Free entropy dimension and the orthogonal free quantum groups

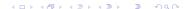
Roland Vergnioux joint work with Michael Brannan

University of Normandy (France)

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Outline

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 - The von Neumann algebra $\mathcal{L}(\mathbb{F}O_n)$
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 - The quantum Cayley graph



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Orthogonal free quantum groups

Wang's algebra defined by generators and relations:

$$\mathcal{A}_o(n) = \langle u_{ij}, 1 \leq i, j \leq n \mid u_{ij} = u_{ij}^*, \ (u_{ij}) \ \text{unitary} \rangle.$$

It comes with a natural "group-like" structure:

$$\Delta: A_o(n) \to A_o(n) \otimes A_o(n), \ u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}.$$

Why "group-like"?

We have $A_o(n) \twoheadrightarrow C(O_n)$, $u_{ij} \mapsto (g \mapsto g_{ij})$ and Δ induces

$$\Delta: C(O_n) \to C(O_n \times O_n), \ \Delta(f)(g,h) = f(gh).$$

One can recover the compact group O_n from $(C(O_n), \Delta)$.

We denote $A_o(n) = C(O_n^+)$, where O_n^+ is a **compact quantum group**.

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Why "group-like"?

We have
$$A_o(n) \twoheadrightarrow C_n = C^*((\mathbb{Z}/2\mathbb{Z})^{*n}), u_{ij} \mapsto \delta_{ij}b_i$$
 and Δ induces

$$\Delta: C_n \to C_n \otimes C_n, \ g \mapsto g \otimes g \ \text{for} \ g \in C^*((\mathbb{Z}/2\mathbb{Z})^{*n}).$$

One can recover
$$(\mathbb{Z}/2\mathbb{Z})^{*n}$$
 as $\{u \in \mathscr{U}(C_n) \mid \Delta(u) = u \otimes u\}$.

We denote $A_o(n) = C^*(\mathbb{F}O_n)$, where $\mathbb{F}O_n$ is a **discrete quantum group**.

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Discrete/Compact quantum groups

A Woronowicz C^* -algebra is a unital C^* -algebra A with *-homomorphism $\Delta: A \to A \otimes A$ (coproduct) such that

- $\bullet \ (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$
- $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

Notation : $A = C^*(\Gamma) = C(\mathbb{G})$.

General theory: Haar state, Peter-Weyl, Tannaka-Krein...

Theorem (Woronowicz)

There exists a unique state $h: C^*(\Gamma) \to \mathbb{C}$ such that $(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = 1 \otimes h$.

- → regular representation $\lambda : C^*(\mathbb{F}) \to B(H)$,
- → reduced Woronowicz C^* -algebra $C^*_{red}(\Gamma) = \lambda(C^*(\Gamma))$,
- ightharpoonup von Neumann algebra $\mathscr{L}(\mathbb{F}) = C^*_{\mathrm{red}}(\mathbb{F})'' \subset B(H)$.



Known results about $\mathscr{L}(\mathbb{F}O_n)$

We restrict to the case $n \ge 3$.

Free probability:

- the elements $(\sqrt{n} u_{ij})_{i,j \leq s}$ are asymptotically free and semi-circular with respect to h as $n \to \infty$ [Banica-Collins 2007, Brannan 2014];
- free entropy dimension of the generators : $\delta_0(u) = 1$ [Brannan-Collins-V. 2012, 2016].

Von Neumann algebra:

- $\mathcal{L}(\mathbb{F}O_n)$ is not injective [Banica 1997]
- it is a full and solid II_1 factor [Vaes-V. 2007]
- it has the HAP and the CBAP [Brannan 2012, Freslon 2013]
- it is strongly solid [Isono 2015, Fima-V. 2015]

Question: is $\mathcal{L}(\mathbb{F}O_n)$ isomorphic to a free group factor?



Known results about $\mathscr{L}(\mathbb{F}O_n)$

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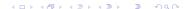
In $\mathcal{L}(F_n)$ with canonical generators a_i one has $\delta_0(a) = n$.

But it is not known whether δ_0 is an invariant of the von Neumann algebra.

