Quantum Boundary Actions and C^*-Simplicity

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Free quantum groups

- [Wang '95 ; W., Van Daele '96]
  Free quantum groups $\mathbb{F}O(Q)$, $\mathbb{F}U(Q)$ for $Q \in GL_N(\mathbb{C})$ given by
  $C^*(\mathbb{F}U(Q)) = C(U^+(Q)) = A_u(Q) = \langle u_{ij} \mid u \text{ and } Q\bar{u}Q^{-1} \text{ unitary}\rangle.$
- [Banica '97] Computation of $\mathrm{Corep}(\mathbb{F}O(Q))$, $\mathrm{Corep}(\mathbb{F}U(Q))$
  $\Rightarrow \mathbb{F}U(Q)$ non amenable, $C^*_{\mathrm{red}}(\mathbb{F}U(Q))$ simple ($N \geq 2$)

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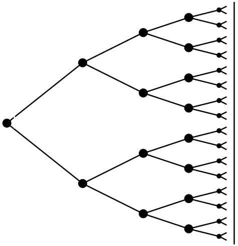
- [V., Vaes '07 ; Vaes, V-Vennet '10] [Biane, Izumi, Neshveyev, Tuset]
  Quantum Gromov Boundaries $\partial_G \mathbb{F}O(Q)$, $\partial_G \mathbb{F}U(Q)$
  $\partial_G \mathbb{F}O(Q)$ realizes the Quantum Poisson and Martin Boundaries
  Amenability of $\mathbb{F}O(Q) \curvearrowright \partial_G \mathbb{F}O(Q) \Rightarrow$ exactness of $\mathbb{F}O(Q)$
- [Kalantar, Kasprzak, Skalski, V. '22] [A.-S., Khosravi '24]
  $\partial \mathbb{F}O(Q)$ is a Quantum Furstenberg Boundary
  $\mathbb{F}O(Q) \curvearrowright \partial_G \mathbb{F}O(Q)$ faithful $\Rightarrow \tau$-KMS states $\subset \{h\}$ on $C^*_{\mathrm{red}}(\mathbb{F}O(Q))$
- [Anderson-Sackaney, V.]
  $\partial \mathbb{F}U(Q)$ is a Quantum Furstenberg Boundary
  $\mathbb{F}U(I_N) \curvearrowright \partial_G \mathbb{F}U(I_N)$ strongly C^*-faithful $\Rightarrow C^*_{\mathrm{red}}(\mathbb{F}U(I_N))$ simple

Classical Case

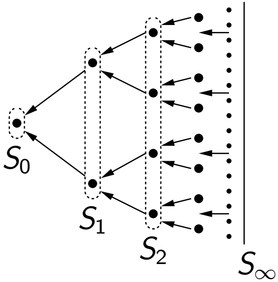


(rooted) tree T → boundary ∂T

Picture: $\partial T = \{0, 1\}^{\mathbb{N}}$
(Cantor "line" on the right)

If $\Gamma \curvearrowright T$ then $\Gamma \curvearrowright \partial T$.

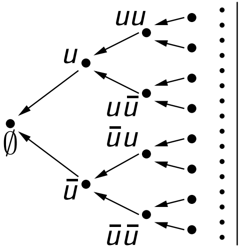
Classical Case



Possible defn: $\partial T = \varprojlim_n (S_n, \psi_n)$
S_n: spheres ; $\psi_n : S_{n+1} \to S_n$
given by tree structure

$\Gamma \curvearrowright \partial T$ maybe not so evident...

Classical Case



This is in fact the *classical* Cayley tree of $\mathbb{F}U(Q)$ with generating set $\{u, \bar{u}\}$.

But there is no action $\mathbb{F}U(Q) \curvearrowright T$.

[Banica '97] $\mathrm{Irr}(\mathbb{F}U(Q))$ is indexed by words in u, $\bar{u}$, so that :

- u is the generating matrix, $\bar{u}$ its conjugate,
- $vu \otimes u \simeq vuu$, $v\bar{u} \otimes u \simeq v \oplus v\bar{u}u$, ...

