Stabilizer subgroups of universal compact quantum groups and the Connes embedding property

Roland Vergnioux
joint work with Benoît Collins and Michael Brannan

University of Normandy (France)

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Outline

Orthogonal free quantum groups

Wang's algebra defined by generators and relations:

$$A_o(n) = \langle u_{ij}, 1 \leq i, j \leq n \mid u_{ij} = u_{ij}^*, \ \ (u_{ij}) \ \text{unitary} \rangle.$$

Consider the discrete group $FO_n = (\mathbb{Z}/2\mathbb{Z})^{*n}$ and the compact group O_n. We have two interesting quotient maps:

$$\begin{aligned} A_o(n) &\to A_o(n)/I \simeq C^*(FO_n) \quad \text{with} \quad I = \langle u_{ij}, i \neq j \rangle, \\ A_o(n) &\to A_o(n)/J \simeq C(O_n) \qquad \text{with} \quad J = \langle [u_{ij}, u_{kl}] \rangle. \end{aligned}$$

We denote $A_o(n) = C^*(\mathbb{F}O_n) = C(O_n^+)$. There is a natural coproduct

$$\Delta : A_o(n) \to A_o(n) \otimes A_o(n), \ u_{ij} \mapsto \textstyle\sum_k u_{ik} \otimes u_{kj}.$$

➙ $\mathbb{F}O_n$ is a discrete quantum group and O_n^+ is a compact quantum group: the "orthogonal free quantum group" and the "universal orthogonal quantum group", dual to each other.

Discrete/Compact quantum groups

A Woronowicz C^*-algebra is a unital C^*-algebra A with $*$-homomorphism $\Delta : A \to A \otimes A$ (coproduct) such that

- $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
- $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

Notation : $A = C^*(\mathbb{\Gamma}) = C(\mathbb{G})$.

Examples :

- G compact group, $A = C(G)$, $\Delta(f) = ((x, y) \mapsto f(xy))$, characterized by commutativity of A ;
- Γ discrete group, $A = C^*(\Gamma)$, $\Delta(g) = g \otimes g$ — but also $A = C^*_{\mathrm{red}}(\Gamma)$, characterized by co-commutativity : $\Sigma\Delta = \Delta$.

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General theory :

- Haar state $h \in C^*(\mathbb{\Gamma})^*$ ➔ GNS representation $\lambda : C^*(\mathbb{\Gamma}) \to B(\ell^2 \mathbb{\Gamma})$,
- $C^*_{\mathrm{red}}(\mathbb{\Gamma}) = \lambda(C^*(\mathbb{\Gamma}))$ and $\mathscr{L}(\mathbb{\Gamma}) = C^*_{\mathrm{red}}(\mathbb{\Gamma})''$,
- trivial representation / co-unit $\epsilon : C^*_{\mathrm{f}}(\mathbb{\Gamma}) = C_{\mathrm{f}}(\mathbb{G}) \to \mathbb{C}$,
- f.-d. corepresentations $v \in M_k(\mathbb{C}) \otimes C(\mathbb{G})$, intertwiners $T \in \mathrm{Hom}_{\mathbb{G}}(v, w) \subset M_{l,k}(\mathbb{C})$.

$\mathbb{\Gamma}$ is called unimodular if h is a trace, amenable if ϵ factors through λ.

Analogies with free group C^*-algebras

$\mathbb{F}O_n$ shares many properties with usual free groups.
On the operator algebraic side:

- $\mathbb{F}O_n$ is non amenable for $n \geq 3$ [Banica 1997];
- $C^*_{\text{red}}(\mathbb{F}O_n)$ is simple, $\mathscr{L}(\mathbb{F}O_n)$ is a full factor [Vaes-V. 2005];
- bi-exactness, rapid decay [V. 2005, 2007], K-amenability [Voigt 2011], a-T-menability [Brannan 2012], weak amenability [Freslon 2013], ...

Analogies with free group C^*-algebras

$\mathbb{F}O_n$ shares many properties with usual free groups.

On the free probability side:

- $\chi_1 = \sum u_{ii}$ is a semicircular variable with respect to h [Banica 1997];
- the elements $(\sqrt{n}\, u_{ij})_{i,j\leq s}$ are asymptotically free and semi-circular with respect to h as $n \to \infty$ [Banica-Collins 2007];
- computation of the spectral measure of u_{ij} with respect to h [Banica-Collins-Zinn-Justin 2009]; ...

