$

Moreover p_{++} and P commute to the natural actions of $C^*(\mathbb{F})$ modulo compact operators. In particular the operator $T(p_{++} + P^*) : \ell^2_\wedge(\mathbb{X}^{(1)}) \oplus \ell^2_\infty(\mathbb{X}^{(1)}) \to \ell^2(\mathbb{X}^{(0)})$ defines an element $\delta \in KK^{\mathbb{F}}(\mathbb{C}, \mathbb{C})$.

I show moreover that the actions of $\mathbb{F}$ on $\ell^2_\wedge(\mathbb{X}^{(1)})$, $\ell^2_\infty(\mathbb{X}^{(1)})$ and $\ell^2(\mathbb{X}^{(0)})$ are weakly contained in the regular representation. To obtain the K-amenability of free quantum groups it remains to show that $\delta = [\mathrm{id}]$. However this question stays open and the K-amenability of free quantum groups was established only later in the article [8], using results of Voigt about the orthogonal case. On the other hand I give another application of the techniques developed in the article:

Theorem 5 [2, Thm. 8.3] *Let $\mathbb{F}$ be a free quantum group such that the quantum dimensions of the generators of* $\mathrm{Corep}\,\mathbb{F}$ *are different from* 2. *Then $\mathbb{F}$ satisfies the Akemann-Ostrand Property.*

In the article [7] I pursue the study of quantum Cayley graphs associated to free quantum groups. In the case of a usual free group F_N, the Cayley graph X is a tree, and in particular one can associate to each element $g \in F_N$, which is also a vertex in the tree, the sum $c(g)$ of edges along the unique path from g to the origin, considered as an element of $\ell^2_\wedge(X^{(1)})$. It is easy to see that the map $c : F_N \to \ell^2_\wedge(X^{(1)})$ is a cocycle for the natural representation of F_N on $\ell^2_\wedge(X^{(1)})$, moreover the norm $\|c(g)\| = \sqrt{2l(g)}$ goes to $+\infty$ as $l(g) \to +\infty$: the cocycle c is said to be *proper*. This establishes *Haagerup's Property* for the free group F_N.

Motivated by this classical result, I investigate in [7] the notion of *path cocycle* in a quantum Cayley graph $\mathbb{X}$. Consider the natural dense sub-$*$-algebra $\mathbb{C}[\Gamma] \subset C^*(\Gamma)$, the natural dense subspaces $\ell^2_f(\mathbb{X}^{(i)}) \subset \ell^2(\mathbb{X}^{(i)})$ corresponding to functions with finite support, as well as the distinguished cyclic vector $\xi_0 \in \ell^2(\mathbb{X}^{(0)})$ corresponding to the origin of the graph. A *path cocycle* is a linear map $c : \mathbb{C}[\Gamma] \to \ell^2_{\wedge f}(\mathbb{X}^{(1)})$ such that $c(xy) = xc(y) + c(x)\epsilon(y)$ and $T \circ c(x) = x\xi_0 - \epsilon(x)\xi_0$, where $\epsilon : \mathbb{C}[\Gamma] \to \mathbb{C}$ is the co-unit of Γ. In the case of free quantum groups, pursuing the study of $\ell^2_\wedge(\mathbb{X}^{(1)})$ started in [2], I prove existence and uniqueness of path cocycles: hence $\mathbb{X}$ can be interpreted as a quantum tree.

Theorem 6 [7, Crl. 4.2, Thm. 4.4 and Prop. 4.5] *Let $\mathbb{X}$ be the quantum Cayley graph associated to a free quantum group $\mathbb{F}$. Assume that the quantum dimensions of generators of* Corep Γ *are different from* 2. *Then:*

1. *There exists a unique path cocycle $c : \mathbb{C}[\Gamma] \to \ell^2_{\wedge f}(\mathbb{X}^{(1)})$.*
2. *If $\mathbb{F} = \mathbb{F}O(Q)$, this cocycle is bounded.*
3. *If $\mathbb{F} = \mathbb{F}U(Q)$, it is neither proper nor bounded.*