However **strong** 1-boundedness is an invariant [Jung 2007]...

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Free entropy

 (M, τ) : finite von Neumann algebra with fixed trace τ . $H = L^2(M, \tau)$. Fix a tuple of self-adjoint elements $x = (x_1, \dots, x_m) \in M^m$. $\chi(x)$: microstates free entropy $/ \chi^*(x)$: non microstates free entropy.

Properties of χ and χ^* : [Voiculescu]

- $\chi(x) \in \mathbb{R} \cup \{-\infty\}$, $\chi(x_i) = \iint \ln |s t| dx_i(s) dx_i(t) + C$.
- $\chi(x) \leq \chi(x_1) + \cdots + \chi(x_m)$ with equality if x_1, \ldots, x_m are freely independant.
- only for χ : assuming $\chi(x_i) > -\infty$, equality in the previous point implies free independance.

Example of free independance : $x_1 \in L^{\infty}(\Omega_1, p_1), \ldots, x_m \in L^{\infty}(\Omega_m, p_m)$ are free in the reduced free product $M = L^{\infty}(\Omega_1, p_1) * \cdots * L^{\infty}(\Omega_m, p_m)$ with respect to the free product state τ .



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Free entropy dimension. [Voiculescu]

Assume M contains a free family $s = (s_1, \ldots, s_m)$ of (0, 1)-semicircular elements, also free from x. One defines:

$$\delta_0(x) = m - \liminf_{\delta \to 0} \chi(x + \delta s : s) / \ln \delta$$

$$\delta^*(x) = m - \liminf_{\delta \to 0} \chi^*(x + \delta s) / \ln \delta$$

 $\delta_0(x)$ only depends on the algebra generated by x. It is not known whether it only depends on the von Neumann algebra.

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We have the following deep result:

Theorem (Biane-Capitaine-Guionnet 2003)

We have $\chi(x) \leq \chi^*(x)$, hence $\delta_0(x) \leq \delta^*(x)$.



The case of $\mathbb{F}O_n$

Theorem (Jung 2006)

If $W^*(x)$ embeds in R^ω and contains a diffuse subalgebra, then $\delta_0(x) \geq 1$.

In $\mathscr{L}(\mathbb{F}O_n)$, the subalgebra $(\sum u_{ii})'' \simeq L^{\infty}([-2,2])$ is diffuse.

[Brannan-Collins-V. 2014]: $\mathcal{L}(\mathbb{F}O_n)$ embeds in R^{ω} .

Theorem (Connes-Shlyakhtenko 2005)

We have
$$\delta^*(x) \leq \beta_1^{(2)}(A, \tau) - \beta_0^{(2)}(A, \tau) + 1$$
 where $A = \mathbb{C}\langle x \rangle$.

By diffuseness, $\beta_0^{(2)}(A,h) = \beta_0^{(2)}(\mathbb{F}O_n) = 0$ [Kyed].

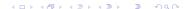
[V. 2012, Kyed-Raum-Vaes-Valvekens 2017]: $\beta_1^{(2)}(\mathbb{F}O_n)=0$.

Corollary

For n > 3 we have $\delta_0(u) = \delta^*(u) = 1$ in $\mathcal{L}(\mathbb{F}O_n)$.

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1-boundedness and von Neumann isomorphisms

Recall that $\delta^*(x) = m - \liminf_{\delta \to 0} \chi^*(x + \delta s) / \ln \delta$.

Hence $\delta^*(x) \le \alpha$ iff $\chi^*(x + \delta s) \le (m - \alpha) \ln \delta + o(\ln \delta)$ as $\delta \to 0$.

One says that x is α -bounded for δ^* if

$$\chi^*(x+\delta s) \le (m-\alpha) \ln \delta + K$$

for small δ and some constant K.

There is a similar notion of α -boundedness for δ_0 [Jung].

Theorem (Jung 2007)

If x is 1-bounded for δ_0 and $\chi(x_i) > -\infty$ for at least one i, then any tuple y of generators of $W^*(x)$ is 1-bounded for δ_0 (hence $\delta_0(y) \leq 1$).