Main result of this talk:

- the generators $u_{ij} \in C^*(\mathbb{F}O_n)$ admit matricial microstates (up to any order and precision) with respect to h (Connes' embedding property)

Strategy:

- $\mathbb{F}O_2$ is amenable, hence $\mathscr{L}(\mathbb{F}O_2) \subset R^\omega$ ➜ induction over n.
- O_n^+ is generated by two copies of O_{n-1}^+.

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   - Main results
2. Stabilizer subgroups
   - Generating subgroups
   - Stabilizer subgroups of O_n^+
   - Idea of the proof
3. Applications
   - Connes' embedding property
   - Free entropy dimension and microstates

Generating subgroups

$\mathbb{G}$ compact quantum group with *full* Woronowicz C^*-algebra $C_{\mathrm{f}}(\mathbb{G})$.

Closed subgroup $\mathbb{H} \subset \mathbb{G}$: compact quantum group with surjective Hopf-$*$-homomorphism $\pi : C_{\mathrm{f}}(\mathbb{G}) \twoheadrightarrow C_{\mathrm{f}}(\mathbb{H})$.

Inner faithful $*$-homomorphism $f : C_{\mathrm{f}}(\mathbb{G}) \to B$: for any factorization

$$f : C_{\mathrm{f}}(\mathbb{G}) \xrightarrow{\pi} C_{\mathrm{f}}(\mathbb{H}) \xrightarrow{g} B$$

with π surjective Hopf-$*$-homomorphism, π is an isomorphism.

Definition

Let $(\mathbb{H}_1, \pi_1)$, $(\mathbb{H}_2, \pi_2)$ but closed subgroups of $\mathbb{G}$. Then $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle$ if $(\pi_1 \otimes \pi_2) \circ \Delta : C_{\mathrm{f}}(\mathbb{G}) \to C_{\mathrm{f}}(\mathbb{H}_1) \otimes C_{\mathrm{f}}(\mathbb{H}_2)$ is inner faithful.

More generally, the subgroup $\mathbb{H} \subset \mathbb{G}$ generated by $\mathbb{H}_1$, $\mathbb{H}_2$ is the *Hopf image* of $(\pi_1 \otimes \pi_2) \circ \Delta$.

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Restriction: $v \in M_n(\mathbb{C}) \otimes C_f(\mathbb{G})$ repr. of $\mathbb{G}$ + $(\mathbb{H}, \pi)$ closed subgroup
➔ $(\mathrm{id} \otimes \pi)(v)$, restricted representation of $\mathbb{H}$.

Proposition

$$\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle \iff \forall v, w \in \mathrm{Rep}(\mathbb{G}) \quad \mathrm{Hom}_{\mathbb{G}}(v, w) = \mathrm{Hom}_{\mathbb{H}_1}(v, w) \cap \mathrm{Hom}_{\mathbb{H}_2}(v, w)$$

Examples

$H_1, H_2 \subset G$ classical compact groups ➜ usual notions.

$\mathbb{G} = \hat{\Gamma}$ dual of discrete group Γ

➜ π_i induced by surjective group morphisms $\pi_i : \Gamma \to \Gamma_i$.

➜ $\hat{\Gamma} = \langle \hat{\Gamma_1}, \hat{\Gamma_2} \rangle \iff \Gamma \to \Gamma_1 \times \Gamma_2$ faithful.

Some subgroups of O_n^+:

- $\rho : C(O_n^+) \to C(O_n)$, $[u_{ij}, u_{kl}] \to 0$.
- $\pi_i : C(O_n^+) \to C(O_{n-1,i}^+) \simeq C(O_{n-1}^+)$, $u_{ii} \to 1$.

Note that $O_{n-1,i}^+ \subset O_n^+$ is the stabilizer of $e_i \in \mathbb{C}^n$.

Theorem

For $n \geq 4$ and $i \neq j$ we have $O_n^+ = \langle O_{n-1,i}^+, O_{n-1,j}^+ \rangle = \langle O_{n-1,i}^+, O_n \rangle$.

