A COCYCLE IN THE ADJOINT REPRESENTATION OF THE ORTHOGONAL FREE QUANTUM GROUPS

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Abstract. We show that the orthogonal free quantum groups are not inner amenable and we construct an explicit proper cocycle weakly contained in the regular representation. This strengthens the result of Vaes and the second author, showing that the associated von Neumann algebras are full II_1-factors and Brannan's result showing that the orthogonal free quantum groups have Haagerup's approximation property. We also deduce Ozawa-Popa's property strong (HH) and give a new proof of Isono's result about strong solidity.

1. Introduction

The orthogonal free quantum groups $\mathbb{F}O_n$ are given by universal unital C^*-algebras $A_o(n) = C^*(\mathbb{F}O_n)$ which were introduced by Wang [Wan95] as follows:

$$A_o(n) = C^*\langle v_{ij}, 1 \leq i,j \leq n \mid v_{ij} = v_{ij}^* \text{ and } (v_{ij}) \text{ unitary}\rangle.$$

Equipped with the coproduct defined by the formula $\Delta(v_{ij}) = \sum v_{ik} \otimes v_{kj}$, this algebra becomes a Woronowicz C^*-algebra [Wor98] and hence corresponds to a compact and a discrete quantum group in duality [Wor98, PW90], denoted respectively O_n^+ and $\mathbb{F}O_n$.

These C^*-algebras have been extensively studied since their introduction, and it has been noticed that the discrete quantum groups $\mathbb{F}O_n$ share many analytical features with the usual free groups F_n from the operator algebraic point of view. Let us just quote two such results that will be of interest for this article:

(1) the von Neumann algebras $\mathcal{L}(\mathbb{F}O_n)$ associated to $\mathbb{F}O_n$, $n \geq 3$, are full II_1 factors [VV07] ;
(2) the von Neumann algebras $\mathcal{L}(\mathbb{F}O_n)$ have Haagerup's approximation property [Bra12].

On the other hand there is one result that yields an operator algebraic distinction between $\mathbb{F}O_n$ and F_n: it was shown in [Ver12] that the first L^2-cohomology group of $\mathbb{F}O_n$ vanishes.

The purpose of this article is to present slight reinforcements and alternate proofs of the results (1), (2) above. Namely we will:

(1) show that the discrete quantum groups $\mathbb{F}O_n$ are not inner amenable (see [MvN43, Eff75] for non abelian free groups);
(2) construct an explicit proper cocycle witnessing Haagerup's property [DFSW13].

Moreover, putting these constructions together we will obtain a new result, namely Property strong (HH) from [OP10b], which is a strengthening of Haagerup's property and corresponds to the existence of a proper cocycle in a representation which is weakly contained in the regular representation. This sheds interesting light on the result from [Ver12] mentioned above, according to which such a proper cocycle cannot live in a representation strongly contained in the regular representation (or multiples of it). Note that the cocycle corresponding to the original proof [Haa79] of Haagerup's property for F_n does live in (multiples of) the regular representation.

Finally we will mention applications to solidity: indeed the constructions of the article and Property strong (HH) allow for a new proof of the strong solidity of $\mathcal{L}(\mathbb{F}O_n)$ using the tools of [OP10a, OP10b]. Notice that strong solidity has already been obtained in [Iso15] using Property AO^+ from [Ver05, VV07] and the tools of [PV14].

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2. Notation

If ζ, ξ are vectors in a Hilbert space H, we denote $\omega_{\zeta,\xi} = (\zeta | \cdot\ \xi)$ the associated linear form on $B(H)$, and we put $\omega_\zeta = \omega_{\zeta,\zeta} \in B(H)_*^+$.

We will use at some places the q-numbers $[k]_q = U_{k-1}(q+q^{-1}) = (q^k - q^{-k})/(q-q^{-1})$, where $(U_k)_{k\in\mathbb{N}}$ are the dilated Chebyshev polynomials of the second kind, defined by $U_0 = 1$, $U_1 = X$ and $XU_k = U_{k-1} + U_{k+1}$.

Following [KV00], a locally compact quantum group $\mathbb{G}$ is given by a Hopf-C^*-algebra $C_0(\mathbb{G})$, with coproduct denoted Δ, satisfying certain axioms including the existence of left and right invariant weights φ, $\varphi' : C_0(\mathbb{G})_+ \to [0, +\infty]$:

$$\varphi((\omega \otimes \mathrm{id})\Delta(f)) = \omega(1)\varphi(f) \quad \text{and} \quad \varphi'((\mathrm{id} \otimes \omega)\Delta(f)) = \omega(1)\varphi'(f),$$

for all $\omega \in C_0(\mathbb{G})_+^*$ and $f \in C_0(\mathbb{G})_+$ such that $\varphi(f) < \infty$. When there is no risk of confusion, we will denote by $\|f\|_2^2 = \varphi(f^*f) \in [0, +\infty]$ the hilbertian norm associated with φ. Using a GNS construction (H, Λ_φ) for φ, one defines the fundamental multiplicative unitary $V \in B(H \otimes H)$, denoted W in [KV00], by putting $V^*(\Lambda_\varphi \otimes \Lambda_\varphi)(f \otimes g) = (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(g)(f \otimes 1))$. Recall that $\mathbb{G}$ is called unimodular, or of Kac type, if $\varphi = \varphi'$.

We will assume for the rest of the article that $\mathbb{G}$ is discrete, meaning for instance that there exists a non-zero vector $\xi_0 \in H$ such that $V(\xi_0 \otimes \zeta) = \xi_0 \otimes \zeta$ for all $\zeta \in H$. In that case one can always identify $C_0(\mathbb{G})$ with its image by the GNS representation on H, moreover it can be reconstructed from V as the norm closure of $(\mathrm{id} \otimes B(H)_*)(V)$, and the coproduct is given by the formula $\Delta(f) = V^*(1 \otimes f)V$.

From V one can also construct the dual algebra $C^*_{\mathrm{red}}(\mathbb{G})$ as the norm closure of $(B(H)_* \otimes \mathrm{id})(V)$, equipped with the coproduct $\Delta(x) = V(x \otimes 1)V^*$ (opposite to the dual coproduct of [KV00]). Since $(\omega_{\xi_0} \otimes \mathrm{id})(V) = 1$, this C^*-algebra is unital, and it is in fact a Woronowicz C^*-algebra [Wor98]. Moreover the restriction of ω_{ξ_0} is the Haar state h of $C^*_{\mathrm{red}}(\mathbb{G})$. When there is no risk of confusion, we will denote $\|x\|_2^2 = h(x^*x)$ the hilbertian norm associated with h. Using $\Lambda_h : x \mapsto x\xi_0$ as a GNS construction for h we have $(\Lambda_h \otimes \Lambda_h)(\Delta(x)(1 \otimes y)) = \Delta(x)V(\xi_0 \otimes y\xi_0) = V(x\xi_0 \otimes y\xi_0)$, so that V coincides with the multiplicative unitary associated to $(C^*_{\mathrm{red}}(\mathbb{G}), \Delta)$ as constructed in [BS93].

The von Neumann algebras associated to $\mathbb{G}$ are denoted $L^\infty(\mathbb{G}) = C_0(\mathbb{G})''$ and $M = \mathcal{L}(\mathbb{G}) = C^*_{\mathrm{red}}(\mathbb{G})''$, they carry natural extensions of the respective coproducts and Haar weights.

Recall that a corepresentation X of V is a unitary $X \in B(H \otimes H_X)$ such that $V_{12}X_{13}X_{23} = X_{23}V_{12}$. There exists a so-called maximal C^*-algebra $C^*(\mathbb{G})$ together with a corepresentation $\mathbb{V} \in M(C_0(\mathbb{G}) \otimes C^*(\mathbb{G}))$ such that any corepresentation X corresponds to a unique $*$-representation $\pi : C^*(\mathbb{G}) \to B(H_X)$ via the formula $X = (\mathrm{id} \otimes \pi)(\mathbb{V})$ [BS93, Corollaire 1.6]. We say that X, π are unitary representations of $\mathbb{G}$. When $X = V$, we obtain the regular representation $\pi = \lambda : C^*(\mathbb{G}) \to C^*_{\mathrm{red}}(\mathbb{G})$. Moreover $C^*(\mathbb{G})$ admits a unique Woronowicz C^*-algebra structure given by $(\mathrm{id} \otimes \Delta)(\mathbb{V}) = \mathbb{V}_{12}\mathbb{V}_{13}$, and λ is then the GNS representation associated with the Haar state h of $C^*(\mathbb{G})$.

On the dual side, a unitary corepresentation of $\mathbb{G}$, or representation of V, is a unitary $Y \in M(K(H_Y) \otimes C^*(\mathbb{G}))$ such that $(\mathrm{id} \otimes \Delta)(Y) = Y_{12}Y_{13}$, corresponding to a $*$-representation of $C_0(\mathbb{G})$ as above. We denote $\mathrm{Corep}(\mathbb{G})$ the category of finite dimensional (f.-d.) unitary corepresentations of $\mathbb{G}$, and we choose a complete set $\mathrm{Irr}\,\mathbb{G}$ of representatives of irreducible ones. In the discrete case, it is known that any corepresentation decomposes in $\mathrm{Irr}\,\mathbb{G}$. We have in particular a $*$-isomorphism $C_0(\mathbb{G}) \simeq c_0\text{-}\bigoplus_{r\in\mathrm{Irr}\,\mathbb{G}} B(H_r)$ such that V corresponds to $\bigoplus_{r\in\mathrm{Irr}\,\mathbb{G}} r$. We denote $p_r \in C_0(\mathbb{G})$ the minimal central projection corresponding to $B(H_r)$, and $C_c(\mathbb{G}) \subset C_0(\mathbb{G})$ the algebraic direct sum of the blocks $B(H_r)$. The $*$-algebra $C_c(\mathbb{G})$, together with the restriction of Δ, is a multiplier Hopf algebra in the sense of [VD94], and we denote its antipode by $\hat{S}$.

On the other hand we denote $\mathbb{C}[\mathbb{G}]_r \subset C^*(\mathbb{G})$ the subspace of coefficients $(\omega \otimes \mathrm{id})(r)$ of an irreducible corepresentation r, and $\mathbb{C}[\mathbb{G}]$ the algebraic direct sum of these subspaces. The coproduct on $C^*(\mathbb{G})$ restricts to an algebraic coproduct on $\mathbb{C}[\mathbb{G}]$, which becomes then a plain Hopf $*$-algebra. Recall also that λ restricts to an injective map on $\mathbb{C}[\mathbb{G}]$, hence we shall sometimes consider $\mathbb{C}[\mathbb{G}]$ as a subspace of $C^*_{\mathrm{red}}(\mathbb{G})$. We will denote by ϵ and S the co-unit and the antipode of $\mathbb{C}[\mathbb{G}]$, and we note that ϵ extends to $C^*(\mathbb{G})$: it is indeed the $*$-homomorphism corresponding to $X = \mathrm{id}_H \otimes \mathrm{id}_{\mathbb{C}}$.

We denote $(f_z)_z$ the Woronowicz characters of $C^*(\mathbb{G})$, satisfying in particular $h(xy) = h(y(f_1 * x * f_1))$, where $\phi * x = (\mathrm{id}\otimes\phi)\Delta(x)$ and $x*\phi = (\phi\otimes\mathrm{id})\Delta(x)$. We put as well $F_v = (\mathrm{id}\otimes f_1)(v) \in B(H_v)$ for any f.-d. unitary corepresentation $v \in B(H_v) \otimes C^*(\mathbb{G})$ and $\mathrm{qdim}\, v = \mathrm{Tr}\, F_v$. Denoting $v_{\zeta,\xi} = (\omega_{\zeta,\xi} \otimes \mathrm{id})(v)$, the Schur orthogonality relations [Wor87, (5.15)] read, for v, w irreducible:

$$h(v^*_{\zeta,\xi} w_{\zeta',\xi'}) = \frac{\delta_{v,w}}{\mathrm{qdim}\, v}(\zeta'|F_v^{-1}\zeta)(\xi|\xi'). \tag{1}$$

Note that in Woronowicz' notation we have $v_{kl} = v_{e_k,e_l}$ if $(e_i)_i$ is a fixed orthonormal basis (ONB) of H_v. Similarly, we have the formula $\varphi(a) = \mathrm{qdim}(v)\,\mathrm{Tr}(F_v a)$ for the left Haar weight φ and $a \in B(H_r) \subset C_0(\mathbb{G})$ [PW90, (2.13)] — note that the coproduct on $C_0(\mathbb{G})$ constructed from V in [PW90] is opposite to ours, so that $\varphi = h_{dR}$ with their notation.

Finally we will use the following analogue of the Fourier transform:

$$\mathcal{F} : C_c(\mathbb{G}) \to C^*_{\mathrm{red}}(\mathbb{G}),\ a \mapsto (\varphi \otimes \mathrm{id})(V(a \otimes 1)).$$

One can show that it is isometric with respect to the scalar products $(a|b) = \varphi(a^*b)$, $(x|y) = h(x^*y)$ for a, $b \in C_c(\mathbb{G})$, x, $y \in C^*_{\mathrm{red}}(\mathbb{G})$ — see also Section 3.1.

In this article we will be mainly concerned with orthogonal free quantum groups. For $Q \in GL_n(\mathbb{C})$ such that $Q\bar{Q} \in \mathbb{C}I_n$, we denote $A_o(Q)$ the universal unital C^*-algebra generated by n^2 elements v_{ij} subject to the relations $Q\bar{v}Q^{-1} = v$ and $v^*v = vv^* = I_n \otimes 1$, where $v = (v_{ij})_{ij} \in M_n(A_o(Q))$ and $\bar{v} = (v^*_{ij})_{ij}$. Equipped with the coproduct Δ given by $\Delta(v_{ij}) = \sum v_{ik} \otimes v_{kj}$, it is a full Woronowicz C^*-algebra, and we denote $\mathbb{G} = \mathbb{F}O(Q)$ the associated discrete quantum group, such that $C^*(\mathbb{F}O(Q)) = A_o(Q)$. In the particular case $Q = I_n$ we denote $A_o(n) = A_o(I_n)$ and $\mathbb{F}O_n = \mathbb{F}O(I_n)$.

It is known from [Ban96] that the irreducible corepresentations of $\mathbb{F}O(Q)$ can be indexed v^k, $k \in \mathbb{N}$ up to equivalence, in such a way that $v^0 = \mathrm{id}_{\mathbb{C}} \otimes 1$, $v^1 = v$, and the following fusion rules hold

$$v^k \otimes v^l \simeq v^{|k-l|} \oplus v^{|k-l|+2} \oplus \cdots \oplus v^{k+l}.$$

Moreover the contragredient of v^k is equivalent to v^k for all k. The classical dimensions $\dim v^k$ satisfy $\dim v^0 = 1$, $\dim v^1 = n$ and $n \dim v^k = \dim v^{k-1} + \dim v^{k+1}$. We denote $\rho \geq 1$ the greatest root of $X^2 - nX + 1$, so that $\dim v^k = [k+1]_\rho$. Similarly we have $\mathrm{qdim}\, v^0 = 1$ and $\mathrm{qdim}\, v^1\, \mathrm{qdim}\, v^k = \mathrm{qdim}\, v^{k-1} + \mathrm{qdim}\, v^{k+1}$. Moreover, if Q is normalized in such a way that $Q\bar{Q} = \pm I_n$, we have $\mathrm{qdim}\, v = \mathrm{Tr}(Q^*Q)$. We denote $q \in]0,1]$ the smallest root of $X^2 - (\mathrm{qdim}\, v)X + 1$, so that $\mathrm{qdim}\, v^k = [k+1]_q$.