In particular if M is generated by a 1-bounded tuple of generators, it is not isomorphic to any free group factor.



Proving 1-boundedness

Consider the algebra of polynomials in m non-commuting variables $\mathbb{C}\langle X \rangle = \mathbb{C}\langle X_1, \dots, X_m \rangle$. There are unique derivations

$$\delta_i: \mathbb{C}\langle X \rangle \to \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle$$

such that $\delta_i(X_j) = \delta_{ij}(1 \otimes 1)$, with the bimodule structure:

$$P \cdot (Q \otimes R) \cdot S = PQ \otimes RS.$$

For
$$P = (P_1, ..., P_I) \in \mathbb{C}\langle X \rangle^I$$
, put $\partial P = (\partial_j P_i) \in \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \otimes B(\mathbb{C}^m, \mathbb{C}^I)$.

Proving 1-boundedness

Consider the algebra of polynomials in m non-commuting variables $\mathbb{C}\langle X \rangle = \mathbb{C}\langle X_1, \ldots, X_m \rangle$. There are unique derivations δ_i such that $\delta_i(X_j) = \delta_{ij}(1 \otimes 1)$. For $P = (P_1, \ldots, P_l) \in \mathbb{C}\langle X \rangle^l$, put $\partial P = (\partial_i P_i) \in \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \otimes B(\mathbb{C}^m, \mathbb{C}^l)$.

Denote $H = L^2(M, \tau)$. Evaluating at X = x one obtains an operator $\partial P(x) \in B(H \otimes H \otimes \mathbb{C}^m, H \otimes H \otimes \mathbb{C}^l)$,

which commutes to the right action $(\zeta \otimes \xi) \cdot (x \otimes y) = \zeta x \otimes y \xi$ of $M \otimes M^{\circ}$ on $H \otimes H$. One considers the Murray-von Neumann dimension:

$$\operatorname{rank} \partial P(x) = \dim_{M \otimes M^{\circ}} \overline{\operatorname{Im}} \ \partial P(x).$$

Theorem (Jung 2016, Shlyakhtenko 2016)

Assume that x satisfies the identities P(x) = 0 and that $\partial P(x)$ is of determinant class. Then x is α -bounded for δ_0 and δ^* , with $\alpha = m - \operatorname{rank} \partial P(x)$.



Relations in $\mathbb{F}O_n$

We take $m=n^2$, $X=(X_{ij})_{ij}\in\mathbb{C}\langle X_{ij}\rangle\otimes M_n(\mathbb{C})$, $x=u=(u_{ij})_{ij}$.

We consider the $I = 2n^2$ canonical relations:

$$P = (P_1, P_2) = (X^tX - 1, XX^t - 1) \in \mathbb{C}\langle X \rangle \otimes M_n(\mathbb{C})^{\oplus 2}.$$

Following [Shlyakhtenko 2016] it is easy to prove that:

Proposition

We have
$$n^2 - \operatorname{rank} \partial P(u) = \beta_1^{(2)}(\mathbb{F}O_n) - \beta_0^{(2)}(\mathbb{F}O_n) + 1 = 1$$
.

Hence if $\partial P(u)$ is of determinant class, Jung–Shlyakhtenko's result allows to conclude that u is 1-bounded.

Determinant class:
$$(h \otimes h \otimes \operatorname{Tr})(\ln_+(\partial P(u)^*\partial P(u))) > -\infty$$
. $\ln_+(t) = \ln(t)$ if $t > 0$, $\ln_+(0) = 0$.



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In the case of a discrete group Γ , this would follow from Lück's determinant conjecture, which holds e.g. if Γ is sofic. In the quantum case, there is no such tool (yet?) to prove the determinant conjecture...