One sees that the cocycles obtained in this way are not proper, in particular this construction does not yield Haagerup's Property. In the orthogonal case, the cocycle is bounded, and this contrasts strongly with the situation for usual free groups. We exploit this fact, using a "university Lemma" and the Property of Rapid Decay, to study the Hochschild groups $H^1_{(2)}(\mathbb{C}[\Gamma]) = H^1(\mathbb{C}[\Gamma], \ell^2(\Gamma))$ and to compute the first L^2-Betti number of orthogonal free quantum groups:

Theorem 7 [7, Thm. 5.1, Crl. 5.2] *Let $Q \in M_N(\mathbb{C})$ be a unitary matrix.*
If $Q\bar{Q} \in \mathbb{C}I_N$ we have $H^1_{(2)}(\mathbb{C}[\mathbb{F}O(Q)]) = 0$ and $\beta^{(2)}_1(\mathbb{F}O(Q)) = 0$. Besides, if $N \geq 2$ we have $\beta^{(2)}_1(\mathbb{F}U(Q)) \neq 0$.

This constitutes the first harmonic analysis result showing a different behaviour for the orthogonal free quantum groups $\mathbb{F}O(Q)$ compared to usual free groups F_N: indeed for the later ones we have $\beta^{(2)}_1(F_N) = N - 1$.

Gromov boundary and applications

In the article [4] we establish several structural results for the von Neumann algebras associated to orthogonal free quantum groups $\mathbb{F}O(Q)$. One of the main tools we use is a quantum analogue of the Gromov boundary of free groups. Denoting S_n the sphere of radius n in the free group F_N with respect to the word length, we have natural maps $S_{n+1} \to S_n$ obtained by deletion of the last letter. The Gromov boundary ∂F_N of F_N is the projective limit of the spheres S_n relatively to these maps, it is a compact, totally disconnected space for the initial topology associated to the projective system. Moreover the action of F_N on itself by left multiplication induces an action by homeomorphisms on ∂F_N which is amenable, and this implies *exactness* of the group F_N.

In the case of $\mathbb{F} = \mathbb{F}O(Q)$ we have a decomposition of the algebra of functions into matrix algebras: $C_0(\mathbb{F}) = \bigoplus_n p_n C_0(\mathbb{F})$ with $p_n C_0(\mathbb{F}) \simeq L(H_n)$, where H_n is the space of the n^{th} irreducible corepresentation $r_n \in \operatorname{Corep} \mathbb{F}$. This decomposition corresponds to the decomposition of F_N into spheres. Moreover we have the fusion rules $r_n \otimes r_1 \simeq r_{n+1} \oplus r_{n-1}$, and the inclusion $H_{n+1} \to H_n \otimes H_1$ induces completely positive maps $p_n C_0(\mathbb{F}) \to p_{n+1} C_0(\mathbb{F})$. We denote then $C(\partial\mathbb{F}) = \varinjlim p_n C_0(\mathbb{F})$ with respect to these maps: this injective limit is *a priori* a subspace of $C_b(\mathbb{F})/C_0(\mathbb{F})$.

Theorem 8 [4, Prop. 3.4, 3.6 et 3.8, Thm. 4.5] *Let $\partial\mathbb{F}$ be the Gromov boundary of an orthogonal free quantum group $\mathbb{F} = \mathbb{F}O(Q)$. Then:*

1. *$C(\partial\mathbb{F})$ is a nuclear unital sub-C^*-algebra of $Q = C_b(\mathbb{F})/C_0(\mathbb{F})$.*
2. *The "right" coaction of $C_0(\mathbb{F})$ on $C_b(\mathbb{F})$, factorized through Q, induces the trivial action of $\mathbb{F}$ on $\partial\mathbb{F}$.*
3. *The "left" coaction of $C_0(\mathbb{F})$ on $C_b(\mathbb{F})$, factorized through Q, induces an action of $\mathbb{F}$ on $\partial\mathbb{F}$.*
4. *This last action is* amenable.