3. The adjoint representation of $\mathbb{F}O_n$

The aim of this section is to prove that the non-trivial part ad° of the adjoint representation of $\mathbb{F}O_n$ factors through the regular representation. Equivalently, we will prove that all states of the form $\omega_\xi \circ \mathrm{ad}^\circ$, with $\xi \in H$, factor through $C^*_{\mathrm{red}}(\mathbb{F}O_n)$. For this we will use the criterion given by Lemma 3.5 below, which is analogous to Theorem 3.1 of [Haa79].

3.1. **Weak containment in the regular representation.** In this section we gather some useful results which are well-known in the classical case. We say that an element $f \in L^\infty(\mathbb{G})$ is a (normalized) positive type function if there exists a state ϕ on $C^*(\mathbb{G})$ such that $f = (\mathrm{id} \otimes \phi)(\mathbb{V})$. The associated multiplier is $M = (\mathrm{id}\otimes\phi)\Delta : \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}]$. The next lemma shows that it extends to a completely positive (CP) map on $C^*_{\mathrm{red}}(\mathbb{G})$ characterized by the identity $(\mathrm{id}\otimes M)(V) = V(f\otimes 1)$. In the locally compact case, the lemma is covered by [Daw12, Theorem 5.2]. For the convenience of the reader we include a short and self-contained proof for the discrete case.

Lemma 3.1. *Let ϕ be a unital linear form on $\mathbb{C}[\mathbb{G}]$ and consider $M = (\mathrm{id} \otimes \phi)\Delta : \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}]$. Then M extends to a CP map on $C^*_{\mathrm{red}}(\mathbb{G})$ if and only if ϕ extends to a state of $C^*(\mathbb{G})$.*

Proof. By Fell's absorption principle $V_{12}\mathbb{V}_{13} = \mathbb{V}_{23}V_{12}\mathbb{V}^*_{23}$, Δ extends to a $*$-homomorphism $\Delta' : C^*_{\mathrm{red}}(\mathbb{G}) \to C^*_{\mathrm{red}}(\mathbb{G}) \otimes C^*(\mathbb{G})$. Hence if ϕ extends to a state of $C^*(\mathbb{G})$, M extends to a CP map $M = (\mathrm{id}\otimes\phi)\Delta'$ on $C^*_{\mathrm{red}}(\mathbb{G})$. For the reverse implication, since $C^*(\mathbb{G})$ is the enveloping C^*-algebra of $\mathbb{C}[\mathbb{G}]$, it suffices to prove that $\phi(xx^*) \geq 0$ for all $x \in \mathbb{C}[\mathbb{G}]$.

We choose a corepresentation $v \in B(H_v) \otimes C^*_{\mathrm{red}}(\mathbb{G})$. Let L be a GNS space for $B(H_v)$, with respect to an arbitrary given state. For $a \in B(H_v)$, resp. $\omega \in B(H_v)^*$, we denote $\hat{a}$, $\hat{\omega}$ the corresponding elements of $L = B(\mathbb{C}, L)$ resp. $L^* = B(L, \mathbb{C})$, and we identify $B(H_v)$ with a subspace of $B(L)$ via left multiplication. Similarly we denote $\hat{v}$ the element of $B(\mathbb{C}, L) \otimes C^*_{\mathrm{red}}(\mathbb{G})$ corresponding to v. We have then $\hat{v}\hat{v}^* \in B(L) \otimes C^*_{\mathrm{red}}(\mathbb{G})$ and $(\hat{\omega} \otimes 1)\hat{v}\hat{v}^*(\hat{\omega}^* \otimes 1) = xx^*$ if $x = (\omega \otimes \mathrm{id})(v)$.

If M is CP, the following element of $B(L) \otimes C^*_{\mathrm{red}}(\mathbb{G})$ is positive:

$$\begin{aligned}(\mathrm{id} \otimes M)(\hat{v}\hat{v}^*) &= (\mathrm{id} \otimes \mathrm{id} \otimes \phi)(\mathrm{id} \otimes \Delta)(\hat{v}\hat{v}^*) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \phi)(v_{12}\hat{v}_{13}\hat{v}_{13}^* v_{12}^*) = v(X \otimes 1)v^*,\end{aligned}$$

where $X = (\mathrm{id} \otimes \phi)(\hat{v}\hat{v}^*) \in B(L)$. We conclude that X is positive, hence $\phi(xx^*) = \hat{\omega}X\hat{\omega}^* \geq 0$ for any $x \in \mathbb{C}[\mathbb{G}]$. □

We will be particularly interested in the case when ϕ factors through $C^*_{\mathrm{red}}(\mathbb{G})$, or equivalently, when ϕ is a weak limit of states of the form $\sum_{i=1}^n \omega_{\xi_i} \circ \lambda$ with $\xi_i \in H$. In that case we say that f, ϕ are weakly associated to λ, or weakly ℓ^2. We have the following classical lemma [Fel63, Lemma 1]:

Lemma 3.2. *Let $\pi : C^*(\mathbb{G}) \to B(K)$ be a $*$-representation. Assume that there exists a subset $X \subset K$ such that $\overline{\mathrm{Span}}\ \pi(\mathbb{C}[\mathbb{G}])X = K$ and $\omega_\xi \circ \pi$ is weakly ℓ^2 for all $\xi \in X$. Then π factors through $C^*_{\mathrm{red}}(\mathbb{G})$.*

Proof. As noted in the original paper of Fell, the proof for groups applies in fact to general C^*-algebras, and in particular to discrete quantum groups. More precisely, take $A = C^*(\mathbb{G})$, $T = \pi$ and $\mathcal{S} = \{\lambda\}$ in [Fel63, Rk 1]. □

On the other hand we say that $\phi \in \mathbb{C}[\mathbb{G}]^*$ is an ℓ^2-form if it is continuous with respect to the ℓ^2-norm on $\mathbb{C}[\mathbb{G}]$, i.e. there exists $C \in \mathbb{R}$ such that $|\phi(x)|^2 \leq Ch(x^*x)$ for all $x \in \mathbb{C}[\mathbb{G}]$. In that case we denote $\|\phi\|_2$ the corresponding norm. Clearly, if ϕ is ℓ^2 then it is weakly ℓ^2.

Although we will not need this in the remainder of this article, the following lemma shows that ℓ^2-forms can also be characterized in terms of the associated "functions" in $C_0(\mathbb{G})$ by means of the left Haar weight. Note that in the unimodular case we have simply $\varphi(ff^*) = \varphi(f^*f) = \|f\|_2^2$.

Lemma 3.3. *Let $\phi \in \mathbb{C}[\mathbb{G}]^*$ be a linear form and put $f = (\mathrm{id} \otimes \phi)(V)$. Then ϕ is an ℓ^2-form if and only if $\varphi(ff^*) < \infty$.*

Proof. We put $p_0 = (\mathrm{id} \otimes h)(V) \in C_c(\mathbb{G})$, which is also the central support of $\hat{\epsilon}$, and we note the following identity (see the Remark after the proof) in the multiplier Hopf algebra $C_c(\mathbb{G})$:

$$\forall f \in C_c(\mathbb{G}) \ (\hat{S} \otimes \varphi)(\Delta(p_0)(1 \otimes f)) = f. \tag{2}$$

From this we can deduce that the scalar product in H implements, via the Fourier transform, the natural duality between $C_c(\mathbb{G})$ and $\mathbb{C}[\mathbb{G}]$ which is given, for $x = (\omega \otimes \mathrm{id})(V) \in \mathbb{C}[\mathbb{G}]$ and $f = (\mathrm{id} \otimes \phi)(V) \in C_c(\mathbb{G})$, by $\langle f, x\rangle = (\omega \otimes \phi)(V) = \phi(x) = \omega(f)$. Indeed we have, using the identities $(\hat{S} \otimes \mathrm{id})(V) = V^*$ and $V_{13}V_{23} = (\Delta \otimes \mathrm{id})(V)$:

$$\begin{aligned}(x\xi_0|\mathcal{F}(f)\xi_0) &= h(x^*\mathcal{F}(f)) = (\varphi \otimes h)((1 \otimes x^*)V(f \otimes 1)) \\ &= (\bar{\omega}\hat{S} \otimes \varphi \otimes h)(V_{13}V_{23}(1 \otimes f \otimes 1)) = (\bar{\omega}\hat{S} \otimes \varphi)(\Delta(p_0)(1 \otimes f)) = \bar{\omega}(f).\end{aligned}$$

This yields, for $x \in \mathbb{C}[\mathbb{G}]$ and ϕ such that $f = (\mathrm{id} \otimes \phi)(V) \in C_c(\mathbb{G})$, the formula $\phi(x) = \omega(f) = (\mathcal{F}(f^*)\xi_0|x\xi_0)$. We get in particular $\|\phi\|_2 = \|\mathcal{F}(f^*)\|_2 = \|f^*\|_2 = \sqrt{\varphi(ff^*)}$.

Now for a general ϕ and $r \in \mathrm{Irr}\,\mathbb{G}$, denote ϕ_r the restriction of ϕ to $\mathbb{C}[\mathbb{G}]_r$. We have $\|\phi\|_2^2 = \sum_{r \in \mathrm{Irr}\,\mathbb{G}} \|\phi_r\|_2^2$, $(\mathrm{id} \otimes \phi_r)(V) = p_r f$ and $\varphi(ff^*) = \sum_{r \in \mathrm{Irr}\,\mathbb{G}} \varphi(ff^*p_r)$, so that the identity $\|\phi\|_2^2 = \varphi(ff^*)$ still holds in $[0, +\infty]$. □

Remark 3.4. The identity (2) is equivalent to the fact that the Fourier transform $\mathcal{F}$ is isometric. Indeed a simple computation yields

$$h(\mathcal{F}(a)^*\mathcal{F}(b)) = \varphi[a^*(\hat{S} \otimes \varphi)(\Delta(p_0)(1 \otimes b))].$$

Note that (2) can be easily proved by standard computations in $C_c(\mathbb{G})$, and this is one convenient way of establishing the fact that $\mathcal{F}$ is isometric.

Now we consider the case of the discrete quantum group $\mathbb{F}O_n$, whose irreducible corepresentations $v^k \in B(H_k) \otimes A_o(n)$ are labeled, up to equivalence, by integers k. Recall that we denote $\mathbb{C}[\mathbb{G}]_k$ the corresponding coefficient subspaces, and $(U_k)_k$ the series of dilated Chebyshev polynomials of the second kind. In [Bra12], Brannan shows that the multiplier $T_s : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C^*_{\mathrm{red}}(\mathbb{F}O_n)$ associated with the functional

$$\tau_s : \mathbb{C}[\mathbb{F}O_n] \to \mathbb{C}, \quad x \in \mathbb{C}[\mathbb{F}O_n]_k \mapsto \frac{U_k(s)}{U_k(n)} \epsilon(x)$$

is a completely positive map as above, for every $s \in]2, n]$ — see also Section 4.1.

Besides it is known that $\mathbb{F}O_n$ satisfies the Property of Rapid Decay. More precisely, for any $k \in \mathbb{N}$ and any $x \in \mathbb{C}[\mathbb{G}]_k \subset C^*_{\mathrm{red}}(\mathbb{G})$ we have $\|x\| \leq C(1+k)\|x\|_2$, where C is a constant depending only on n [Ver07]. On the other hand we denote by $l : \mathbb{C}[\mathbb{G}] \to \mathbb{C}$ the length form on $\mathbb{F}O_n$, whose restriction to $\mathbb{C}[\mathbb{G}]_k$ coincides by definition with $k\epsilon$, and we recall that the convolution of two linear forms ϕ, $\psi \in \mathbb{C}[\mathbb{G}]^*$ is defined by $\phi * \psi = (\phi \otimes \psi)\Delta$. It is well-known that the convolution exponential $e^{\phi} = \sum \phi^{*n}/n!$ is well-defined on $\mathbb{C}[\mathbb{G}]$, and in the case of l (or any other central form) we have simply $e^{-\lambda l}(x) = e^{-\lambda k}\epsilon(x)$ if $x \in \mathbb{C}[\mathbb{G}]_k$.

Using the multipliers T_s and Property RD we can follow the proof of [Haa79, Theorem 3.1] and deduce the following lemma:

Lemma 3.5. *A state ϕ on $C^*(\mathbb{F}O_n)$ factors through $C^*_{\mathrm{red}}(\mathbb{F}O_n)$ if and only if $e^{-\lambda l} * \phi$ is an ℓ^2-form for all $\lambda > 0$.*

Proof. We denote by ϕ_k the restriction of ϕ to the f.-d. subspace $\mathbb{C}[\mathbb{G}]_k$. Since these subspaces are pairwise orthogonal with respect to h, we have $\|\phi\|_2^2 = \sum \|\phi_k\|_2^2$. Now for every $s < n$ there is a $\lambda > 0$ such that $U_k(s)/U_k(n) \leq e^{-\lambda k}$ for all k. We can then write

$$\|\tau_s * \phi\|_2^2 = \sum \|(\tau_s * \phi)_k\|_2^2 = \sum \frac{U_k(s)^2}{U_k(n)^2} \|\phi_k\|_2^2 \leq \sum e^{-2\lambda k} \|\phi_k\|_2^2 = \|e^{-\lambda l} * \phi\|_2^2.$$

Now if $e^{-\lambda l} * \phi$ is ℓ^2 for all $\lambda > 0$, we conclude that this is also the case of $\tau_s * \phi$ for all $s \in]2, n[$. In particular $\tau_s * \phi$ is weakly ℓ^2 for all s. We have $(\tau_s * \phi)(x) \to \phi(x)$ as $s \to n$ for all $x \in \mathbb{C}[\mathbb{G}]$, and $\tau_s * \phi$ is a state for all s, hence $\tau_s * \phi \to \phi$ weakly and it follows that ϕ is weakly ℓ^2.