Brauer diagrams

$P(k,l)$: set of partitions of k upper points and l lower points into pairs
$NCP(k,l) \subset P(k,l)$: partitions that can be represented by a planar diagram with noncrossing strings

Let $H = \mathbb{C}^n$ and associate to $p \in P(k,l)$ the linear map $T_p : H^{\otimes k} \to H^{\otimes l}$:

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_j \begin{pmatrix} i_1 \ldots i_k \\ p \\ j_1 \ldots j_l \end{pmatrix} e_{j_1} \otimes \cdots \otimes e_{j_l},$$

where the middle symbol is 1 if all blocs in p join pairs of equal indices, and 0 if not.

Then:

- $\mathrm{Hom}_{O_n}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Span}\{T_p \mid p \in P(k,l)\}$ [Brauer],
- $\mathrm{Hom}_{O_n^+}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Span}\{T_p \mid p \in NCP(k,l)\}$ [Banica].

A lemma of linear algebra

Denote $TCP(k,l) \subset P(k,l)$ the subset of diagrams where crossings are allowed only with lines that are connected to an upper point. Then:

Lemma

$$\mathrm{Hom}_{O^+_{n-1,i}}(1, u^{\otimes k}) = \mathrm{Span}\{T_p(e_i \otimes \cdots \otimes e_i) \mid s \leq k, p \in TCP(s,k)\}$$

Put $\xi_s = e_1 \otimes \cdots \otimes e_1 \otimes e_2 + e_1 \otimes \cdots \otimes e_2 \otimes e_1 + \cdots + e_2 \otimes e_1 \otimes \cdots \otimes e_1 \in H^{\otimes s}$.

Lemma

We have $\mathrm{Hom}_{O^+_{n-1,i}}(1, u^{\otimes k}) \cap \mathrm{Hom}_{O^+_{n-1,j}}(1, u^{\otimes k}) = \mathrm{Hom}_{O^+_n}(1, u^{\otimes k})$ *iff the family of vectors* $\{T_p(\xi_s) \mid 1 \leq s \leq k, p \in TCP(s,k)\}$ *is linearly independant.*

A lemma of linear algebra

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Lemma

We have $\mathrm{Hom}_{O^+_{n-1,i}}(1, u^{\otimes k}) \cap \mathrm{Hom}_{O^+_{n-1,j}}(1, u^{\otimes k}) = \mathrm{Hom}_{O^+_n}(1, u^{\otimes k})$ *iff the family of vectors* $\{T_p(\xi_s) \mid 1 \leq s \leq k, p \in TCP(s,k)\}$ *is linearly independant.*

Lemma

If $n \geq 4$*, the independance property of the previous lemma is true for any* k*. As a result* $O^+_n = \langle O^+_{n-1,i}, O^+_{n-1,j} \rangle$*.*

Moreover we have strong numerical evidence of:

Conjecture

The same is true for $n = 3$*.*

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Connes' embedding property

Let R^ω be an ultrapower of the hyperfinite II_1 factor.
For A unital C^*-algebra, define

$$CEP(A) = \{\tau : A \to \mathbb{C} \text{ tracial state} \mid \pi_\tau(A)'' \hookrightarrow R^\omega \text{ tracially}\},$$

where π_τ is the GNS representation.

For $\mathbb{\Gamma}$ *unimodular* discrete quantum group: $CEP(\mathbb{\Gamma}) = CEP(C_{\mathrm{f}}^*(\mathbb{\Gamma}))$.
We say that $\mathbb{\Gamma}$ is **hyperlinear** if $h \in CEP(\mathbb{\Gamma})$, i.e. if its von Neumann algebra $\mathscr{L}(\mathbb{\Gamma})$ embeds tracially in R^ω.

Proposition

- *If* $\tau_1, \tau_2 \in CEP(\mathbb{\Gamma})$ *then* $\tau_1 * \tau_2 = (\tau_1 \otimes \tau_2) \circ \Delta \in CEP(\mathbb{\Gamma})$.
- *If* $\tau_n \to \tau$ *pointwise and* $\tau_n \in CEP(\mathbb{\Gamma})$ *then* $\tau \in CEP(\mathbb{\Gamma})$.

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- *If* $\tau_n \to \tau$ *pointwise and* $\tau_n \in CEP(\Gamma)$ *then* $\tau \in CEP(\Gamma)$.

