Discrete Quantum Groups

[Woronowicz, van Daele]

Setting

Two dual Hopf C^*-algebras (S, δ), $(\widehat{S}_{\text{red}}, \widehat{\delta})$

Densely defined co-units and antipodes

$$\begin{array}{ccccc} [c_0(\Gamma)] & S & \xrightarrow{\mathcal{F}} & \widehat{S}_{\text{red}} & [C^*_{\text{red}}(\Gamma)] \\ & \text{GNS} \downarrow h_R & & \widehat{h} \downarrow \text{GNS} & \\ & H & \xrightarrow{\simeq} & H & [\ell^2(\Gamma)] \end{array}$$

Structure of (S, δ)

$S = \bigoplus_\alpha B(H_\alpha) =: \bigoplus_\alpha p_\alpha S$, $\quad \mathcal{S} = \text{alg-}\bigoplus p_\alpha S$

Index set $\mathcal{I} := \{\alpha\}$, fd representations of S

Haar weights h_L, h_R defined on $\mathcal{S}$

$\forall\ a \in p_\alpha S$ $h_L(a) = m_\alpha \text{Tr } (F_\alpha^{-1} a)$ and

$h_R(a) = m_\alpha \text{Tr } (F_\alpha a)$

with $F_\alpha \in B(H_\alpha)_+$ st $\text{Tr } F_\alpha = \text{Tr } F_\alpha^{-1} =: m_\alpha$

The Property of Rapid Decay

[cf Haagerup, Jolissaint]

Length on (S, δ)

It is an unbounded multiplier $L \in S^\eta$ st

$$L \geq 0,\ \varepsilon(L) = 0,\ \kappa(L) = L$$

$$\delta(L) \leq 1 \otimes L + L \otimes 1.$$

$p_n \in M(S)$: spectral proj of L for $[n, n+1[$.

Sobolev norms

For $a \in \mathcal{S}$ we put $||a||_2 := h_R(a^* a)^{1/2}$ and

$$||a||_{2,s} := ||(1+L)^s a||_2.$$

Let $H_L^s \subset H$ be the associated completions.

Definition / Proposition

Let L be a central length on (S, δ).

We say that (S, δ, L) has Property RD if

$$\exists\ C, s \in \mathbb{R}_+ \quad \forall a \in \mathcal{S} \quad ||\mathcal{F}(a)|| \leq C ||a||_{2,s}$$

$$\iff \quad H_L^\infty := \textstyle\bigcap_{s \geq 0} H_L^s \subset \widehat{S}_{\text{red}} \text{ inside } H$$

$$\iff \quad \exists\ P \in \mathbb{R}[X] \quad \forall k, l, n\ \forall a \in p_n \mathcal{S}$$

$$||p_l \mathcal{F}(a) p_k|| \leq P(n) ||a||_2.$$

The amenable case

Growth

We say that (S, δ, L) has polynomial growth if

$$\exists\ P \in \mathbb{R}[X] \quad \forall n \in \mathbb{N} \quad h_R(p_n) \leq P(n)$$

Proposition

(S, δ, L) amenable + RD $\Rightarrow$ polynomial growth

(S, δ, L) polynomial growth $\Rightarrow$ Prop RD

Example

Duals of connected compact Lie groups have Property RD. In fact in this case

$$H_L^\infty \subset \widehat{S}_{\text{red}} \iff C^\infty(G) \subset C(G).$$

Proposition

(S, δ) not unimodular $\Rightarrow$ not polynomial growth

The dual of $SU_q(N)$ does not have RD.

A necessary condition

If (S, δ, L) has RD there exists $P \in \mathbb{R}[X]$ st for any inclusion $\gamma \subset \beta \otimes \alpha$ without multiplicity

$$\forall a \in p_\alpha S \quad ||p_\gamma \mathcal{F}(a) p_\beta|| \leq P(|\alpha|) ||a||_2.$$

Proposition

This condition is equivalent to requiring, for any $a \in L(H_\alpha)$, $b \in L(H_\beta)$:

$$||\delta(p_\gamma)(b \otimes a)\delta(p_\gamma)||_2 \leq \sqrt{\frac{m_\gamma}{m_\beta m_\alpha}}\ P(|\alpha|)\ ||b \otimes a||_2$$

NB : $\delta(p_\gamma)$ is the projection onto $H_\gamma \subsetneqq H_\beta \otimes H_\alpha$.