Conversely, assume that ϕ factors through $C^*_{\mathrm{red}}(\mathbb{G})$. Using Property RD we can write for any $x \in \mathbb{C}[\mathbb{G}]_k \subset C^*_{\mathrm{red}}(\mathbb{G})$:

$$|\phi(x)| \leq \|x\| \leq C(1+k)\|x\|_2,$$

hence $\|\phi_k\|_2 \leq C(1+k)$. This clearly implies that $\|e^{-\lambda l} * \phi\|_2$ is finite for every $\lambda > 0$. □

3.2. **The adjoint representation.** Recall that we denote by $\lambda : C^*(\mathbb{G}) \to B(H)$ the GNS representation of the Haar state h, and consider the corresponding right regular representation $\rho : C^*(\mathbb{G}) \to B(H)$, $x \mapsto U\lambda(x)U$ given by the unitary $U(\Lambda_h(x)) = \Lambda_h(f_1 * S(x)) = \Lambda_h(S(x * f_{-1}))$. We have in particular $\rho(v_{ij})\Lambda_h(x) = \Lambda_h(x(f_1 * v_{ji}^*))$, if $v_{ij} = v_{e_i,e_j}$ are the coefficients of a unitary corepresentation in an ONB. Recall that $[UC^*_{\mathrm{red}}(\mathbb{G})U, C^*_{\mathrm{red}}(\mathbb{G})] = 0$.

The adjoint representation of $\mathbb{G}$ is $\mathrm{ad} : C^*(\mathbb{G}) \to B(H)$, $x \mapsto \sum \lambda(x_{(1)})\rho(x_{(2)})$. Here $\Delta : x \mapsto \sum x_{(1)} \otimes x_{(2)}$ is the coproduct from $C^*(\mathbb{G})$ to $C^*_{\mathrm{red}}(\mathbb{G}) \otimes_{\max} C^*_{\mathrm{red}}(\mathbb{G})$. We have $(\mathrm{id} \otimes \mathrm{ad})(v) = (\mathrm{id} \otimes \lambda)(v)(\mathrm{id} \otimes \rho)(v)$ if v is a corepresentation of $\mathbb{G}$. Here are two explicit formulae, for $x \in \mathbb{C}[\mathbb{G}]$ and v_{ij} coefficient of a unitary corepresentation in an ONB:

$$\mathrm{ad}(x)\Lambda_h(y) = \sum \Lambda_h(x_{(1)} y (f_1 * S(x_{(2)}))),$$
$$\mathrm{ad}(v_{ij})\Lambda_h(y) = \sum \Lambda_h(v_{ik} y (f_1 * v_{jk}^*)).$$

The corepresentation of V associated to ρ is $W = (1 \otimes U)V(1 \otimes U)$, and the one associated to ad is $A = VW$. Note that we have, in the notation of [BS93], $W = \Sigma\tilde{V}\Sigma$. In particular [BS93, Proposition 6.8] shows that $W(1 \otimes f)W^* = \sigma\Delta(f)$ for $f \in C_0(\mathbb{G})$. This means that the multiplicative unitary W^* is associated to the discrete quantum group $\mathbb{G}^{\text{co-op}}$.

Lemma 3.6. *The canonical line $\mathbb{C}\xi_0 \subset H$ is invariant for the adjoint representation* $\mathrm{ad} : C^*(\mathbb{G}) \to B(H)$ *if and only if $\mathbb{G}$ is unimodular. In that case, ξ_0 is a fixed vector for* ad.

Proof. For $x \in \mathbb{C}[\mathbb{G}]$ one can compute $\mathrm{ad}(x)\xi_0 = \sum x_{(1)}(f_1 * S(x_{(2)}))\xi_0$, which equals in the unimodular case $\sum x_{(1)}S(x_{(2)})\xi_0 = \epsilon(x)\xi_0$. So in that case ξ_0 is fixed.

Furthermore, for $v \in \mathrm{Irr}\,\mathbb{G}$ the previous computation together with Woronowicz' orthogonality relations lead to

$$(\xi_0|\,\mathrm{ad}(v_{ij})\xi_0) = \sum_k h(v_{ik}(f_1 * v_{jk}^*)) = \sum_k h((v_{jk}^* * f_{-1})v_{ik})$$
$$= \sum_k \frac{\delta_{ji}\delta_{kk}}{\mathrm{qdim}\, v} = \frac{\dim v}{\mathrm{qdim}\, v}\epsilon(v_{ij}).$$

In particular, denoting $\phi : x \mapsto (\xi_0|\,\mathrm{ad}(x)\xi_0)$, we see that $(\mathrm{id} \otimes \phi)(v) = (\dim v/\,\mathrm{qdim}\, v)\mathrm{id}$. Now if the line $\mathbb{C}\xi_0$ is invariant, ϕ is a character and since v is unitary we must have $\dim v = \mathrm{qdim}\, v$ for all $v \in \mathrm{Irr}\,\mathbb{G}$. This happens exactly when $\mathbb{G}$ is unimodular. □

Definition 3.7. If $\mathbb{G}$ is a unimodular discrete quantum group, we denote ad° the restriction of the adjoint representation of $C^*(\mathbb{G})$ to the subspace $H^\circ = \xi_0^\perp \subset H$.

Our aim in the remainder of the section is to show that ad° factors through λ in the case of $\mathbb{G} = \mathbb{F}O_n$, $n \geq 3$ — or more generally for unimodular orthogonal free quantum groups. We will need some estimates involving coefficients of irreducible corepresentations of $\mathbb{G}$.

In order to carry on the computations, we will need to be more precise about the irreducible corepresentations of $\mathbb{F}O(Q)$. We denote by $v = v^1 \in B(\mathbb{C}^n) \otimes A_o(Q)$ the fundamental corepresentation. Then we introduce recursively the irreducible corepresentation $v^k \in B(H_k) \otimes A_o(Q)$ as the unique subcorepresentation of $v^{\otimes k}$ not equivalent to any v^l, $l < k$. In this way we have $H_k \subset H_1^{\otimes k}$, with $H_0 = \mathbb{C}$ and $H_1 = \mathbb{C}^n$. We denote by $P_k \in B(H_1^{\otimes k})$ the orthogonal projection onto H_k. Recall that each irreducible corepresentation of $A_o(Q)$ is equivalent to exactly one v^k, and that we have the equivalence of corepresentations

$$H_k \otimes H_l \simeq H_{|k-l|} \oplus H_{|k-l|+2} \oplus \cdots \oplus H_{k+l-2} \oplus H_{k+l}.$$

It is known that $\dim H_k = [k+1]_\rho = (\rho^{k+1} - \rho^{-k-1})/(\rho - \rho^{-1})$. Note in particular that we have $D_1\rho^k \leq \dim H_k \leq D_2\rho^k$ with constants $0 < D_1 < D_2$ depending only on n.

Let us also denote by $Q_r^k \in L(H_1^{\otimes k})$ the orthogonal projection onto the sum of all subspaces equivalent to H_r, so that $P_k = Q_k^k$. If p_r is the minimal central projection of $C_0(\mathbb{F}O_n)$ associated with v^r, then Q_r^k also corresponds to $\Delta^{k-1}(p_r)$ via the natural action of $C_0(\mathbb{F}O_n)$ on H_1.

Note that the subspace $\mathrm{Im}\, Q_r^{a+b+c}(P_{a+b} \otimes P_c)$ (resp. $\mathrm{Im}\, Q_r^{a+b+c}(P_a \otimes P_{b+c})$) corresponds to the unique subcorepresentation of $v^{a+b} \otimes v^c$ (resp. $v^a \otimes v^{b+c}$) equivalent to v^r, when it is non-zero. When $r = a+b+c$ both spaces coincide with H_{a+b+c}. On the other hand one can show that these subspaces of $H_1^{\otimes a+b+c}$ are pairwise "far from each other" when $r < a+b+c$ and b is big. More precisely, Lemma A.4 of [VV07] shows that

$$\|(P_{a+b} \otimes P_c)Q_r^{a+b+c}(P_a \otimes P_{b+c})\| \leq C_1 q^b$$

for some constant $C_1 > 0$ depending only on q. Indeed when r varies up to $a+b+c-2$ the maps on the left-hand side live in pairwise orthogonal subspaces of $H_1^{\otimes a+b+c}$ and sum up to $(P_{a+b} \otimes P_c)(P_a \otimes P_{b+c}) - P_{a+b+c}$.

Since we are interested in ad°, we assume in the remainder of this section that $\mathbb{F}O(Q)$ is unimodular : equivalently, Q is a scalar multiple of a unitary matrix, or $q\rho = 1$.

In the following lemma we give an upper estimate for $|(\omega_\zeta \otimes \mathrm{Tr}_k)((P_l \otimes P_k)Q_r^{k+l}\Sigma_{lk})|$, where ζ is any vector in H_l, Tr_k is the trace on $B(H_k)$, and $\Sigma_{lk} : H_l \otimes H_k \to H_k \otimes H_l$ is the flip map. Note that $\|(P_l \otimes P_k)Q_r^{k+l}\Sigma_{lk}\| \leq 1$ so that $(\dim H_k)\|\zeta\|^2$ is a trivial upper bound, which grows exponentially with k. In the lemma we derive an upper bound which is polynomial in k (and even constant for $r < k+l$).

Note that it is quite natural to consider maps like $(P_l \otimes P_k)Q_r^{k+l}\Sigma_{lk}$ when studying the adjoint representation of $A_o(n)$. A non-trivial upper bound for the norm $\|(P_1 \otimes P_k)Q_{k-1}^{k+1}\Sigma_{1k}\|$ was given in [VV07, Lemma 7.11], but it does not imply our tracial estimate below.

We choose a fixed vector $T_1 = H_1 \to H_1$ with norm $\sqrt{\dim H_1}$ — T_1 is unique up to a phase, and we have $(\mathrm{id}_1 \otimes T_1^*)(T_1 \otimes \mathrm{id}_1) = \mu \mathrm{id}_1$ with $\mu = \pm 1$. We choose then fixed vectors $T_m \in H_m \otimes H_m$

inductively by putting $T_m = (P_m \otimes P_m)(\mathrm{id}_{m-1} \otimes T_1 \otimes \mathrm{id}_{m-1})T_{m-1}$. If $(e_s^m)_s$ is an ONB of H_m, we have $T_m = \sum e_s^m \otimes \bar{e}_s^m$ where $(\bar{e}_s^m)_s$ is again an ONB of H_m, and $\|T_m\| = \sqrt{\dim H_m}$. Finally, we denote more generally

$$T_{ab}^m : H_1^{\otimes a-m} \otimes H_1^{\otimes b-m} \to H_1^{\otimes a+b},\ \zeta \otimes \xi \mapsto \sum \zeta \otimes e_s^m \otimes \bar{e}_s^m \otimes \xi.$$

Lemma 3.8. *Let $(e_p^k)_p$ be an ONB of H_k. There exists a constant $C \geq 1$, depending only on n, such that we have, for any $0 \leq m \leq l \leq k$, $l \neq 0$, $\zeta \in H_l$ and $r = k-l, k-l+2, \dots, k+l-2m-2$:*

$$\Big|\sum_p (T_{lk}^{m*}(\zeta \otimes e_p^k) | Q_r^{k+l-2m} T_{kl}^{m*}(e_p^k \otimes \zeta))\Big| \leq C^l \|\zeta\|^2,$$

$$\Big|\sum_p (T_{lk}^{m*}(\zeta \otimes e_p^k) | P_{k+l-2m} T_{kl}^{m*}(e_p^k \otimes \zeta))\Big| \leq kC^l \|\zeta\|^2 \text{ if } l \neq 2m,$$

$$\leq (Ck)^l \|\zeta\|^2 \text{ if } l = 2m.$$

Proof. In this proof we denote $d_k = \dim H_k = \mathrm{qdim}\, v^k$ and $\mathrm{id}_k \in B(H_1^{\otimes k})$ the identity map. Let us denote by S_r^k, S_+^k the sums in the statement as well as $S^k = S_+^k + \sum_r S_r^k$. We first prove the estimate for S_r^k: using the Lemma A.4 from [VV07] recalled above, with $a = c = l - m$ and $b = k - l$, we can write

$$\begin{aligned}
|S_r^k| &= \Big|\sum_p (T_{lk}^{m*}(\zeta \otimes e_p^k) | Q_r^{k+l-2m} T_{kl}^{m*}(e_p^k \otimes \zeta))\Big| \\
&\leq \sum_p |(T_{lk}^{m*}(\zeta \otimes e_p^k) | (P_{l-m} \otimes P_{k-m}) Q_r^{k+l-2m} (P_{k-m} \otimes P_{l-m}) T_{kl}^{m*}(e_p^k \otimes \zeta))| \\
&\leq \|(P_{l-m} \otimes P_{k-m}) Q_r^{k+l-2m} (P_{k-m} \otimes P_{l-m})\| \times \|T_m\|^2 \times \sum_p \|\zeta\|^2 \|e_p^k\|^2 \\
&\leq C_1 q^{k-l} d_k d_m \ \|\zeta\|^2 \leq C_1 D_2^2 q^{-l-m} \|\zeta\|^2.
\end{aligned}$$

The case of S_+^k is more involved. First notice that $|S_+^k - S^k| = |\sum_r S_r^k| \leq (l-m) C_1 D_2^2 q^{-l-m} \|\zeta\|^2$ according to the estimate above. This shows that the estimate for S_+^k is equivalent to the same estimate for S^k with a possibly different constant. In the case when $2m < l$ we will prove the estimate for S_+^k using an induction over k. We first perform the following transformation using scalar products in $H_{l-m} \otimes H_{k-l} \otimes H_{l-m}$:

$$\begin{aligned}
S^k &= \sum_{p,q,i,j} (T_{lk}^{m*}(\zeta \otimes e_p^k) | e_i^{l-m} \otimes e_q^{k-l} \otimes e_j^{l-m}) \times (e_i^{l-m} \otimes e_q^{k-l} \otimes e_j^{l-m} | T_{kl}^{m*}(e_p^k \otimes \zeta)) \\
&= \sum_{p,q,i,j,s,t} (\zeta \otimes e_p^k | e_i^{l-m} \otimes e_s^m \otimes \bar{e}_s^m \otimes e_q^{k-l} \otimes e_j^{l-m}) \times \\
&\qquad\qquad \times (e_i^{l-m} \otimes e_q^{k-l} \otimes e_t^m \otimes \bar{e}_t^m \otimes e_j^{l-m} | e_p^k \otimes \zeta) \\
&= \sum_{q,i,j,s,t} (\zeta | e_i^{l-m} \otimes e_s^m) \times (e_i^{l-m} \otimes e_q^{k-l} \otimes e_t^m | P_k(\bar{e}_s^m \otimes e_q^{k-l} \otimes e_j^{l-m})) \times (\bar{e}_t^m \otimes e_j^{l-m} | \zeta) \\
&= \sum_{q,s,t} (T_{l,k-l+2m}^{m*}(\zeta \otimes \bar{e}_s^m \otimes e_q^{k-l} \otimes e_t^m) | P_k T_{k-l+2m,l}^{m*}(\bar{e}_s^m \otimes e_q^{k-l} \otimes e_t^m \otimes \zeta)).
\end{aligned}$$