As in the case of usual discrete groups, this result implies again the Akemann-Ostrand Property for $\mathbb{F}O(Q)$, without restriction on the quantum dimension of r_1, as well as the exactness of $C^*_r(\mathbb{F}O(Q))$. We give also another argument that allows to deduce exactness from the monoidal equivalence between $\mathbb{F}O(Q)$ and the dual of a quantum group $SU_q(2)$. Finally, we show that the Gromov boundary $\partial\mathbb{F}O(Q)$ identifies to the *Martin boundary* of the quantum random walk on $\mathbb{F}O(Q)$ generated by the fundamental corepresentation r_1.

On the other hand we investigate the structure of the von Neumann algebra $\mathscr{L}(\mathbb{F}) = C^*_r(\mathbb{F})'' \subset B(\ell^2(\mathbb{F}))$ for $\mathbb{F} = \mathbb{F}O(Q)$. The main tool of this study is an operator $P : \ell^2(\mathbb{F}) \to \ell^2(\mathbb{F})$ which is the quantum analogue of "conjugation by generators": in the case of $F_N = \langle a_i \rangle$, one would consider

$$P(f) = (2N)^{-1} \sum \left(f(a_i^{-1} \cdot a_i) + f(a_i \cdot a_i^{-1}) \right).$$

In the classical case, the analytic properties of P are studied using the combinatorics of conjugation in the free group. In the quantum case, we use rather computations in the category $\operatorname{Corep}\mathbb{F}$, and more precisely, we establish asymptotic properties of the geometry of fusion rules between irreducible corepresentations. From this investigation we can deduce the following results:

Theorem 9 [4, Thm. 7.1] *Let $Q \in M_N(\mathbb{C})$, $N \geq 3$, be a matrix such that $Q\bar{Q} = \pm I_N$ and $\|Q\|^2 \leq \operatorname{Tr}(Q^*Q)/\sqrt{5}$. We denote $\mathbb{F} = \mathbb{F}O(Q)$ and $\Lambda \subset \mathbb{R}^*_+$ the subgroup generated by the spectrum of $(Q^*Q)\otimes(Q^*Q)^{-1}$. Then the von Neumann algebra $\mathscr{L}(\mathbb{F})$ is a full and solid factor. If Q is unitary, $\mathscr{L}(\mathbb{F})$ is type II_1. If $\Lambda = \lambda^{\mathbb{Z}}$ with $\lambda \in]0,1[$, $\mathscr{L}(\mathbb{F})$ is type III_λ, and in the remaining cases $\mathscr{L}(\mathbb{F})$ is type III_1.*

Note that the assumption $\|Q\|^2 \leq \operatorname{Tr}(Q^*Q)/\sqrt{5}$, which is imposed by technical limitations of our proof, means that Q cannot be "too far" from a unitary matrix. The solidity of $\mathscr{L}(\mathbb{F})$, which follows from Akemann-Ostrand Property and exactness, implies the fact that $\mathscr{L}(\mathbb{F})$ is a *prime factor*: if $\mathscr{L}(\mathbb{F}) \simeq M_1\bar{\otimes}M_2$, necessarily M_1 or M_2 is type I. Combining the study of the operator P and the Property of Rapid Decay, we prove as well the simplicity of $C^*_r(\mathbb{F}O(Q))$ when Q^*Q is sufficiently close to I_N.

Quantum Bass-Serre trees and applications

The article [1] presents results from my Ph.D. Thesis concerning equivariant KK-theory of C^*-algebras with respect to actions of locally compact quantum groups, in the framework introduced by Baaj and Skandalis in 1989. We extend to the quantum case several results of the classical theory, including the Green-Julg isomorphism, the construction of a descent morphism and various characterizations of K-amenability.