The inclusions $w \subset v \otimes u$, $w \subset v \otimes \bar{u}$ correspond to edges in the tree.

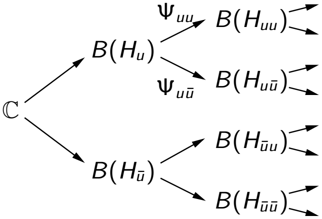
Quantum Case

For each $v \in I := \mathrm{Irr}(\mathbb{F}U(Q))$ we have a f.d. Hilbert space H_v. We get:

$$\ell^\infty(\mathbb{F}U(Q)) \simeq \bigoplus_{v\in I} B(H_v).$$

From $\psi_{vu} : H_{vu} \hookrightarrow H_v \otimes H_u$ we get:

$$\Psi_{vu} : B(H_v) \to B(H_{vu}), a \mapsto \psi^*_{vu}(a \otimes \mathrm{id})\psi_{vu}.$$



We define

$$C(S_n) := \bigoplus_{|v|=n} B(H_v),$$
$$\Psi_n = \bigoplus_{|w|=n+1} \Psi_w,$$
$$C(\partial_G(\mathbb{F}U(Q)) = \varinjlim_n (C(S_n), \Psi_n).$$

A priori it is only an operator space.

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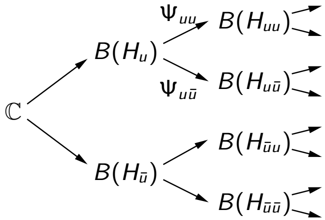

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Theorem (Vaes, Vander Vennet '10)

$C(\partial_G(\mathbb{F}U(Q))$ is a unital sub-C^*-algebra of $\ell^\infty(\mathbb{F}U(Q))/c_0(\mathbb{F}U(Q))$.
The coproduct yields $\alpha : C(\partial_G(\mathbb{F}U(Q)) \to M(c_0(\mathbb{F}U(Q)) \otimes C(\partial_G(\mathbb{F}U(Q)))$.

Moreover, putting $\mathrm{qTr}_n = \sum_{|v|=n} \mathrm{qTr}_v$ and $\mathrm{qtr}_n = \mathrm{qTr}_n / \mathrm{qTr}_n(1)$,
$\omega = \varinjlim_n \mathrm{qtr}_n$ is a well-defined state s.t. $(\mathrm{qtr}_1 \otimes \omega)\alpha = \omega$ (stationary).

Quantum Furstenberg Boundaries

Γ-boundary: $\Gamma \curvearrowright X$ compact which is minimal and strongly proximal.
Example: $F_N \curvearrowright \partial F_N$.

Definition [KKSV22]. A quantum action $\mathbb{\Gamma} \curvearrowright A$ unital C^*-algebra is a $\mathbb{\Gamma}$-boundary if any $\mathbb{\Gamma}$-equivariant UCP map $F : A \to B$ to any B is automatically completely isometric (CI).
Theorem [KKSV22]. There is a universal $\mathbb{\Gamma}$-boundary, denoted $\partial_F \mathbb{\Gamma}$.

Theorem (Anderson-Sackaney, V.)

If $N \geq 3$, $\partial_G \mathbb{F}U(Q)$ is an $\mathbb{F}U(Q)$-boundary.

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Proposition (KKSV22)

Take $\mu \in \ell^\infty(\mathbb{\Gamma})^+_*$. If there exists a unique μ-stationary state $\nu \in A^*$, and if $P_\nu := (\mathrm{id} \otimes \nu)\alpha : A \to \ell^\infty(\mathbb{\Gamma})$ is CI, then A is a $\mathbb{\Gamma}$-boundary.

P_ω completely isometric: known since [VVV10].

Theorem (Habbestad, Hataishi, Neshveyev '22)

The state ω is the unique $D(\mathbb{F}U(Q))$-stationary state (wrt $\mathrm{qtr}_1 \otimes h$) on $A = C(\partial_G \mathbb{F}U(Q))$. Hence $\partial_G \mathbb{F}U(Q)$ is a $D(\mathbb{F}U(Q))$-boundary.