One can factor $(P_{k-l+m} \otimes \mathrm{id}_{l-m})$ out of P_k and let it move to the right of $T_{k-l+2m,l}^{m*}$. Similarly, one can factor $(\mathrm{id}_{l-m} \otimes P_{k-l+m})$ out, let it move through $T_{k-l+2m,l}^{m*}$ on the left-hand side of the scalar product, and take it back to the right-hand side thank to the sum over q and t (by cyclicity of the trace). This yields:

$$\begin{aligned}
S^k = \sum_{q,s,t} &(T_{l,k-l+2m}^{m*}(\zeta \otimes \bar{e}_s^m \otimes e_q^{k-l} \otimes e_t^m) \ | \\
&\qquad | \ P_k T_{k-l+2m,l}^{m*}((P_{k-l+m} \otimes \mathrm{id}_m)(\mathrm{id}_m \otimes P_{k-l+m})(\bar{e}_s^m \otimes e_q^{k-l} \otimes e_t^m) \otimes \zeta)),
\end{aligned}$$

which can be compared to:

$$\begin{aligned}
S_+^{k-l+2m} &= \sum_p (T_{l,k-l+2m}^{m*}(\zeta \otimes e_p^{k-l+2m}) | P_k T_{k-l+2m,l}^{m*}(e_p^{k-l+2m} \otimes \zeta)) \\
&= \sum_{q,s,t} (T_{l,k-l+2m}^{m*}(\zeta \otimes \bar{e}_s^m \otimes e_q^{k-l} \otimes e_t^m) | P_k T_{k-l+2m,l}^{m*}(P_{k-l+2m}(\bar{e}_s^m \otimes e_q^{k-l} \otimes e_t^m) \otimes \zeta)).
\end{aligned}$$

More precisely, using again the Lemma A.4 from [VV07] with $a = c = m$ and $b = k - l$ we get

$$|S^k - S_+^{k-l+2m}| \le \sum_{q,s,t} \|T_m\|^2 \|(P_{k-l+m} \otimes \mathrm{id}_m)(\mathrm{id}_m \otimes P_{k-l+m}) - P_{k-l+2m}\| \|\zeta\|^2$$
$$\le \sum_{q,s,t} \|T_m\|^2\ C_1 q^{k-l} \|\zeta\|^2 \le C_1 d_m^3 d_{k-l} q^{k-l} \|\zeta\|^2 \le C_1 D_2^4 q^{-3m} \|\zeta\|^2.$$

Using our previous estimate for the $l - m - 1$ terms S_r^k we obtain a recursive inequation for S_+^k:

$$|S_+^k| = |S^k - \sum_r S_r^k| \le |S^k| + C_1 D_2^2 (l-m) q^{-l-m} \|\zeta\|^2$$
$$\le |S_+^{k-l+2m}| + C_1 (D_2^2 l + D_2^4) q^{-3l} \|\zeta\|^2.$$

For $k \le 2l$ we use the trivial estimate $|S_+^k| \le d_k \|T_m\|^2 \|\zeta\|^2 \le D_2^2 q^{-3l} \|\zeta\|^2$. Since $l - 2m > 0$ an easy induction on k yields $|S_+^k| \le klC^l \|\zeta\|^2$, which implies the estimate of the statement up to a change of the constant C.

For the case $2m > l$ we consider the vector $\bar\zeta = (\zeta^* \otimes \mathrm{id})T_l$, which satisfies $(\mathrm{id} \otimes T_m^*)(\zeta \otimes \mathrm{id}) = \mu^{l-m}(\bar\zeta^* \otimes \mathrm{id})(\mathrm{id} \otimes T_{l-m})$ and $(T_m^* \otimes \mathrm{id})(\mathrm{id} \otimes \zeta) = \mu^{l-m}(\mathrm{id} \otimes \bar\zeta^*)(T_{l-m} \otimes \mathrm{id})$. We transform S^k as follows:

$$\begin{aligned}
S^k &= \mathrm{Tr}\left((\zeta^* \otimes \mathrm{id}_k)(\mathrm{id}_{l-m} \otimes T_m \otimes \mathrm{id}_{k-m})(\mathrm{id}_{k-m} \otimes T_m^* \otimes \mathrm{id}_{l-m})(\mathrm{id}_k \otimes \zeta)\right) \\
&= \mathrm{Tr}\left((\mathrm{id}_m \otimes T_{l-m}^* \otimes \mathrm{id}_{k-m})(\bar\zeta \otimes \mathrm{id}_{k+l-2m})(\mathrm{id}_{k+l-2m} \otimes \bar\zeta^*)(\mathrm{id}_{k-m} \otimes T_{l-m} \otimes \mathrm{id}_m)\right) \\
&= \mathrm{Tr}\left((\mathrm{id}_{k+l-2m} \otimes \bar\zeta^*)(\mathrm{id}_{k-m} \otimes T_{l-m} \otimes \mathrm{id}_m)(\mathrm{id}_m \otimes T_{l-m}^* \otimes \mathrm{id}_{k-m})(\bar\zeta \otimes \mathrm{id}_{k+l-2m})\right), \\
\bar S^k &= \mathrm{Tr}\left((\bar\zeta^* \otimes \mathrm{id}_{k+l-2m})(\mathrm{id}_m \otimes T_{l-m} \otimes \mathrm{id}_{k-m})(\mathrm{id}_{k-m} \otimes T_{l-m}^* \otimes \mathrm{id}_m)(\mathrm{id}_{k+l-2m} \otimes \bar\zeta)\right).
\end{aligned}$$

As a result, $\bar S^k$ can be obtained from S^{k+l-2m} by replacing ζ with $\bar\zeta$ and m with $l - m$, and we can apply the case $2m < l$ to obtain the result — observe that $k + l - 2m \le k$ and $\|\bar\zeta\| = \|\zeta\|$.

If l is even, we still have to deal with the case $2m = l$, which is the subject of the next lemma. □

Lemma 3.8bis. *Let $(e_p^k)_p$ be an ONB of H_k. There exists a constant $C \ge 1$, depending only on n, such that we have, for any $\zeta, \xi \in H_{2m}$ and $0 < 2m = l \le k$:*

$$\Big|\sum_p ((\mathrm{id}_m \otimes T_m^* \otimes \mathrm{id}_{k-m})(\zeta \otimes e_p^k) | (\mathrm{id}_{k-m} \otimes T_m^* \otimes \mathrm{id}_m)(e_p^k \otimes \xi))\Big| \le (Ck)^l \|\zeta\| \|\xi\|.$$

Proof. We need in fact to prove a more general statement, where m is allowed to take two different values $\bar m$, $m \in \mathbb{N}^*$ on the left and on the right of the scalar product, with $\bar m + m \le k$. For $\zeta \in H_{2\bar m}$, $\xi \in H_{2m}$ we denote

$$S^k(\zeta, \xi) = \sum_p ((\mathrm{id}_{\bar m} \otimes T_{\bar m}^* \otimes \mathrm{id}_{k-\bar m})(\zeta \otimes e_p^k) | (\mathrm{id}_{k-m} \otimes T_m^* \otimes \mathrm{id}_m)(e_p^k \otimes \xi)).$$

We first transform this quantity by writing the trace of $B(H_k)$ as the restriction to a corner of the trace of $B(H_{k-1} \otimes H_1)$:

$$\begin{aligned}
S^k(\zeta, \xi) &= \sum_{q,s} ((\mathrm{id}_{\bar m} \otimes T_{\bar m}^* \otimes \mathrm{id}_{k-\bar m-1})(\zeta \otimes e_q^{k-1}) \otimes e_s^1 | (\mathrm{id}_{k-m} \otimes T_m^* \otimes \mathrm{id}_m)(P_k(e_q^{k-1} \otimes e_s^1) \otimes \xi)) \\
&= \sum_q ((\mathrm{id}_{\bar m} \otimes T_{\bar m}^* \otimes \mathrm{id}_{k-\bar m-1})(\zeta \otimes e_q^{k-1}) | (\mathrm{id}_{k-m} \otimes T_m^* \otimes \mathrm{id}_{m-1})(P_k \otimes \mathrm{id}_{2m-1})(e_q^{k-1} \otimes \tilde\xi)),
\end{aligned}$$

where $\tilde\xi = \sum (e_s^1 \otimes \mathrm{id}_{2m-1} \otimes e_s^{1*})(\xi) \in H_1 \otimes H_{2m-1}$ has the same norm as ξ. Observe that one has $(e_q^{k-1} \otimes \tilde\xi) = (\mathrm{id}_k \otimes P_m \otimes \mathrm{id}_{m-1})(e_q^{k-1} \otimes \tilde\xi)$ so that one can replace T_m^* on the right-hand side of the scalar product by $T_{m-1}^*(\mathrm{id}_{m-1} \otimes T_1^* \otimes \mathrm{id}_{m-1})$ or $T_1^*(\mathrm{id}_1 \otimes T_{m-1}^* \otimes \mathrm{id}_1)$.

Then we use the following generalization of Wenzl's induction formula, which can be found e.g. in [VV07, Equation (7.4)]:

$$P_k = (P_{k-1} \otimes \mathrm{id}_1) + \sum_{i=1}^{k-1} (-\mu)^{k-i} \frac{d_{i-1}}{d_{k-1}} (\mathrm{id}_{i-1} \otimes T_1 \otimes \mathrm{id}_{k-i-1} \otimes T_1^*)(P_{k-1} \otimes \mathrm{id}_1).$$

Substituting P_k in the formula for $S^k(\zeta, \xi)$ above, we see that most terms vanish. The terms corresponding to $k - m + 1 \le i \le k - 1$ vanish because in that case both legs of T_1 in the factor $(\mathrm{id}_{k-m} \otimes T_m^* \otimes \mathrm{id}_{m-1})(\mathrm{id}_{i-1} \otimes T_1 \otimes \mathrm{id}_{k-i-1} \otimes T_1^* \otimes \mathrm{id}_{2m-1})$ hit the left leg of T_m which lies in

H_m. The terms corresponding to $\bar m+1 \le i \le k-m-1$ vanish because in that case the vector T_1 goes through $(\mathrm{id}_{k-m}\otimes T_m^*\otimes \mathrm{id}_{m-1})$ and hits e_q^{k-1} on the left of the scalar product. Finally, the terms corresponding to $1\le i\le \bar m-1$ vanish because the vector T_1 hits ζ on the left of the scalar product.

Int the term with $(P_{k-1}\otimes \mathrm{id}_1)$ we replace T_m^* by $T_{m-1}^*(\mathrm{id}_{m-1}\otimes T_1^*\otimes \mathrm{id}_{m-1})$ on the right-hand side of the scalar product and obtain

$$\sum_q ((\mathrm{id}_{\bar m}\otimes T_{\bar m}^*\otimes \mathrm{id}_{k-\bar m-1})(\zeta\otimes e_q^{k-1}) | (\mathrm{id}_{k-m}\otimes T_{m-1}^*\otimes \mathrm{id}_{m-1})(e_q^{k-1}\otimes \xi'))$$

where $\xi' = (T_1^*\otimes \mathrm{id}_{2m-2})(\tilde\xi) = \sum(\bar e_s^{1*}\otimes \mathrm{id}_{2m-2}\otimes e_s^{1*})(\xi) \in H_{2m-2}$. We recognize $S^{k-1}(\zeta,\xi')$ and we note that $\|\xi'\|\le d_1\|\xi\|$. We remark also that when $m=1$ we have $\xi'=0$: indeed in the Kac case $\Sigma_{11}T_1$ is proportional to T_1, and $T_1^*(H_2)=\{0\}$.

In the term $i=k-m$, the right-hand side of the scalar product reads:

$$\begin{aligned} A &= (\mathrm{id}_{k-m}\otimes T_1^*\otimes \mathrm{id}_{m-1})(\mathrm{id}_{k-m+1}\otimes T_{m-1}^*\otimes \mathrm{id}_m)\times \\ &\qquad\times (\mathrm{id}_{k-m-1}\otimes T_1\otimes \mathrm{id}_{3m-2})(\mathrm{id}_{k-2}\otimes T_1^*\otimes \mathrm{id}_{2m-1})(e_q^{k-1}\otimes\tilde\xi) \\ &= \mu(\mathrm{id}_{k-m-1}\otimes T_{m-1}^*\otimes \mathrm{id}_m)(\mathrm{id}_{k-2}\otimes T_1^*\otimes \mathrm{id}_{2m-1})(e_q^{k-1}\otimes\tilde\xi) \\ &= \mu(\mathrm{id}_{k-m-1}\otimes T_m^*\otimes \mathrm{id}_m)(e_q^{k-1}\otimes\tilde\xi). \end{aligned}$$

Now we decompose $\tilde\xi = \xi'' + Q_{2m-2}^{2m}(\tilde\xi)$, where $\xi'' = P_{2m}(\tilde\xi)$. Summing over q, the terms corresponding to ξ'' yield the quantity $S^{k-1}(\zeta,\xi'')$ and we note that $\|\xi''\|\le\|\xi\|$.

On the other hand, by Wenzl's recursion formula, $Q_{2m-2}^{2m}(\tilde\xi) = \tilde\xi - P_{2m}(\tilde\xi)$ decomposes as a linear combination of vectors of the form $(\mathrm{id}_j\otimes T_1\otimes \mathrm{id}_{2m-2-j})(\xi')$. Again, the contributions with $j\ne m-1$ vanish, either because T_1 hits the right leg of T_m^*, or because it hits e_q^{k-1} on the left-hand side of the scalar product (if $k>\bar m+m$). The corresponding term in A, without the multiplicative factor $(-1)^m d_{m-1}/d_{2m-1}$, is

$$\begin{aligned} A' &= \mu(\mathrm{id}_{k-m-1}\otimes T_m^*\otimes \mathrm{id}_m)(\mathrm{id}_{k+m-2}\otimes T_1\otimes \mathrm{id}_{m-1})(e_q^{k-1}\otimes\xi') \\ &= \mu(\mathrm{id}_{k-m-1}\otimes T_1^*\otimes \mathrm{id}_m)(\mathrm{id}_{k-m}\otimes T_{m-1}^*\otimes \mathrm{id}_{m+1})(\mathrm{id}_{k+m-2}\otimes T_1\otimes \mathrm{id}_{m-1})(e_q^{k-1}\otimes\xi') \\ &= (\mathrm{id}_{k-m}\otimes T_{m-1}^*\otimes \mathrm{id}_{m-1})(e_q^{k-1}\otimes\xi'), \end{aligned}$$

and summing the scalar products over q we recognize again $S^{k-1}(\zeta,\xi')$.