Let $(\mathbb{H}_1, \pi_1)$, $(\mathbb{H}_2, \pi_2)$ be subgroups of $\mathbb{G}$.
Denote $h_i = h_{\mathbb{H}_i} \circ \pi_i : C^*_\mathrm{f}(\mathbb{G}) \to \mathbb{C}$ and $h = h_\mathbb{G} : C^*_\mathrm{f}(\mathbb{G}) \to \mathbb{C}$.

Proposition

We have $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle$ *iff* $h = \lim (h_1 * h_2)^{*n}$ *pointwise.*

Corollary

If $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle$ *and* $\hat{\mathbb{H}}_1$, $\hat{\mathbb{H}}_2$ *are hyperlinear, then* $\hat{\mathbb{G}}$ *is hyperlinear.*

Hyperlinearity of $\mathbb{F}O_n$

Corollary

If $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle$ and $\hat{\mathbb{H}}_1$, $\hat{\mathbb{H}}_2$ are hyperlinear, then $\hat{\mathbb{G}}$ is hyperlinear.

Recall that $\mathbb{F}O_n = \hat{O}_n^+$ and $O_n^+ = \langle O_{n-1,i}^+, O_{n-1,j}^+ \rangle$ for $n \geq 4$.
Moreover $\mathbb{F}O_2$ is hyperlinear because it is amenable.
→ $\mathbb{F}O_n$ hyperlinear for all n **if** $O_3^+ = \langle O_{2,i}^+, O_{2,j}+ \rangle$.

Bypass to avoid the use of the conjecture at $n = 3$:

Lemma (after A. Chirvasitu)

We have $O_4^+ = \langle O_2^+ \hat{*} O_2^+, O_4 \rangle$.

Altogether:

Theorem

$\mathbb{F}O_n$ is hyperlinear for all $n \neq 3$.

Free entropy dimension

Denote by δ_0 Voiculescu's modified free entropy dimension.

Consequence of Connes' embedding property: we can apply Jung's "hyperfinite monotonicity" result. Since $\mathscr{L}(\mathbb{F}O_n)$ contains diffuse von Neumann subalgebras this yields:

Corollary

For the generators u_{ij} of $\mathscr{L}(\mathbb{F}O_n)$, $n \neq 3$, we have $1 \leq \delta_0(u_{ij})$.

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Corollary

For the generators u_{ij} of $\mathscr{L}(\mathbb{F}O_n)$, $n \neq 3$, we have $1 \leq \delta_0(u_{ij})$.

On the other hand we have an upper bound coming from ℓ^2-Betti numbers. More precisely

$$\delta_0(u_{ij}) \leq \delta^*(u_{ij}) \leq \beta_1^{(2)}(\mathbb{F}O_n) - \beta_0^{(2)}(\mathbb{F}O_n) + 1$$

by [Biane-Capitaine-Guionnet] and [Connes-Shlyakhtenko].

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by [Biane-Capitaine-Guionnet] and [Connes-Shlyakhtenko]. Moreover

Theorem (V. 2012)

We have $\beta_1^{(2)}(\mathbb{F}O_n) = 0$ for all $n \geq 3$.

Since $\mathbb{F}O_n$ is infinite we have $\beta_0^{(2)}(\mathbb{F}O_n) = 0$ [Kyed] and finally

Corollary

For the generators u_{ij} of $\mathscr{L}(\mathbb{F}O_n)$, $n \neq 3$, we have $\delta_0(u_{ij}) = 1$.

What next?

Explicit matricial models

- $O_2^+ = SU_{-1}(2)$ has an exact 2×2 random matrix model.
- our inductive strategy is also explicit.

→ explicit asymptotic random matrix model for the generators of $\mathbb{F}O_n$.
In progress: find a more natural random matrix model

Generation result at $n = 3$
Recall Di Francesco's determinant formula : $\det G_{2k}(n) = \prod_{l=1}^{k} U_l(n)^{a_{k,l}}$, where $G_{2k}(n)$ is the Gram matrix of the vectors T_p, $p \in NCP(0, 2k)$, and the U_l are Chebyshev polynomials.
We have a similar conjectural formula for the vectors $T_p(e_i)$, $p \in TCP(1, 2k + 1)$ (involving other polynomials V_l) which is much stronger than the needed independence property.
In progress: proof of this formula.