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The state ω is the unique $\mathbb{F}U(Q)$-stationary state (wrt qtr_1) on $A = C(\partial_G \mathbb{F}U(Q))$. Hence $\partial_G \mathbb{F}U(Q)$ is an $\mathbb{F}U(Q)$-boundary.

Topological Freeness

Theorem [Kalantar, Kennedy '17]. $C^*_{\mathrm{red}}(\Gamma)$ is simple iff Γ admits a topologically free Γ-boundary iff $\partial_F\Gamma$ is free.

Topological freeness: $\forall g \neq e$, X^g has empty interior.

Definition (ASV)

$\mathbb{\Gamma} \curvearrowright A$ is topologically free if any $F \in M(c_0(\mathbb{\Gamma}) \otimes A)$ such that $\forall a \in A$ $F(1 \otimes a) = \alpha(a)F$ must lie in $p_0 \otimes A$ (p_0 support of the co-unit).

Theorem (ASV)

- $\mathbb{\Gamma} \curvearrowright A$ *top. free* $\Rightarrow$ *faithful.*
- $\mathbb{\Gamma} \curvearrowright \partial_F\mathbb{\Gamma}$ *top. free* $\Rightarrow$ $C^*_{\mathrm{red}}\mathbb{\Gamma}$ *simple.*

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Problems:

- A $\mathbb{\Gamma}$-boundary with $\mathbb{\Gamma} \curvearrowright A$ top. free $\Rightarrow$ $\mathbb{\Gamma} \curvearrowright \partial_F \mathbb{\Gamma}$ top. free ?
- Quantum examples of topologically free $\mathbb{\Gamma}$-boundaries ?
- $\mathbb{\Gamma} \curvearrowright \partial_F \mathbb{\Gamma}$ topologically free $\Rightarrow$ $\mathbb{\Gamma}$ unimodular.

Strong C^*-faithfulness

Remark. For $\Gamma \curvearrowright X$ compact minimal, the action is topologically free iff it is strongly faithful: $\forall F \subset \Gamma \setminus \{e\}$ finite $\exists x \in X \; \forall g \in F \; gx \neq x$.

Denote $PZ^\circ(\mathbb{\Gamma})$ the set of central projections $p \in c_0(\mathbb{\Gamma}) \cap \mathrm{Ker}(\epsilon)$.

Definition (ASV)

$\mathbb{\Gamma} \curvearrowright A$ is strongly C^*-faithful if $\forall p \in PZ^\circ(\mathbb{\Gamma}) \; \forall \eta > 0 \; \exists k \in \mathbb{N}$ $\exists a \in (A \otimes M_k(\mathbb{C}))_+$ s.t. $\|(p \otimes a)(\alpha \otimes \mathrm{id})(a)\| < \eta \|a\|$.

$\mathbb{\Gamma} = \Gamma$ and $A = C(X)$: can take $k = 1$, $\eta = 0$ and then, for $p = \mathbb{1}_F$, $(p \otimes a)(\alpha \otimes \mathrm{id})(a) = 0$ means $a \cdot \alpha_g(a) = 0$ for all $g \in F$.

$\mathbb{\Gamma} = \Gamma$, A general: strictly stronger that strong faithfulness.

Strong C^*-faithfulness

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Theorem (ASV)

- *A strongly C^*-faithful and $A \hookrightarrow B$ $\Rightarrow$ B strongly C^*-faithful.*
- *A strongly C^*-faithful $\mathbb{\Gamma}$-boundary $\Rightarrow$ $C^*_{\mathrm{red}}\mathbb{\Gamma}$ simple.*

Proposition (ASV)

For $Q = I_N$, $\partial_G \mathbb{F}U(Q)$ is strongly C^*-faithful.

Remark. Simplicity of $C^*_{\mathrm{red}}(\mathbb{F}U(Q))$ already known [Banica].

Open Questions

- Is $\partial_G \mathbb{F}O(Q)$ strongly C^*-faithful?
  Simplicity of $C^*_{\mathrm{red}}(\mathbb{F}O(Q))$ is known only with restrictions on Q.
- C^*-simplicity $\Rightarrow$ topological freeness? Strong C^*-faithfulness?

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Thank You!