Finally for $i=\bar m$ we let the vector T_1 from Wenzl's formula go to the left-hand side of the tensor product. Noting that $T_m^*(\mathrm{id}_m\otimes T_1^*\otimes \mathrm{id}_m) = T_{m+1}^*$ on $H_{m+1}\otimes H_1^{\otimes m+1}$ and $T_1^*(\mathrm{id}_1\otimes T_{\bar m}^*\otimes \mathrm{id}_1) = T_{\bar m+1}^*$ on $H_{\bar m+1}\otimes H_{\bar m+1}$ we obtain

$$((\mathrm{id}_{\bar m-1}\otimes T_{\bar m+1}^*\otimes \mathrm{id}_{k-\bar m-2})(\zeta\otimes e_q^{k-1}) | (\mathrm{id}_{k-m-1}\otimes T_{m+1}^*\otimes \mathrm{id}_{m-2})(e_q^{k-1}\otimes\tilde\xi)).$$

In absolute value, the sum over q of these terms is less than

$$\dim H_{k-1}\|T_{\bar m+1}\|\|T_{m+1}\|\|\zeta\|\|\xi\| = d_{k-1}\sqrt{d_{\bar m+1}d_{m+1}}\|\zeta\|\|\xi\|.$$

Putting everything together we have obtained

$$\begin{aligned} |S^k(\zeta,\xi)| &\le |S^{k-1}(\zeta,\xi')| + \tfrac{d_{k-m-1}}{d_{k-1}}\,|S^{k-1}(\zeta,\xi'')| + \tfrac{d_{k-m-1}d_{m-1}}{d_{k-1}d_{2m-1}}\,|S^{k-1}(\zeta,\xi')| + \\ &\qquad + \tfrac{d_{\bar m-1}}{d_{k-1}}\,d_{k-1}\sqrt{d_{\bar m+1}d_{m+1}}\,\|\zeta\|\|\xi\| \\ &\le |S^{k-1}(\zeta,\xi'')| + 2|S^{k-1}(\zeta,\xi')| + D_2^2 q^{-2(m+\bar m)}\|\zeta\|\|\xi\|. \end{aligned}$$

This allows to prove, by induction over $m+\bar m$, that we have $|S^k(\zeta,\xi)|\le D_2^2(3q^{-2}k)^{m+\bar m}\|\zeta\|\|\xi\|$ for all $\zeta\in H_{2\bar m}$, $\xi\in H_{2m}$ and all $k\ge\bar m+m$. We initialize the induction at $m+\bar m=2$, i.e. $m=\bar m=1$. Then $\xi'=0$, and by an easy induction over $k\ge 2$ the inequation above yields $|S^k(\zeta,\xi)|\le D_2^2q^{-4}k\|\zeta\|\|\xi\|$. Now if the result holds at $m+\bar m-1$, and assuming that $m>1$, we apply the induction hypothesis to $S^{k-1}(\zeta,\xi')$ in the inequation above and get

$$\begin{aligned} |S^k(\zeta,\xi)| &\le |S^{k-1}(\zeta,\xi'')| + D_2^2(2(3q^{-2}(k-1))^{m+\bar m-1} + q^{-2(m+\bar m)})\|\zeta\|\|\xi\| \\ &\le |S^{k-1}(\zeta,\xi'')| + D_2^2(3q^{-2})^{m+\bar m}(k-1)^{m+\bar m-1}\|\zeta\|\|\xi\|. \end{aligned}$$

The required estimate results then from an easy induction over k, using the trivial upper bound $d_k\sqrt{d_m d_{\bar m}}\|\zeta\|\|\xi\|$ to initialize at $k = m+\bar m$. Finally, if $m = 1$ but $\bar m > 1$, we proceed similarly but transform the left-hand side of the scalar product instead of the right-hand side. $\square$

From Lemma 3.8 one can deduce a similar estimate concerning coefficients of corepresentations. Given ONB's $(e^k_i)_i$, $(e^l_a)_a$ of H_k, H_l, we denote by v^k_{ij}, v^l_{ab} the associated coefficients of v^k, v^l.

Lemma 3.9. *There exist numbers C_l, depending only on n and l, such that for all $k \geq l > 0$ we have*

$$\sum_{ij}\Big|\sum_p(\Lambda_h(v^k_{jp}v^l_{ab})|\Lambda_h(v^l_{ab}v^k_{ip}))\Big|^2 \leq C_l k^{2l}q^k.$$

Proof. Let us recall some facts from the Woronowicz-Peter-Weyl theory in the Kac case. Since $v^k\otimes v^l \simeq v^{|k-l|}\oplus\cdots\oplus v^{k+l-2}\oplus v^{k+l}$, the product of coefficients $v^k_{jp}v^l_{ab}$ decomposes as a sum of coefficients of v^r, $r = |k-l|, \ldots, k+l-2, k+l$. More precisely the decomposition corresponds to projecting $e^k_j\otimes e^l_a$ and $e^k_p\otimes e^l_b$ onto the subspace equivalent to H_r. Besides, the scalar product of two coefficients $(\omega_{x,y}\otimes\mathrm{id})(w)$, $(\omega_{x',y'}\otimes\mathrm{id})(w)$ of an irreducible corepresentation w corresponds to the scalar product $(x'\otimes y|x\otimes y')$ up to a factor $(\dim w)^{-1}$, according to (1).

To compute the products $v^k_{jp}v^l_{ab}$, $v^l_{ab}v^k_{ip}$ we need isometric intertwiners, unique up to a phase:

$$\phi^{k,l}_r : H_r\to H_k\otimes H_l,\ \phi^{l,k}_r : H_r\to H_l\otimes H_k.$$

By irreducibility of r, these morphisms are respectively proportional to the maps $(P_k\otimes P_l)T^m_{kl}P_r$ and $(P_l\otimes P_k)T^m_{lk}P_r$, where $r = (k+l)-2m$ and $H_r\subset(H_{k-m}\otimes H_{l-m})\cap(H_{l-m}\otimes H_{k-m})\subset H_1^{\otimes r}$. These two maps have the same norm $N^{k,l}_m$, which is given by the following formula from [Ver07, Lemma 4.8]:

$$\left(N^{k,l}_m\right)^2 = \frac{\mathrm{qdim}\, v^k}{\mathrm{qdim}\, v^{k-m}}\prod_{q=1}^m\Big(1-\frac{\mathrm{qdim}\, v^{k-m}\,\mathrm{qdim}\, v^{l-m-1}}{\mathrm{qdim}\, v^{k-q+1}\,\mathrm{qdim}\, v^{l-q}}\Big)$$

and is known, again by [Ver07, Lemma 4.8], to be controlled as follows:

$$E_1\frac{\mathrm{qdim}\, v^k\,\mathrm{qdim}\, v^l}{\mathrm{qdim}\, v^{k+l-2m}} \leq \left(N^{k,l}_m\right)^4 \leq E_2\frac{\mathrm{qdim}\, v^k\,\mathrm{qdim}\, v^l}{\mathrm{qdim}\, v^{k+l-2m}}$$

with $0 < E_1 < E_2$ independent of k, l, m. In particular, since m takes only a finite number of values when l is fixed, and since $(D_1/D_2)q^t \leq \mathrm{qdim}\, v^k/\,\mathrm{qdim}\, v^{k+t} \leq (D_2/D_1)q^t$ for all k, t, there exist numbers $0 < F_{l,1} < F_{l,2}$ such that $F_{l,1}\leq N^{k,l}_m \leq F_{l,2}$ for all k, l, m.

Now we compute the scalar product $(\Lambda_h(v^k_{jp}v^l_{ab})|\Lambda_h(v^l_{ab}v^k_{ip}))$ as the sum over $r = k+l-2m$ of the following terms:

$$\begin{aligned}(\Lambda_h(v^k_{jp}v^l_{ab})|p_r\Lambda_h(v^l_{ab}v^k_{ip})) &=\\ &= \frac{1}{\dim H_r}(\phi^{l,k*}_r(e^l_a\otimes e^k_i)|\phi^{k,l*}_r(e^k_j\otimes e^l_a))(\phi^{k,l*}_r(e^k_p\otimes e^l_b)|\phi^{l,k*}_r(e^l_b\otimes e^k_p))\\ &= \frac{1}{(N^{k,l}_m)^2\dim H_r}(\phi^{l,k*}_r(e^l_a\otimes e^k_i)|\phi^{k,l*}_r(e^k_j\otimes e^l_a))(T^{m*}_{kl}(e^k_p\otimes e^l_b)|P_rT^{m*}_{lk}(e^l_b\otimes e^k_p))\end{aligned}$$

Denoting C^r_{ij} the sum of these terms over p, we can use Lemma 3.8 to obtain:

$$\begin{aligned}|C^r_{ij}| &\leq \frac{1}{(N^{k,l}_m)^2\dim H_r}|(\phi^{l,k*}_r(e^l_a\otimes e^k_i)|\phi^{k,l*}_r(e^k_j\otimes e^l_a))|\times\Big|\sum_p(T^{m*}_{kl}(e^k_p\otimes e^l_b)|P_rT^{m*}_{lk}(e^l_b\otimes e^k_p))\Big|\\ &\leq \frac{(Ck)^l}{(N^{k,l}_m)^2\dim H_r}|(\phi^{l,k*}_r(e^l_a\otimes e^k_i)|\phi^{k,l*}_r(e^k_j\otimes e^l_a))|.\end{aligned}$$

Now we sum over i and j to get to the stated estimate. Since r takes $l+1$ values we have

$$\begin{aligned}\sum_{ij}\Big|\sum_r C^r_{ij}\Big|^2 &\leq (l+1)\sum_{ijr}|C^r_{ij}|^2\\ &\leq \frac{(l+1)(Ck)^{2l}}{(N^{k,l}_m)^4(\dim H_r)^2}\sum_{ijr}|(\phi^{l,k*}_r(e^l_a\otimes e^k_i)|\phi^{k,l*}_r(e^k_j\otimes e^l_a))|^2.\end{aligned}$$

We recognize on the right-hand side the squares of the Hilbert-Schmidt norms of the maps $(\omega_{e_a^l}\otimes\mathrm{id})(\phi_r^{l,k}\phi_r^{k,l*}\Sigma_{l,k}) \in B(H_k)$, which are dominated by $\dim H_k$ because the corresponding operator norms are less than 1. Since $r \geq k-l$ we have $(\dim H_k)/(\dim H_r)^2 \leq D_2 D_1^{-2} q^{k-2l}$ and we obtain

$$\sum_{ij}\Big|\sum_r C_{ij}^r\Big|^2 \leq \frac{(l+1)^2(Ck)^{2l}}{(N_m^{k,l})^4}\frac{\dim H_k}{(\dim H_r)^2} \leq (C^{2l}D_2D_1^{-2}F_{l,1}^{-4}(l+1)^2q^{-2l})\,k^{2l}q^k.$$

□

Theorem 3.10. *Consider an orthogonal free quantum group $\mathbb{F}O(Q)$ which is unimodular. Then the representation* ad° *of $C^*(\mathbb{F}O(Q))$ factors through λ.*

Proof. By Lemma 3.2 it suffices to prove that the states $\phi = \omega_\xi \circ \mathrm{ad} : x \mapsto (\xi|\,\mathrm{ad}(x)\xi)$ on $A_o(n)$ are weakly associated to the regular representation, for a set of vectors ξ spanning a dense subspace of H°. In particular, we can assume that ξ is a coefficient of some non-trivial irreducible corepresentation, and we will take in fact $\xi = \Lambda_h(v_{ab}^{m*})$.

We already saw that $\|\phi\|_2^2 = \sum \|\phi_k\|_2^2$, where ϕ_k is the restriction of ϕ to the subspace $\mathbb{C}[\mathbb{G}]_k = \mathrm{Span}\{v_{ij}^k\}$. Moreover in the unimodular case $(\sqrt{\dim H_k}v_{ij}^k)_{ij}$ is an ONB of $\mathbb{C}[\mathbb{G}]_k$ with respect to the ℓ^2 norm, and hence $\|\phi_k\|_2^2 = \dim H_k \sum |\phi(v_{ij}^k)|^2$. Using the fact that h is a trace, we have

$$\begin{aligned}\phi(v_{ij}^k) &= \sum_p(\Lambda_h(v_{ab}^{m*})|\Lambda_h(v_{ip}^k v_{ab}^{m*}v_{jp}^{k*})) = \sum_p(\Lambda_h(v_{ip}^{k*}v_{ab}^{m*})|\Lambda_h(v_{ab}^{m*}v_{jp}^{k*}))\\ &= \sum_p(\Lambda_h(v_{jp}^k v_{ab}^m)|\Lambda_h(v_{ab}^m v_{ip}^k)).\end{aligned}$$

Now we can use Lemma 3.9 and we obtain $\|\phi_k\|_2^2 \leq \dim(H_k)C_m k^{2m}q^k \leq D_2 C_m k^{2m}$. In particular it is clear now that $\|e^{-\lambda l} * \phi\|_2^2 \leq D_2 C_m \sum e^{-2\lambda k}k^{2m} < \infty$ for all $\lambda > 0$, and so ϕ is weakly associated to the regular representation by Lemma 3.5. □

3.3. **Inner amenability.** The notion of inner amenability for locally compact quantum groups has been defined in [GNI13]. We will only consider this notion for discrete quantum groups. Recall that, when $\mathbb{G}$ is a discrete quantum group, we denote by $p_0 \in L^\infty(\mathbb{G})$ the minimal central projection corresponding to the trivial corepresentation. Following Effros [Eff75] we define inner amenability as follows.

Definition 3.11. A discrete quantum group $\mathbb{G}$ is called *inner amenable* if there exists a state $m \in L^\infty(\mathbb{G})^*$ such that $m(p_0) = 0$ and

$$m((\mathrm{id}\otimes\omega)\Delta(f)) = m((\omega\otimes\mathrm{id})\Delta(f)) \quad \text{for all} \quad \omega \in L^\infty(\mathbb{G})_*,\ f \in L^\infty(\mathbb{G}).$$

Remark 3.12. Our terminology is different from [GNI13] where they call *strictly inner amenable* a discrete quantum group satisfying Definition 3.11. Note however that, according to [GNI13, Remark 3.1(c)], all discrete quantum groups are inner amenable in the sense of [GNI13, Definition 3.1]. In the classical case, the following theorem is proved in [Eff75]. The quantum case is more involved and we use some techniques from [Tom06].

Theorem 3.13. *Let $\mathbb{G}$ be a unimodular discrete quantum group. The following are equivalent.*

(1) *$\mathbb{G}$ is inner amenable.*
(2) *The trivial representation $\epsilon : C^*(\mathbb{G}) \to \mathbb{C}$ is weakly contained in* ad°.

Moreover, if $\mathbb{G}$ is countable and $\mathcal{L}(\mathbb{G})$ has property Gamma then $\mathbb{G}$ is inner amenable.

Proof. $1 \Rightarrow 2$. The proof of this implication is similar to the one of the implication $1 \Rightarrow 2$ in [Tom06, Theorem 3.8], and moreover we are in the unimodular case. Let us give a sketch of the proof. Putting $A = (\mathrm{id}\otimes\mathrm{ad})(\mathbb{V}) = VW$, it is known that ϵ is weakly contained in ad° if and only if there exists a net of unit vectors $\xi_n \in H^\circ = \xi_0^\perp$ such that $\|A(\eta\otimes\xi_n) - \eta\otimes\xi_n\| \to 0$ for all $\eta \in H$.