In the second part of the article I investigate K-amenability of amalgamated free products of discrete quantum groups, using a quantum analogue of the Bass-Serre tree. Recall that for an amalgamated free product $\Gamma = \Gamma_0 *_\Lambda \Gamma_1$ of usual discrete groups, the Bass-Serre tree has $\Gamma/\Gamma_0 \sqcup \Gamma/\Gamma_1$ as set of vertices and Γ/Λ as set of edges, with the natural projections as target and source maps t, s. The result of Julg and Valette mentionned above applies equally to this situation and implies that an amalgamated free product of amenable discrete groups is K-amenable.

In the quantum case, after proving some useful results about subgroups and quotients of discrete quantum groups, I construct an element $\gamma \in KK^{\mathbb{\Gamma}}(\mathbb{C},\mathbb{C})$, for $\mathbb{\Gamma} = \mathbb{\Gamma}_0 *_{\mathbb{\Lambda}} \mathbb{\Gamma}_1$: to this purpose I use a quantum analogue of the Julg-Valette operator associated to the Bass-Serre tree of the free product, which connects the Hilbert spaces $\ell^2(\mathbb{X}^{(0)}) = \ell^2(\mathbb{\Gamma}/\mathbb{\Gamma}_0) \oplus \ell^2(\mathbb{\Gamma}/\mathbb{\Gamma}_1)$ and $\ell^2(\mathbb{X}^{(1)}) = \ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$. Then one proves:

Theorem 10 [1, Thm. 3.3] *Let $\gamma \in KK^{\mathbb{\Gamma}}(\mathbb{C},\mathbb{C})$ be the element of KK-theory induced by the Julg-Valette operator associated to an amalgamated free product $\mathbb{\Gamma} = \mathbb{\Gamma}_0 *_{\mathbb{\Lambda}} \mathbb{\Gamma}_1$ of discrete quantum groups.*

1. *We have $\gamma = [\mathrm{id}]$ in $KK^{\mathbb{\Gamma}}(\mathbb{C},\mathbb{C})$.*
2. *If $\mathbb{\Gamma}_0$, $\mathbb{\Gamma}_1$ are amenable, then the representations of $\mathbb{\Gamma}$ on $\ell^2(\mathbb{\Gamma}/\mathbb{\Gamma}_0)$, $\ell^2(\mathbb{\Gamma}/\mathbb{\Gamma}_1)$ and $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$ are weakly contained in the regular representation.*

Therefore, an amalgamated free product of amenable discrete quantum groups is K-amenable.

In the article [8] we establish the *strong Baum-Connes Property* for unitary free quantum groups $\mathbb{F}U(Q)$, which allows to compute their K-theory. We base on work by Voigt in 2011 who proves that orthogonal free quantum groups $\mathbb{F}O(Q)$ satisfy this property: via the inclusions $\mathbb{F}U(Q) \subset \mathbb{Z} * \mathbb{F}O(Q)$ established by Banica when $Q\bar{Q} \in \mathbb{C}I_N$, it suffices then to prove the stability of the Baum-Connes Property under passage to free products and subgroups — more precisely we consider a particular type of subgroups that we call *divisible*.

For the stability under free products, we use again the quantum Bass-Serre tree $\mathbb{X}$ associated to $\mathbb{\Gamma} = \mathbb{\Gamma}_0 * \mathbb{\Gamma}_1$. Following ideas of Kasparov and Skandalis, we associate to $\mathbb{X}$ a C^*-algebra $\mathscr{P} \subset C_0(\mathbb{R}) \otimes K(\ell^2(\mathbb{X}^{(0)}) \oplus \ell^2(\mathbb{X}^{(1)}))$ obtained by assembling the algebras $C_0(\mathbb{X}^{(i)})$ and the operators R, S with support conditions over $\mathbb{R}$. We check that $\mathscr{P}$ is stable with respect to the natural representation of $\mathbb{\Gamma}$, and the inclusion of $\Sigma\mathscr{P} = C_0(\mathbb{R}) \otimes \mathscr{P}$ in $C_0(\mathbb{R}^2) \otimes K(\ell^2(\mathbb{X}^{(0)}) \oplus \ell^2(\mathbb{X}^{(1)}))$ induces then a *Dirac element* $\alpha \in KK^{\mathbb{\Gamma}}(\Sigma\mathscr{P}, \mathbb{C})$. Combining this construction with the one of the γ element above, we obtain as well $\beta \in KK^{\mathbb{\Gamma}}(\mathbb{C}, \Sigma\mathscr{P})$ such that $\beta \otimes \alpha = \gamma = [\mathrm{id}] \in KK^{\mathbb{\Gamma}}(\mathbb{C}, \mathbb{C})$.