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Computation of $\partial P(u)$

Identify $M_n(\mathbb{C}) \simeq p_1 H = \operatorname{Span}\{u_{ij}\xi_0\} \subset H$. Then $u \in C^*_{\operatorname{red}}(\mathbb{F}O_n) \otimes M_n(\mathbb{C})$ acts by left mult. on $H \otimes p_1 H$. If $S: H \to H$ is the antipode, we have in $B(H \otimes H \otimes p_1 H)$:

$$\partial P_1(u) = (1 \otimes S \otimes S)u_{23}(1 \otimes S \otimes 1) + u_{13}^* \partial P_2(u) = (1 \otimes S \otimes S)u_{23}^*(1 \otimes S \otimes S) + u_{13}(1 \otimes 1 \otimes S)$$

Proposition

We have $\partial P_1(u)^*\partial P_1(u) = \partial P_2(u)^*\partial P_2(u)$ and it is unitarily conjugated to $(2+2\operatorname{Re}\Theta)\otimes 1\in \mathcal{B}(H\otimes p_1H\otimes H)$, where $\Theta=(S\otimes 1)u(S\otimes S)\in \mathcal{B}(H\otimes p_1H)$.

Fact: Θ is the reversing operator of the quantum Cayley graph of $\mathbb{F}O_n!$

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Decomposing the quantum Cayley graph

Classical case

One has $H = \ell^2(\Gamma) = \overline{\operatorname{span}}\{g, g \in \Gamma\}$, $p_1 H = \ell^2(\Lambda)$ with $\Lambda = \Lambda^{-1} \subset \Gamma$. The "fundamental matrix" is $u = \operatorname{diag}(\lambda(g))_{g \in \Lambda}$ and $S(g) = g^{-1}$. Then:

$$\Theta(g\otimes h)=(S\otimes 1)u(S\otimes S)(g\otimes h)=gh\otimes h^{-1}.$$

Interpretation: $(g, h) \rightarrow$ edge connecting g to gh in the Cayley graph $(gh, h^{-1}) \rightarrow$ reversed edge

In particular: $\Theta^2 = 1$ hence $H \otimes p_1 H = \operatorname{Ker}(\Theta - 1) \oplus \operatorname{Ker}(\Theta + 1)$.

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Quantum case

Quantum Cayley graph given by $\ell^2(\mathbb{X}^{(0)}) = H$, $\ell^2(\mathbb{X}^{(1)}) = H \otimes p_1H$, the boundary operator $E = V : \ell^2(\mathbb{X}^{(1)}) \to \ell^2(\mathbb{X}^{(0)}) \otimes \ell^2(\mathbb{X}^{(0)})$, the reversing operator Θ .

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In particular: $\Theta^2 = 1$ hence $H \otimes p_1 H = \operatorname{Ker}(\Theta - 1) \oplus \operatorname{Ker}(\Theta + 1)$.

Quantum case

We have $\Theta^2 \neq 1$, $\operatorname{Ker}(\Theta - 1) \oplus \operatorname{Ker}(\Theta + 1) \subsetneq H \otimes p_1 H$. The description of $\operatorname{Ker}(\Theta \pm 1)$ was an essential tool in the proof of $\beta_1^{(2)}(\mathbb{F}O_n) = 0$.

Theorem

On $\operatorname{Ker}(\Theta-1)^{\perp} \cap \operatorname{Ker}(\Theta+1)^{\perp}$, $\operatorname{Re}(\Theta) \simeq \bigoplus \operatorname{Re}(r_{\alpha})$ is an infinite direct sum of real parts of weighted right shifts r_{α} .

Conclusion

Lemma

For any right shift r with weights in]0,1], $2 + 2 \operatorname{Re} r$ is of determinant class with respect to the specific state coming from $h \otimes \operatorname{Tr}$.

Finally one can apply Jung-Shlyakhtenko's result:

Corollary

The generating matrix u is 1-bounded in $\mathcal{L}(\mathbb{F}O_n)$. $\mathcal{L}(\mathbb{F}O_n)$ is not isomorphic to a free group factor.

Next questions...

- Is there a group Γ such that $\mathscr{L}(\mathbb{F}O_n) \simeq \mathscr{L}(\Gamma)$?
- What about $\mathcal{L}(\mathbb{F}U_n)$? Recall that $\mathcal{L}(\mathbb{F}U_2) \simeq \mathcal{L}(F_2)$.