Let J_φ and J_h be the modular conjugations of φ and h respectively, with respect to the GNS constructions as in Section 2. It is known that $(J_h\otimes J_\varphi)V(J_h\otimes J_\varphi) = V^*$, and since $U = J_\varphi J_h = J_h J_\varphi$ in the discrete case we also have $(J_h\otimes J_\varphi)W(J_h\otimes J_\varphi) = W^*$. Moreover $W = (1\otimes U)V(1\otimes U) = (J_h\otimes J_h)V^*(J_h\otimes J_h)$, and [KV03, Proposition 2.15] yields the second formula below:

$$\begin{aligned}V^*(\Lambda_\varphi(g)\otimes\Lambda_\varphi(f)) &= (\Lambda_\varphi\otimes\Lambda_\varphi)(\Delta(f)(g\otimes 1)) \text{ and}\\ W(\Lambda_\varphi(g)\otimes\Lambda_\varphi(f)) &= (\Lambda_\varphi\otimes\Lambda_\varphi)(\sigma\Delta(f)(g\otimes 1)),\end{aligned} \tag{3}$$

for all $f,g\in\mathcal{N}_\varphi$, and σ the flip map on $L^\infty(\mathbb{G})\,\bar{\otimes}\,L^\infty(\mathbb{G})$. Note that for the second formula to hold in the non-unimodular case one has to replace Λ_φ by a GNS construction for the right Haar weight φ'.

We use also the identification $C_0(\mathbb{G})=\bigoplus_\alpha B(H_\alpha)$, where α runs over $\operatorname{Irr}\mathbb{G}$, and we recall that $\varphi=\sum_\alpha \dim(H_\alpha)\operatorname{Tr}_\alpha$ in this identification. We denote $(e^\alpha_{ij})_{ij}$ the matrix units of $B(H_\alpha)$ associated to a chosen ONB $(e^\alpha_i)_i$ of H_α. Recall finally the notation $\omega_{\zeta,\xi}\in B(H)_*$ for ζ, $\xi\in H$. In what follows we restrict these linear forms to $L^\infty(\mathbb{G})$ and we convolve them according to Δ. Now we have:

Claim 1. *Let $f=f^*\in\mathcal{N}_\varphi\subset L^\infty(\mathbb{G})$ and put $X=\sigma\Delta(f^2)-\Delta(f^2)=\sum_{\alpha,i,j}e^\alpha_{ij}\otimes X(\alpha)_{ij}$. For all $\alpha\in\operatorname{Irr}(\mathbb{G})$ we have:*

$$\|\omega_{\Lambda_\varphi(f)}*\omega_{J_h\Lambda_\varphi(e^\alpha_{i1}),J_h\Lambda_\varphi(e^\alpha_{j1})}-\omega_{J_h\Lambda_\varphi(e^\alpha_{i1}),J_h\Lambda_\varphi(e^\alpha_{j1})}*\omega_{\Lambda_\varphi(f)}\|\geq\varphi(e^\alpha_{11})\varphi(|X(\alpha)_{ji}|).$$

Proof of Claim 1. Let z be the adjoint of the phase of $X(\alpha)_{ji}$ and write

$$\begin{aligned}\omega_{\Lambda_\varphi(f)}*\omega_{J_h\Lambda_\varphi(e^\alpha_{i1}),J_h\Lambda_\varphi(e^\alpha_{j1})}(z)-\omega_{J_h\Lambda_\varphi(e^\alpha_{i1}),J_h\Lambda_\varphi(e^\alpha_{j1})}*\omega_{\Lambda_\varphi(f)}(z)\\ =\left(\omega_{J_h\Lambda_\varphi(e^\alpha_{i1}),J_h\Lambda_\varphi(e^\alpha_{j1})}\otimes\omega_{\Lambda_\varphi(f)}\right)(\sigma\Delta(z)-\Delta(z)).\end{aligned}$$

Using the formulas $\sigma\Delta(z)=W(1\otimes z)W^*$, $\Delta(z)=V^*(1\otimes z)V$, $(J_h\otimes J_\varphi)V(J_h\otimes J_\varphi)=V^*$, $(J_h\otimes J_\varphi)W(J_h\otimes J_\varphi)=W^*$ and Equations (3), one obtains, as in the proof of [Tom06, Lemma 3.14], the formula:

$$\omega_{\Lambda_\varphi(f)}*\omega_{J_h\Lambda_\varphi(e^\alpha_{i1}),J_h\Lambda_\varphi(e^\alpha_{j1})}(z)-\omega_{J_h\Lambda_\varphi(e^\alpha_{i1}),J_h\Lambda_\varphi(e^\alpha_{j1})}*\omega_{\Lambda_\varphi(f)}(z)=\varphi(e^\alpha_{11})\varphi(|X(\alpha)_{ji}|).$$

Since z is a partial isometry, the result follows. □

Claim 2. *Let $f\in\mathcal{N}_\varphi\cap L^\infty(\mathbb{G})_+$. Then, for all $\alpha\in\operatorname{Irr}(\mathbb{G})$, one has:*

$$\|A(\Lambda_\varphi(e^\alpha_{ij})\otimes\Lambda_\varphi(f))-\Lambda_\varphi(e^\alpha_{ij})\otimes\Lambda_\varphi(f)\|^2\leq(\varphi\otimes\varphi)(|X(\alpha)|).$$

Proof of Claim 2. The proof is the same as [Tom06, Lemma 3.16]. Define $Y=\sigma\Delta(f)-\Delta(f)$. By Equations (3) the left hand side of the inequality of the Claim is

$$\begin{aligned}\|(\Lambda_\varphi\otimes\Lambda_\varphi)(\sigma\Delta(f)(e^\alpha_{ij}\otimes 1))-(\Lambda_\varphi\otimes\Lambda_\varphi)(\Delta(f)(e^\alpha_{ij}\otimes 1))\|^2=\|(\Lambda_\varphi\otimes\Lambda_\varphi)(Y(e^\alpha_{ij}\otimes 1))\|^2\\ =(\varphi\otimes\varphi)((e^\alpha_{ji}\otimes 1)Y^2(e^\alpha_{ij}\otimes 1))=(\varphi\otimes\varphi)(Y^2(e^\alpha_{ii}\otimes 1))=(\varphi\otimes\varphi)(Y(e^\alpha_{ii}\otimes 1)Y).\end{aligned}$$

Let p_α be the minimal central projection in $L^\infty(\mathbb{G})$ corresponding to α. Since $e^\alpha_{ii}\leq p_\alpha$ and by using the Powers-Størmer inequality we get

$$\begin{aligned}(\varphi\otimes\varphi)(Y(e^\alpha_{ii}\otimes 1)Y)\leq(\varphi\otimes\varphi)(Y(p_\alpha\otimes 1)Y)=\|(\Lambda_\varphi\otimes\Lambda_\varphi)(\sigma\Delta(f)(p_\alpha\otimes 1)-\Delta(f)(p_\alpha\otimes 1))\|^2\\ \leq\|\omega_{(\Lambda_\varphi\otimes\Lambda_\varphi)(\sigma\Delta(f)(p_\alpha\otimes 1))}-\omega_{(\Lambda_\varphi\otimes\Lambda_\varphi)(\Delta(f)(p_\alpha\otimes 1))}\|=(\varphi\otimes\varphi)(|X(\alpha)|).\end{aligned}$$

This concludes the proof of Claim 2. □

We can now finish the proof of $1\Rightarrow 2$. Let $m\in L^\infty(\mathbb{G})^*$ be a state such that $m(p_0)=0$ and $m((\mathrm{id}\otimes\omega)\Delta(f))=m((\omega\otimes\mathrm{id})\Delta(f))$ for all $\omega\in L^\infty(\mathbb{G})_*$, $f\in L^\infty(\mathbb{G})$. By the weak* density of the normal states $\omega\in L^\infty(\mathbb{G})_*$ such that $\omega(p_0)=0$ in the set of states $\mu\in L^\infty(\mathbb{G})^*$ such that $\mu(p_0)=0$, there exists a net of normal states $(\omega_n)_n$ such that $\omega_n(p_0)=0$ for all n and $\omega_n*\omega-\omega*\omega_n\to 0$ weak* for all $\omega\in L^\infty(\mathbb{G})_*$.

By the standard convexity argument, we may and will assume that $\|\omega_n*\omega-\omega*\omega_n\|\to 0$. Now, since $L^\infty(\mathbb{G})$ is standardly represented on H, and by a straightforward cut-off argument, we can assume that $\omega_n=\omega_{\Lambda_\varphi(f_n)}$ with $f_n\in L^\infty(\mathbb{G})_+\cap C_c(\mathbb{G})$, $p_0f_n=0$ and $\|f_n\|_2=1$. Applying Claim 1 (instead of [Tom06, Lemma 3.14]), [Tom06, Lemma 3.15] and Claim 2 (instead of [Tom06, Lemma 3.16]), we obtain

$$\|A(\Lambda_\varphi(e^\alpha_{ij})\otimes\Lambda_\varphi(f_n))-\Lambda_\varphi(e^\alpha_{ij})\otimes\Lambda_\varphi(f_n)\|\to 0$$

for all α, i, j. Putting $\xi_n=\Lambda_\varphi(f_n)\in H^\circ$, this easily implies that $\|A(\eta\otimes\xi_n)-\eta\otimes\xi_n\|\to 0$ for all $\eta\in H$, hence ϵ factors through ad°.

$2 \Rightarrow 1$. Take a net of unit vectors $\xi_n \in H^\circ$ such that $\|A(\eta \otimes \xi_n) - \eta \otimes \xi_n\| \to 0$ for all $\eta \in H$, and put $\widetilde{\xi}_n = J_\varphi \xi_n$. Then $\widetilde{\xi}_n$ has norm one, it is orthogonal to ξ_0 for all n and we have for all $\eta \in H$:

$$\begin{aligned}\|W^*(\eta \otimes \widetilde{\xi}_n) - V(\eta \otimes \widetilde{\xi}_n)\| &= \|W(J_h\eta \otimes \xi_n) - V^*(J_h\eta \otimes \xi_n)\| \\ &= \|A(J_h\eta \otimes \xi_n) - J_h\eta \otimes \xi_n\| \to 0.\end{aligned}$$

Let $m \in L^\infty(\mathbb{G})^*$ be a weak* accumulation point of the net of states $(\omega_{\widetilde{\xi}_n})_n$. One has:

$$\begin{aligned}m((\omega_\eta \otimes \mathrm{id})\Delta(f)) &= \lim\langle (1 \otimes f)V(\eta \otimes \widetilde{\xi}_n), V(\eta \otimes \widetilde{\xi}_n)\rangle = \lim\langle (1 \otimes f)W^*(\eta \otimes \widetilde{\xi}_n), W^*(\eta \otimes \widetilde{\xi}_n)\rangle \\ &= m((\omega_\eta \otimes \mathrm{id})\sigma\Delta(f)) = m((\mathrm{id} \otimes \omega_\eta)\Delta(f)) \quad \text{for all} \quad \eta \in H,\ f \in L^\infty(\mathbb{G}).\end{aligned}$$

Moreover, $m(p_0) = \lim\langle p_0\widetilde{\xi}_n, \widetilde{\xi}_n\rangle = 0$.

Finally, suppose that $\mathbb{G}$ is a countable unimodular discrete quantum group such that $\mathcal{L}(\mathbb{G})$ has property Gamma. To show that the co-unit is weakly contained in ad° we follow the proof of Effros [Eff75]. Write $\mathrm{Irr}(\mathbb{G}) = \{u^k \mid k \in \mathbb{N}\}$ where $u^k \in B(H_k) \otimes C^*(\mathbb{G})$. Given $n \in \mathbb{N}$, choose a unitary $u_n \in \mathcal{L}(\mathbb{G})$ such that $h(u_n) = 0$ and

$$\|u_n\lambda(u^k_{ij}) - \lambda(u^k_{ij})u_n\|_2 < \frac{1}{n\max\{\dim(u^l)^2 \mid l \leq n\}}$$

for all $k = 1, \dots, n$ and $1 \leq i, j \leq \dim u^k$. From this inequality, it is easy to check that, for all $k \leq n$ and all $\eta \in H_k$, one has

$$\|(\mathrm{id} \otimes \lambda)(u^k)^*(1 \otimes u_n^*)(\eta \otimes \xi_0) - (1 \otimes u_n^*)(\mathrm{id} \otimes \lambda)(u^k)^*(\eta \otimes \xi_0)\| < \frac{1}{n}\|\eta\|.$$

Recall that ξ_0 is a fixed vector for ad. Hence, for all $\eta \in H_k$, one has $(\mathrm{id} \otimes \mathrm{ad})(u^k)(\eta \otimes \xi_0) = \eta \otimes \xi_0$. Moreover, since the representations λ and ρ commute we find, with $\xi_n = u_n^*\xi_0$, $k \leq n$ and $\eta \in H_k$,

$$\begin{aligned}&\|(\mathrm{id} \otimes \mathrm{ad})(u^k)(\eta \otimes \xi_n) - (\eta \otimes \xi_n)\| = \\ &\qquad = \|(\mathrm{id} \otimes \mathrm{ad})(u^k)(1 \otimes u_n^*)(\mathrm{id} \otimes \mathrm{ad})(u^k)^*(\eta \otimes \xi_0) - (\eta \otimes \xi_n)\| \\ &\qquad = \|(\mathrm{id} \otimes \lambda)(u^k)(1 \otimes u_n^*)(\mathrm{id} \otimes \lambda)(u^k)^*(\eta \otimes \xi_0) - (1 \otimes u_n^*)(\eta \otimes \xi_0)\| \\ &\qquad < \frac{1}{n}\|\eta\|.\end{aligned}$$

Since $\xi_n \in H^\circ = \xi_0^\perp$, it follows that the co-unit is weakly contained in ad°. □

Corollary 3.14 (cf. [VV07])**.** *For $n \geq 3$ the discrete quantum group $\mathbb{F}O_n$ is not inner amenable, and in particular the von Neumann algebra $\mathcal{L}(\mathbb{F}O_n)$ is a full factor.*

Proof. For $n \geq 3$ it is known that $\mathbb{F}O_n$ is not amenable, hence λ does not weakly contain ϵ. On the other hand by Theorem 3.10 the representation ad° is weakly contained in λ. Consequently ϵ is not weakly contained in ad°, hence $\mathbb{F}O_n$ is not inner amenable and $\mathcal{L}(\mathbb{F}O_n)$ is full by Theorem 3.13. □

4. Property (HH) for $\mathcal{L}(\mathbb{F}O_n)$

4.1. **A Deformation.** Recall that $\Delta : A_o(n) \to A_o(n) \otimes A_o(n)$ factors to $\Delta' : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C^*_{\mathrm{red}}(\mathbb{F}O_n) \otimes A_o(n)$ by Fell's absorption principle. On the other hand, for any element $g \in O_n$ we have a character $\omega_g : A_o(n) \to \mathbb{C}$ defined by putting $\omega_g(v_{ij}) = g_{ij}$ and using the universal property of $A_o(n)$. It is easy to check that $(\omega_g \otimes \omega_h)\Delta = \omega_{gh}$ and that $\omega_e = \epsilon$, where e is the unit of O_n and ϵ is the co-unit of $A_o(n)$.