For the strong Baum-Connes Property in the quantum framework we use the approach to the Baum-Connes conjecture developed by Meyer and Nest — in particular we work in the category $KK^{\mathbb{\Gamma}}$ triangulated by mapping cones. We denote $TI_{\mathbb{\Gamma}}$ the full subcategory whose objects are of the form $C_0(\mathbb{\Gamma}) \otimes A$, with trivial action on A, and $\langle TI_{\mathbb{\Gamma}} \rangle$ the localizing subcategory generated by $TI_{\mathbb{\Gamma}}$. We say that $\mathbb{\Gamma}$ satisfies the (torsion-free) strong Baum-Connes Property if $\langle TI_{\mathbb{\Gamma}} \rangle = KK^{\mathbb{\Gamma}}$: this implies K-amenablity and, in the classical case, the validity of the Baum-Connes conjecture with coefficients. Then we show the stability of this Property under free products using the preceeding constructions — in fact one needs to check that all these constructions are equivariant with respect to the actions of the *Drinfel'd double* $D\mathbb{\Gamma}$.

Theorem 11 [8, Thm. 6.6] *Let $\mathbb{\Gamma} = \mathbb{\Gamma}_0 * \mathbb{\Gamma}_1$ be a free product of discrete quantum groups, and $\mathbb{X}$ the associated Bass-Serre tree. We denote α, β, γ the elements of $D\mathbb{\Gamma}$-equivariant KK-theory associated to $\mathbb{X}$. If $\mathbb{\Gamma}_0$ and $\mathbb{\Gamma}_1$ satisfy the strong Baum-Connes Property, then $\Sigma\mathscr{P} \in \langle TI_{\mathbb{\Gamma}} \rangle$ and $\alpha \otimes \beta = [\mathrm{id}] \in KK^{\mathbb{\Gamma}}(\Sigma\mathscr{P}, \Sigma\mathscr{P})$. As a result, $\mathbb{\Gamma}$ satisfies the strong Baum-Connes Property.*

Corollary 12 [8, Thm. 6.9] *The free quantum groups $\mathbb{F} = \mathbb{F}U(P_1) * \cdots * \mathbb{F}U(P_k) * \mathbb{F}O(Q_1) * \cdots * \mathbb{F}O(Q_l)$, with $Q_i \bar{Q}_i$ scalar, satisfy the strong Baum-Cones Property. In particular they are K-amenable.*

The strong Baum-Connes Property allows to effectively compute the K-theory groups. More precisely, a length 2 resolution of $A \in KK^{\mathbb{\Gamma}}$

$$0 \to C_1 \to C_0 \to A \to 0$$

by objects $C_i \in TI_{\mathbb{\Gamma}}$, exact in *non-equivariant* K-theory, induces a cyclic exact sequence that computes the K-theory of the crossed product $A \rtimes \mathbb{\Gamma}$. In the case when $\mathbb{\Gamma} = \mathbb{F}U(Q)$ is a unitary free quantum group, it is easy to construct such a resolution of $A = \mathbb{C}$, and we can deduce:

Theorem 13 [8, Thm. 7.2] *Let $Q \in GL_N(\mathbb{C})$ and $\mathbb{F} = \mathbb{F}U(Q)$. We have $K_0(C_r^*(\mathbb{F})) = \mathbb{Z}$, generated by the class of the unit, and $K_1(C_r^*(\mathbb{F})) = \mathbb{Z}^2$, generated by the classes of the fundamental corepresentation and its conjugate.*