Corollary

Non-unimodular DQG cannot have RD.
(Consider $\varepsilon \subset \bar{\alpha} \otimes \alpha$.)

Free quantum groups

[Wang, van Daele, Banica]

Recall that

$C^*(F_N) = C^*(1, u_i \mid \forall i \quad u_i u_i^* = u_i^* u_i = 1)$

One puts for $Q \in GL(N, \mathbb{C})$

$A_u(Q) = C^*(1, u_{ij} \mid U$ and $Q\bar{U}Q^{-1}$unitary)

$A_o(Q) = C^*(1, u_{ij} \mid U = Q\bar{U}Q^{-1}$unitary)

These are compact quantum groups whose duals are called the free quantum groups.

In the orthogonal case (with $\bar{Q}Q \in \mathbb{C}$id)

$\mathcal{I} \simeq \mathbb{N}$ with $U \simeq \alpha_1$, $\bar{\alpha}_k \simeq \alpha_k$ and

$\alpha_k \otimes \alpha_l \simeq \alpha_{|k-l|} \oplus \alpha_{|k-l|+2} \oplus \cdots \oplus \alpha_{k+l}$.

In the unitary case

$\mathcal{I} \simeq \{$words on $u, \bar{u}\}$ with $U \simeq u$, $\overline{wu} \simeq \bar{u}\bar{w}$,

$wu \otimes uw' \simeq wuuw'$, $wu \otimes \bar{u}w' \simeq wu\bar{u}w' \oplus w \otimes w'$.

(S, δ) unimodular $\Longleftrightarrow Q \in \mathbb{C}U(N)$.

If $N \geq 3$, m_α grows exponentially with $|\alpha|$.

Free quantum groups

Haagerup proved that F_N has Property RD.

Proposition
For the duals of $A_o(Q)$ and $A_u(Q)$, the necessary condition of Slide 5 is sufficient.

Theorem
If $Q \in \mathbb{C}U(N)$, the duals of $A_o(Q)$ and $A_u(Q)$ have Property RD.

(requires a finer description of the representation theory than just the semi-ring structure)

Application to K-theory

[cf Jolissaint, Lafforgue]

L word length on finitely generated (S, δ).
D : Dom$D \subset B(H) \to B(H)$ derivation by L.

Proposition
We have $\widehat{S}_{\text{red}} \cap \text{Dom} D^k \subset H_L^k$.
If (S, δ, L) has RD we have $H_L^{k+s} \subset \widehat{S}_{\text{red}} \cap \text{Dom} D^k$.

Corollary
H_L^∞ is a dense subalgebra of $\widehat{S}_{\text{red}}$, stable under holomorphic functional calculus.
The inclusion induces an isomorphism

$$K_*(H_L^\infty) \xrightarrow{\sim} K_*(\widehat{S}_{\text{red}}).$$

Similarly, for s big enough, H_L^s is a dense subalgebra of $\widehat{S}_{\text{red}}$ which is stable under holomorphic functional calculus.

Another Application

$U \in M_N \otimes \widehat{S}_{\mathrm{red}}$ fundamental corepr. of $A_o(I_N)$.

Consider the operator

$$\begin{aligned} \Psi : \widehat{S}_{\mathrm{red}} &\to \widehat{S}_{\mathrm{red}} \\ x &\mapsto (\mathrm{Tr} \otimes \mathrm{id})(U^*(1 \otimes x)U)/N \end{aligned}$$

Proposition

There exists $\lambda < 1$ such that

$$\forall x \in \widehat{S}_{\mathrm{red}} \quad \widehat{h}(x) = 0 \;\Rightarrow\; ||\Psi(x)||_2 \leq \lambda ||x||_2.$$

Corollary For $A_o(I_N)$,

1. $\widehat{S}''_{\mathrm{red}}$ is a full factor.
2. $\widehat{S}_{\mathrm{red}}$ is simple with a unique tracial state.