Combining these objects we get $*$-homomorphisms $\alpha_g = (\mathrm{id} \otimes \omega_g) \circ \Delta' : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C^*_{\mathrm{red}}(\mathbb{F}O_n)$ such that $\alpha_g \circ \alpha_k = \alpha_{gk}$, $\alpha_e = \mathrm{id}$ and $h \circ \alpha_g = h$. In particular each α_g is an automorphism of $C^*_{\mathrm{red}}(\mathbb{F}O_n)$ and we have got an action of O_n on $C^*_{\mathrm{red}}(\mathbb{F}O_n)$ by trace preserving automorphisms. Note that there is also an action α' of O_n^{op} on $C^*_{\mathrm{red}}(\mathbb{F}O_n)$ given by $\alpha'_g = (\omega_g \otimes \mathrm{id}) \circ \Delta''$, where $\Delta'' : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to A_o(n) \otimes C^*_{\mathrm{red}}(\mathbb{F}O_n)$ is the homomorphism analogous to Δ'.

Let $n \geq 3$. Denote $M = \mathcal{L}(\mathbb{F}O_n)$ the von Neumann algebra of $\mathbb{F}O_n$ and $\tilde{M} = M \mathbin{\bar{\otimes}} M$. We identify M inside $\tilde{M}$ via the unital normal faithful trace preserving $*$-homomorphism $\iota := \Delta$. Denote by E the trace-preserving conditional expectation from $\tilde{M}$ to $\iota(M)$. We let O_n act on $\tilde{M}$ by putting $A_g = (\alpha_g \otimes \mathrm{id}) : \tilde{M} \to \tilde{M}$ for all $g \in O_n$.

Proposition 4.1 (cf. [Bra12]). *Denote $(U_k)_k$ the dilated Chebyshev polynomials of the second kind and consider, for each $s \in \mathbb{R}$, the densely defined map $T_s : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C^*_{\mathrm{red}}(\mathbb{F}O_n)$ with domain $\mathbb{C}[\mathbb{F}O_n]$ such that $T_s = \frac{U_k(s)}{U_k(n)}\mathrm{id}$ on $\mathbb{C}[\mathbb{F}O_n]_k$ for all $k \in \mathbb{N}$.*

Then for each $g \in O_n$ we have $E \circ A_g \circ \iota = T_s$ where $s = \mathrm{Tr}(g)$. In particular, for such s the map T_s extends to a trace preserving completely positive map on $C^*_{\mathrm{red}}(\mathbb{F}O_n)$.

Proof. For $r \in \mathbb{N}$, denote v^r_{ij} the coefficients of the r^{th} irreducible corepresentation v^r of $\mathbb{F}O_n$, with respect to a given ONB of the corresponding space. Using the orthogonality relations (1), it is easy to check that $E(v^r_{ij} \otimes v^s_{kl}) = \delta_{rs}\delta_{jk}v^r_{il}/U_r(n)$, where $U_r(n) = \dim v^r$.

On the other hand, denoting u^r the image of v^r as a representation of O_n, we have by definition $\omega_g(v^r_{ij}) = u^r_{ij}(g)$. The character $\chi_r = \mathrm{Tr} \circ u^r = \sum_k u^r_{kk}$ of u^r is given by $\chi_r(g) = U_r(\mathrm{Tr}\, g)$: indeed by the fusion rule $u^1 \otimes u^r \simeq u^{r-1} \oplus u^{r+1}$ these characters satisfy the recursion relation of the Chebyshev polynomials U_r, and we have $\chi_1(g) = \mathrm{Tr}(g)$. Now it suffices to compute:

$$E \circ A_g \circ \Delta(v^r_{ij}) = \sum_{k,l} E(v^r_{ik}u^r_{kl}(g) \otimes v^r_{lj}) = \sum_k v^r_{ij}\frac{u^r_{kk}(g)}{U_r(n)} = v^r_{ij}\frac{U_r(\mathrm{Tr}\, g)}{U_r(n)},$$

and we recognize the definition of T_s for $s = \mathrm{Tr}(g)$. □

Example 4.2. Define $g_t = \mathrm{id}_{n-2} \oplus R_t \in O_n$ where

$$R_t = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Denoting $A_t = A_{g_t}$, we get in this way a 1-parameter group of automorphisms of $\tilde{M}$ such that $E \circ A_t \circ \iota = T_s$ with $s = n - 2 + 2\cos t \in [n-4, n]$.

Fix $0 < t_0 < \frac{\pi}{3}$. For all $0 < t < t_0$ one has $2 < 1 + 2\cos(t_0) < \mathrm{Tr}(g_t) < n$. By [Bra12, Proposition 4.4.1], there exists a constant $C > 0$ such that

$$\frac{U_k(\mathrm{Tr}(g_t))}{U_k(n)} \le C\left(\frac{\mathrm{Tr}(g_t)}{n}\right)^k$$

for all $0 < t < t_0$ and all $k \in \mathbb{N}$. Hence, for $0 < t < t_0$, the map $T_s = E \circ A_t \circ \iota$ is L^2-compact, and this is how Brannan obtains the Haagerup approximation property in [Bra12].

Finally, consider $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $k = \mathrm{id}_{n-2} \oplus S \in O_n$ and $B = \alpha_k \otimes \alpha'_k \in \mathrm{Aut}(\tilde{M})$. We clearly have $B^2 = \mathrm{id}$, $BA_t = A_{-t}B$, and one can check on the generators v_{ij} that $B \circ \Delta = \Delta$ so that B restricts to the identity on $\iota(M)$. In other words, our deformation can by reversed by an inner involution of $\mathrm{Aut}(\tilde{M})$ respecting the bimodule structure.

Remark 4.3. The constructions and results of this Section 4 remain valid for the other unimodular orthogonal free quantum groups $\mathbb{F}O(Q)$. Up to isomorphism, the only missing cases are $\mathbb{F}O(Q_{2n})$ with $Q_{2n} = \mathrm{diag}(Q_2, \ldots, Q_2)$ and $Q_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. But these discrete quantum groups all admit the dual of $SU(2)$ as a commutative quotient, and one can build a deformation using the same matrices R_t as in Example 4.2.

On the other hand, in the non unimodular case there is no trace preserving conditional expectation $E : M \bar{\otimes} M \to \Delta(M)$, so that the arguments of Proposition 4.1 do not apply anymore. Recall however that Haagerup's Property still holds in that case, as shown in [DCFY14].

Let us give some applications of the preceding construction. It is known that Haagerup's property is equivalent to the existence of a "proper conditionally negative type function" and/or of a proper cocycle in some representation. Denoting τ_s the linear form on $\mathbb{C}[\mathbb{F}O_n]$ given by $\tau_s(v^r_{ij}) = \delta_{ij}U_r(s)/U_r(n)$, it follows from the proposition above and Lemma 3.1 that τ_s extends to a state of $C^*(\mathbb{F}O_n)$ for $s \le n$. Differentiating at $s = n$ we obtain a conditionally negative form $\psi : \mathbb{C}[\mathbb{F}O_n] \to \mathbb{C}$ given by $\psi(v^r_{ij}) = \delta_{ij}U'_r(n)/U_r(n)$ — this was also observed in [CFK14, Corollary 10.3], where all ad-invariant conditionally negative forms on $\mathbb{C}[\mathbb{F}O_n]$ are classified. We have indeed, for all $x \in \mathrm{Ker}\,\epsilon \subset \mathbb{C}[\mathbb{F}O_n]$:

$$\psi(x^*x) = \lim_{s \to n} \frac{\tau_s(x^*x) - \epsilon(x^*x)}{s - n} = \lim_{s \to n} \frac{\tau_s(x^*x)}{s - n} \le 0.$$

Moreover the following lemma (cf. also [CFK14]) shows that ψ is indeed proper, and more precisely that the associated function $\Psi = (\mathrm{id} \otimes \psi)(V)$ behaves like the "naive" length function $L = \sum r p_r$ at infinity:

Lemma 4.4. *We have* $\frac{U_r'(n)}{U_r(n)} = \frac{r}{\sqrt{n^2-4}} + O(1)$ *as* $r \to \infty$.

Proof. We have the well-known formula $U_r(n) = (q^{r+1} - q^{-r-1})/(q - q^{-1})$, where $q = \frac{1}{2}(n - \sqrt{n^2-4})$. Differentiating first with respect to q, we get

$$\frac{U_r'(n)}{U_r(n)} = \frac{r(1-q^{-2}) - 2q^{-2} + 2q^{-2r-2} + rq^{-2r-2}(1-q^{-2})}{(q-q^{-1})(1-q^{-2r-2})} \times \frac{\mathrm{d}q}{\mathrm{d}n}.$$

Noticing that $\mathrm{d}q/\mathrm{d}n = -q/\sqrt{n^2-4}$, we obtain finally

$$\frac{U_r'(n)}{U_r(n)} = \frac{1}{\sqrt{n^2-4}} \left(r + \frac{2}{1-q^{-2}} + o(1) \right).$$

□

4.2. **Property (HH).** It is then natural to ask also for the explicit construction of a proper cocycle establishing Haagerup's property. Notice that by [Ver12] such a cocycle cannot live in the regular representation or a finite multiple of it. Now an explicit cocycle can easily be obtained by differentiating the 1-parameter group of automorphisms above. More precisely, for any $X \in \mathfrak{o}_n$, $X \neq 0$, we define $\delta_X = d_X A_g \circ \iota : \mathbb{C}[\mathbb{F}O_n] \to \tilde{M}$. Denoting $u^r \in B(H_r) \otimes C(O_n)$ the image of v^r, we have explicitly

$$\delta_X(v_{ij}^r) = \sum_{k,l} v_{ik}^r d_X u_{kl}^r \otimes v_{lj}^r.$$

Here $d_X u_{kl}^r$ is the differential of $u_{kl}^r : O_n \to \mathbb{C}$, evaluated on the tangent vector X at e. Since $A_g \circ \iota$ is a $*$-homomorphism, we get indeed derivations with respect to the $\mathbb{C}[\mathbb{F}O_n]$-bimodule structure coming from the embedding $\iota = \Delta$, and in fact we get a Lie algebra map $(X \mapsto \delta_X)$ from $\mathfrak{o}_n$ to the space of derivations $\mathrm{Der}(\mathbb{C}[\mathbb{F}O_n], \tilde{M})$.

We moreover denote $f_X = (\mathrm{id} \otimes \delta_X)(V)$, which is an unbounded multiplier of the Hilbert $C_0(\mathbb{F}O_n)$-module $C_0(\mathbb{F}O_n) \otimes \tilde{M}$. Then $f_X^* f_X$ is an unbounded multiplier of $C_0(\mathbb{F}O_n)$, the "conditionally negative" type function associated to δ_X. The following Lemma shows in particular that $f_X^* f_X$ corresponds to the conditionally negative form ψ obtained by differentiating the family of states τ_s.

Lemma 4.5. *The derivation δ_X takes its values in the orthogonal complement $\tilde{M}^\circ$ of $\iota(M)$ with respect to $h \otimes h$. Moreover we have*

$$f_X^* f_X = \mathrm{Tr}(X^* X) \times (\mathrm{id} \otimes \psi)(V).$$

In particular for any $X \in \mathfrak{o}_n$, $X \neq 0$, the derivation δ_X is proper in the sense that $p_r f_X^* f_X \geq c_r p_r$ for any r, with $c_r \to \infty$.

Proof. The beginning of the proof is quite general and probably well-known to experts.

We choose a 1-parameter subgroup $(g_t)_t$ of O_n such that $g_0' = X$, we put $A_t = A_{g_t}$, $s(t) = \mathrm{Tr}(g_t)$ and we differentiate the identity $E \circ A_t \circ \iota = T_{s(t)}$ between linear maps on $\mathbb{C}[\mathbb{F}O_n]$ from Proposition 4.1. We obtain $\mathrm{Tr}(g_t') \times T_{s(t)}' = E \circ A_t' \circ \iota$, where T_s' is the derivative with respect to s, and the other derivatives are relative to t. In particular for $t = 0$ this yields $E \circ \delta_X = E \circ d_X A_g \circ \iota = \mathrm{Tr}(X) \times T_n'$. Since $\mathrm{Tr}\, X = 0$ for $X \in \mathfrak{o}_n$ we obtain $E \circ \delta_X = 0$, hence $\delta_X(\mathbb{C}[\mathbb{F}O_n]) \subset \tilde{M}^\circ$.

Now we differentiate once more at $t = 0$, obtaining $\mathrm{Tr}(g_0'') \times T_n' = E \circ A_0'' \circ \iota$ as linear maps on $\mathbb{C}[\mathbb{F}O_n]$. Since $(g_t)_t$ is a 1-parameter group and $X^* + X = 0$, we have $g_0'' = g_0'^2 = -X^* X$ in $M_n(\mathbb{R})$. Similarly, since $(A_t)_t$ is a 1-parameter group of trace-preserving automorphisms we have $A_0'' = A_0'^2 = -A_0'^* A_0'$, where $A_0'^*$ denotes the adjoint of A_0' with respect to the hilbertian structure of $\tilde{M}$. Moreover we have $E = \iota^*$ at the hilbertian level, so that $\delta_X^* \delta_X = (A_0' \iota)^* (A_0' \iota) = -E \circ A_0'' \circ \iota = \mathrm{Tr}(X^* X) \times T_n'$.

The last identity can also be written $(\delta_X(a) | \delta_X(b)) = \mathrm{Tr}(X^* X) h(a^* T_n'(b))$ for all $a, b \in \mathbb{C}[\mathbb{F}O_n]$. As a result we obtain

$$f_X^* f_X = (\mathrm{id} \otimes \delta_X)(V)^* (\mathrm{id} \otimes \delta_X)(V) = \mathrm{Tr}(X^* X) \times (\mathrm{id} \otimes h)(V^* (\mathrm{id} \otimes T_n')(V))$$

as unbounded multipliers — in other words, the identity above makes sense in the f.-d. algebra $p_rC_0(\mathbb{F}O_n) \simeq B(H_r)$ for any r. But by definition of ψ we have $T'_n = (\mathrm{id} \otimes \psi) \circ \Delta$, and using the identity $(\mathrm{id} \otimes \Delta)(V) = V_{12}V_{13}$ we can write

$$f_X^* f_X = \mathrm{Tr}(X^*X) \times (\mathrm{id} \otimes h \otimes \psi)(V_{12}^* V_{12} V_{13}) = \mathrm{Tr}(X^*X) \times (\mathrm{id} \otimes \psi)(V).$$

Finally we have $(p_r \otimes \psi)(V) = c_r p_r$ with $c_r = U'_r(n)/U_r(n)$ by the computation of ψ before Lemma 4.4, and the properness results from that lemma. □

Recall the construction of the bimodule K_π associated to a $*$-representation $\pi : C^*(\mathbb{G}) \to B(H_\pi)$. We put $K_\pi = H \otimes H_\pi$ where H is the GNS space of the Haar state h. The space K_π is endowed with two representations, $\tilde{\pi} = (\lambda \otimes \pi)\Delta' : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to B(K_\pi)$ and $\tilde{\rho} = \lambda^{\mathrm{op}} \otimes 1 : C^*_{\mathrm{red}}(\mathbb{F}O_n)^{\mathrm{op}} \to B(K_\pi)$, where we put $\lambda^{\mathrm{op}}(x)\Lambda_h(y) = \Lambda_h(yx)$.

Lemma 4.6. *The space $L^2(\tilde{M})$, viewed as an M-M-bimodule via the left and right actions of $\iota(M) = \Delta(M)$, is isomorphic to the M-M-bimodule K_π naturally associated with the adjoint representation $\pi = \mathrm{ad}$ of $\mathbb{G}$. The trivial part $\mathbb{C}\xi_0$ of the adjoint representation corresponds to the trivial sub-bimodule $\iota(M) \subset \tilde{M}$.*

Proof. We consider the unitary $V(1 \otimes U) : L^2(\tilde{M}) = H \otimes H \to H \otimes H$. Recall that, in the Kac case, $V(\Lambda_h \otimes \Lambda_h)(x \otimes y) = (\Lambda_h \otimes \Lambda_h)(\Delta(x)(1 \otimes y))$ and $U\Lambda_h(y) = \Lambda_h(S(y))$ for x, $y \in \mathbb{C}[\mathbb{G}]$, so that

$$V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)(x \otimes y) = (\Lambda_h \otimes \Lambda_h)(\Delta(x)(1 \otimes S(y))).$$

Recall that $\mathrm{ad}(z)\Lambda_h(y) = \Lambda_h(z_{(1)}yS(z_{(2)}))$. Then the left module structure reads:

$$\begin{aligned} V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)(\Delta(z)(x \otimes y)) &= (\Lambda_h \otimes \Lambda_h)((z_{(1)} \otimes z_{(2)})\Delta(x)(1 \otimes S(y)S(z_{(3)}))) \\ &= (\lambda \otimes \mathrm{ad})\Delta(z)V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)(x \otimes y). \end{aligned}$$

On the other hand we compute:

$$\begin{aligned} V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)((x \otimes y)\Delta(z)) &= (\Lambda_h \otimes \Lambda_h)(\Delta(x)(z_{(1)} \otimes z_{(2)})(1 \otimes S(z_{(3)})S(y))) \\ &= (\Lambda_h \otimes \Lambda_h)(\Delta(x)(z \otimes S(y))) \\ &= (\lambda^{\mathrm{op}}(z) \otimes \mathrm{id})V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)(x \otimes y). \end{aligned}$$

This shows that $V(1 \otimes U)$ yields an isomorphism $L^2(\tilde{M}) \simeq K_{\mathrm{ad}}$ as M-M-bimodules. Moreover, putting $x = y = 1$ we see that $V(1 \otimes U)(\Lambda_h \otimes \Lambda_h)\iota(M) = \Lambda_h(M) \otimes \xi_0$. □

Recall now from [OP10b] that a discrete group G has Property strong (HH) if it admits a proper cocycle with values in a representation weakly contained in the regular representation. These notions make sense in the quantum case, and combining the lemma above with Theorem 3.10 and Lemma 4.5 we obtain:

Corollary 4.7. *The discrete quantum groups $\mathbb{F}O_n$ have Property strong (HH).*

Remark 4.8. Lemma 4.6 works also at the level of the derivations δ_X and yields a formula for the associated cocycles with values in the adjoint representation $\pi = \mathrm{ad}$. More precisely, a simple computation shows that $V(1 \otimes U)\delta_X$ is of the form $(\Lambda_h \otimes c_X)\Delta$, with $c_X : \mathbb{C}[\mathbb{F}O_n] \to H$ given by $c_X = \Lambda_h m(d_X\alpha_g \otimes S)\Delta$, or in other terms:

$$c_X(v^r_{ij}) = \sum (d_X u^r_{kl}) \times \Lambda_h(v^r_{ik} v^{r*}_{jl}).$$

The fact that $V(1 \otimes U)\delta_X$ is a derivation implies that c_X is a cocycle, i.e. $c_X(xy) = \pi(x)c_X(y) + c_X(x)\epsilon(y)$ for x, $y \in \mathbb{C}[\mathbb{F}O_n]$. Denoting $g_X = (\mathrm{id} \otimes c_X)(V)$ the unbounded multiplier of the Hilbert module $C_0(\mathbb{F}O_n) \otimes H$ associated with c_X, we have $f_X^* f_X = g_X^* g_X$. Hence Lemma 4.5 also shows that the cocycle c_X is proper.

For the generators v_{ij} of $\mathbb{C}[\mathbb{F}O_n]$ we have $c_X(v_{ij}) = \sum X_{kl}\Lambda_h(v_{ik}v_{jl})$. For a particular choice of X one gets e.g. $c(v_{ij}) = \Lambda_h(v_{i1}v_{j2} - v_{i2}v_{j1})$, and the other values of c can be deduced recursively using the cocycle relation.

Remark 4.9. In the classical case, Property (HH) clearly implies the Property $\mathcal{QH}_{\mathrm{reg}}$ of [CS13], which is in turn equivalent, for exact groups, to bi-exactness and to Property $(\mathrm{AO})^+$, see [PV14, Proposition 2.7] and [BO08, Chapter 15]. In the quantum case, it is proved in [Iso13] that the existence of a "sufficiently nice" boundary action implies Property $(\mathrm{AO})^+$, but the connection with quasi-cocycles remains to be explored. Note that Property $(\mathrm{AO})^+$ for $\mathbb{F}O_n$ is established in [Ver05].

4.3. **Strong solidity.** We first recall the following result due to Ozawa, Popa [OP10a] and Sinclair [Sin11].

Theorem 4.10. *Let M be a tracial von Neumann algebra which is weakly amenable and admits the following deformation property: there exists a tracial von Neumann algebra $\tilde{M}$, a trace preserving inclusion $M \subset \tilde{M}$ and a one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of trace-preserving automorphisms of $\tilde{M}$ such that*

- $\lim_{t\to 0} \|\alpha_t(x) - x\|_2 = 0$ *for all* $x \in M$.
- ${}_M\Big(L^2(\tilde{M}) \ominus L^2(M)\Big)_M \prec {}_M\big(L^2(M) \otimes L^2(M)\big)_M$.
- $E_M \circ \alpha_t$ *is compact on* $L^2(M)$ *for all* t *small enough.*

Then for any diffuse amenable von Neumann subalgebra $P \subset M$ we have that $\mathcal{N}_M(P)''$ is amenable — in other words M is strongly solid.

Proof. If $K = L^2(\tilde{M}) \ominus L^2(M)$ is (strongly) contained in an amplification of the coarse bimodule, this is a particular case of [OP10a, Theorem 4.9], with $k = 1$, $Q = Q_1 = \mathbb{C}$ and $\mathcal{G} = \mathcal{N}_M(P)$. Indeed in that case "compactness over Q" for $E_M \circ \alpha_t$ means that its extension to $L^2(M)$ is compact, $L^2\langle M, e_{Q_1}\rangle$ is the coarse bimodule $L^2(M) \otimes L^2(M)$, so that the conclusions of [OP10a, Proposition 4.8] hold also in our case. On the other hand $P \not\preceq_M Q$ means that P is diffuse. Moreover if M is weakly amenable and P is amenable, then the action of $\mathcal{G} = \mathcal{N}_M(P)$ on P is weakly compact by [Oza12, Theorem B]. Hence the hypotheses of [OP10a, Theorem 4.9] are satisfied, and we can conclude that $N = \mathcal{N}_M(P)''$ is amenable relative to Q inside M, which just means that N is amenable in our case.

Note that the proof of [OP10a, Theorem 4.9] shows the existence, for all non-zero central projection $p \in M$, all $F \subset \mathcal{N}_M(P)$ finite and all $\epsilon > 0$ the existence of a vector $\zeta \in K \otimes L^2(M)$ such that $\|p\zeta\|_2 \geq \|p\|_2/8$, $\|[u \otimes \bar{u}, \zeta]\|_2 < \epsilon/2$ for all $u \in F$ and $\|x\zeta\|_2 \leq \|x\|_2$ for all $x \in M$. In particular one can then show as in [Sin11, Theorem 3.1] that K is left amenable over $\mathcal{N}_M(P)'' \subset M$. Now, if K is only weakly contained in the coarse bimodule, [Sin11, Theorem 3.2] still allows to conclude that $\mathcal{N}_M(P)''$ is amenable. ☐

Now by Proposition 4.1, Lemma 4.6 and Theorem 3.10 one can apply the preceding Theorem to the deformation of $\mathcal{L}(\mathbb{F}O_n)$ presented in Section 4.1. Moreover the weak amenability assumption is satisfied by [Fre13, Theorem 6.1] — in fact Freslon also proves that the Cowling-Haagerup constant of $\mathbb{F}O_n$ equals 1, so that the invocation of [Oza12] in the proof of Theorem 4.10 is not necessary in this case. Finally we obtain:

Theorem 4.11. *For all $n \geq 3$, the II_1 factor $\mathcal{L}(\mathbb{F}O_n)$ is strongly solid.*

Note that this result was already proved in [Iso15] using the more recent approach from [PV14] to strong solidity, and the strong Akemann-Ostrand Property, established in [Ver05, Theorem 8.3] for free quantum groups. Finally, it seems likely that strong solidity can also be deduced from Property strong (HH) as in [OP10b, Corollary B], by adapting Peterson's techniques to the proper derivation $\delta_X : \mathbb{C}[\mathbb{F}O_n] \to \tilde{M}$ in the quantum setting. Notice in particular that $\Delta_X = \delta_X^* \delta_X$ is a "central multiplier" of $\mathbb{C}[\mathbb{F}O_n]$ by the proof of Lemma 4.5, so that the passage from the classical to the quantum setting is probably straightforward.

References

[Ban96] T. Banica. Théorie des représentations du groupe quantique compact libre O(n). *C. R. Acad. Sci. Paris Sér. I Math.*, 322(3):241–244, 1996.

[BO08] N. P. Brown and N. Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[Bra12] M. Brannan. Approximation properties for free orthogonal and free unitary quantum groups. *J. Reine Angew. Math.*, 672:223–251, 2012.

[BS93] S. Baaj and G. Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres. *Ann. Sci. École Norm. Sup. (4)*, 26(4):425–488, 1993.

[CFK14] F. Cipriani, U. Franz, and A. Kula. Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory. *J. Funct. Anal.*, 266(5):2789–2844, 2014.

[CS13] I. Chifan and T. Sinclair. On the structural theory of II_1 factors of negatively curved groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(1):1–33 (2013), 2013.

[Daw12] M. Daws. Completely positive multipliers of quantum groups. *Internat. J. Math.*, 23(12):1250132, 2012.

[DCFY14] K. De Commer, A. Freslon, and M. Yamashita. CCAP for Universal Discrete Quantum Groups. *Comm. Math. Phys.*, 331(2):677–701, 2014.

[DFSW13] M. Daws, F. Fima, A. Skalski, and S. White. The Haagerup property for locally compact quantum groups. Preprint, arXiv:1303.3261, to appear in *J. Reine Angew. Math.*, 2013.

[Eff75] E. G. Effros. Property Γ and inner amenability. *Proc. Amer. Math. Soc.*, 47:483–486, 1975.

[Fel63] J.M.G Fell. Weak containment and Kronecker products of group representations. *Pacific J. Math.*, 13(2):503–510, 1963.

[Fre13] A. Freslon. Examples of weakly amenable discrete quantum groups. *J. Funct. Anal.*, 265(9):2164–2187, 2013.

[GNI13] M.R. Ghanei and R. Nasr-Isfahani. Inner amenability of locally compact quantum groups. *Internat. J.. Math.*, 24(7), 2013.

[Haa79] U. Haagerup. An example of a nonnuclear C^*-algebra, which has the metric approximation property. *Invent. Math.*, 50(3):279–293, 1978/79.

[Iso13] Y. Isono. On bi-exactness of discrete quantum groups. Preprint, arXiv:1308.5103, to appear in *Int. Math. Res. Not. IMRN*, 2013.

[Iso15] Yusuke Isono. Examples of factors which have no Cartan subalgebras. *Trans. Amer. Math. Soc.*, 367(11):7917–7937, 2015.

[KV00] J. Kustermans and S. Vaes. Locally compact quantum groups. *Ann. Sci. École Norm. Sup. (4)*, 33(6):837–934, 2000.

[KV03] J. Kustermans and S. Vaes. Locally compact quantum groups in the von Neumann algebraic setting. *Math. Scand.*, 92(1):68–92, 2003.

[MvN43] J.J. Murray and J. von Neumann. On rings of operators I.V. *Ann. of Math.*, 44:716–808, 1943.

[OP10a] N. Ozawa and S. Popa. On a class of II_1 factors with at most one Cartan subalgebra. *Ann. of Math. (2)*, 172(1):713–749, 2010.

[OP10b] N. Ozawa and S. Popa. On a class of II_1 factors with at most one Cartan subalgebra, II. *Amer. J. Math.*, 132(3):841–866, 2010.

[Oza12] N. Ozawa. Examples of groups which are not weakly amenable. *Kyoto J. Math.*, 52(2):333–344, 2012.

[PV14] S. Popa and S. Vaes. Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups. *J. Reine Angew. Math.*, 694:215–239, 2014.

[PW90] P. Podleś and S. L. Woronowicz. Quantum deformation of Lorentz group. *Comm. Math. Phys.*, 130(2):381–431, 1990.

[Sin11] T. Sinclair. Strong solidity of group factors from lattices in $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$. *J. Funct. Anal.*, 260(11):3209–3221, 2011.

[Tom06] R. Tomatsu. Amenable discrete quantum groups. *J. Math. Soc. Japan*, 58(4):949–964, 2006.

[VD94] A. Van Daele. Multiplier Hopf algebras. *Trans. Amer. Math. Soc.*, 342(2):917–932, 1994.

[Ver05] R. Vergnioux. Orientation of quantum Cayley trees and applications. *J. Reine Angew. Math.*, 580:101–138, 2005.

[Ver07] R. Vergnioux. The property of rapid decay for discrete quantum groups. *J. Operator Theory*, 57(2):303–324, 2007.

[Ver12] R. Vergnioux. Paths in quantum Cayley trees and L^2-cohomology. *Adv. Math.*, 229(5):2686–2711, 2012.

[VV07] S. Vaes and R. Vergnioux. The boundary of universal discrete quantum groups, exactness, and factoriality. *Duke Math. J.*, 140(1):35–84, 2007.

[Wan95] S. Wang. Free products of compact quantum groups. *Comm. Math. Phys.*, 167(3):671–692, 1995.

[Wor87] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987.

[Wor98] S. L. Woronowicz. Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, pages 845–884. North-Holland, Amsterdam, 1998.

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