THE PROPERTY OF RAPID DECAY FOR DISCRETE QUANTUM GROUPS

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ABSTRACT. We introduce the Property of Rapid Decay for discrete quantum groups by equivalent characterizations that generalize the classical ones. We investigate examples, proving in particular the Property of Rapid Decay for unimodular free quantum groups. We finally check that the applications to the K-theory of the reduced group C^*-algebras carry over to the quantum case.

KEYWORDS: *quantum groups, rapid decay, group C^*-algebras.*

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1. INTRODUCTION

Let $\Gamma = F_N$ be the free group on N generators, and denote by $l(\gamma)$ the length of a reduced word $\gamma \in \Gamma$. In the founding paper [9], Haagerup proved that the norm of the reduced group C^*-algebra $C_r^*(\Gamma)$ can be controlled by Sobolev ℓ^2-norms associated to l:

$$\forall x \in \mathbb{C}\Gamma \; ||x||_{C_r^*\Gamma} \leq C \, ||x||_{2,s} := C \left(\sum_{\gamma \in \Gamma} |(1 + l(\gamma))^s x_\gamma|^2 \right)^{1/2}$$

for suitable constants C, $s > 0$. This is remarkable since on the other hand the norm of $C_r^*(\Gamma)$ always dominates the (non-weighted) ℓ^2-norm on $\mathbb{C}\Gamma$. Moreover the norm of $C_r^*(\Gamma)$ is a non-trivial and very interesting data, whereas the Sobolev norms are easily computable.

In fact this phenomenon occurs in many more cases and we say that a discrete group Γ endowed with a length function l has the *Property of Rapid Decay* (Property RD) if the inequality above is satisfied on $\mathbb{C}\Gamma$ for some constants C, s. This general notion was introduced and studied by Jolissaint in [12], where many examples are also presented.

Among the various applications of this property, let us mention the one concerning K-theory, which is interesting from the point of view of noncommutative geometry: thanks to the control on the norm of $C_r^*(\Gamma)$ provided by Property RD one can prove that the K-theory of $C_r^*(\Gamma)$ equals the K-theory of certain dense convolution subalgebras of rapidly decreasing functions on Γ [11]. This fact was notably used by V. Lafforgue in his approach to the Baum-Connes conjecture via Banach KK-theory [14, 15].

The foundations of the theory of discrete quantum groups are now very well understood [25, 8, 20, 27], and it is a general motivation to figure out whether the classical analytic properties of discrete groups are still relevant in the quantum framework. In this article we address this question for the Property of Rapid Decay.

As we will see, there is a quite natural way to extend the definition of Property RD to discrete quantum groups, and one can prove that the applications to K-theory still work in the general case. The natural candidates for interesting quantum examples are the free quantum groups of Wang-Banica [21, 3], and we will show that the unimodular ones indeed have Property RD: this is the quantum analogue of the founding result of Haagerup.

The first section of this article summarizes definitions and facts about discrete quantum groups that are needed in the sequel. In the second section we give a definition of Property RD for discrete quantum groups, establishing the equivalence between various characterizations that generalize the classical ones.

In the third section, we investigate the main known classes of examples. Amenable discrete quantum groups have Property RD **iff** they have polynomial growth and this is the case of duals of connected compact Lie groups, see Section 4.1. On the other hand it turns out that Property RD implies unimodularity: this results from a necessary condition that we introduce in Section 4.2. In Section 4.3 we address the case of the free quantum groups. The previous necessary condition is then sufficient — in the unitary case, this is an adaptation of the classical proof of Haagerup —, and it is true in the unimodular case — this is the "purely quantum" part of the proof.

Finally we prove in the fourth section that the application to K-theory mentioned above still holds in the quantum case. We deal with the case of the Banach algebras $\hat{H}_L^s$, with s big enough, and also with $\hat{H}_L^\infty$, which is a smaller dense subalgebra but not a Banach algebra anymore.

Let us conclude this introduction with a remark. In the original paper of Haagerup the Property of Rapid Decay was used in conjunction with the fact that l is conditionnally of negative type to show that $C_r^*(F_N)$ has the Metric Approximation Property — although it does not have the Completely Positive Approximation Property since F_N is not amenable. In the case of the free quantum groups however one can show that the natural length, which satisfies Property RD, is not conditionnally of negative type in the appropriate sense.

2. NOTATION

In this section we introduce the notation and results about discrete quantum groups that are needed in the article. Having in mind the study of the Property of Rapid Decay, it is natural to use the framework of Hopf C^*-algebras [18]. The global situation to be considered consists in two "dual" Hopf C^*-algebras (S, δ) and $(\hat{S}, \hat{\delta})$ related by a multiplicative unitary $V \in M(\hat{S} \otimes S)$ of discrete type. In fact this discrete situation can be axiomatized equally at the level of S, $\hat{S}$ or V:

- $(\hat{S}, \hat{\delta})$ is a bisimplifiable unital Hopf C^*-algebra [25, 27].
- V is a regular multiplicative unitary with a unique co-fixed line [1].
- (S, δ) is the completion of a multiplier Hopf $*$-algebra which is a direct sum of matrix algebras [19, 20].

Notice also that these notions fit in the theory of locally compact quantum groups [13], yielding exactly the discrete case. Moreover, they include many families of interesting examples, such as discrete groups, duals of compact Lie groups and their q-deformations, unitary and orthogonal free quantum groups. References and basic facts about these examples will be presented along the text of the article as they are investigated.

To be more precise, and since it will be convenient for Property RD to have the C^*-algebras S and $\hat{S}$ represented on the same Hilbert space right from the begining, let us start from the multiplicative unitary. It is a unitary element of $B(H \otimes H)$ for some Hilbert space H, such that $V_{12}V_{13}V_{23} = V_{23}V_{12}$ on $H \otimes H \otimes H$. Regularity means that the norm closure of $(\mathrm{id} \otimes B(H)_*)(\Sigma V)$ coincides with $K(H)$, where Σ is the flip operator, here on $H \otimes H$. A co-fixed vector is an element $e \in H$ such that $V(\zeta \otimes e) = \zeta \otimes e$ for all $\zeta \in H$. The Hopf C^*-algebra (S, δ) can then be defined by

$$S = \overline{\mathrm{Lin}}\{(\omega \otimes \mathrm{id})(V) \mid \omega \in B(H)_*\} \text{ and}$$
$$\forall a \in S \ \delta(a) = V(a \otimes 1)V^*.$$

It follows from the work of Baaj, Skandalis [1] and Podleś, Woronowicz [16] that (S, δ) is indeed a Hopf C^*-algebra whith left and right Haar weights h_L, h_R.

The involutive monoidal category $\mathcal{C}$ of finite-dimensional unitary representations of S is a major tool when investigating in detail the properties of S and $\hat{S}$ [26]. We will denote by H_α the space of the representation $\alpha \in \mathcal{C}$. Choosing a complete set $\mathrm{Irr}\,\mathcal{C}$ of representatives of the irreducible representations, we have

$$S \simeq \bigoplus_{\alpha \in \mathrm{Irr}\,\mathcal{C}} L(H_\alpha).$$

We will denote by $\mathcal{S}$ the algebraic direct sum of the matrix algebras $L(H_\alpha)$, by $p_\alpha \in S$ the central support of $\alpha \in \mathrm{Irr}\,\mathcal{C}$, and by $\mathcal{H}$ the algebraic direct sum of the f.-d. subspaces $p_\alpha H$.

The category of f.-d. representations of S can be endowed with a tensor product $\alpha \otimes \beta := (\alpha \otimes \beta) \circ \delta$ and with a conjugation which we describe now. Let

(e_i) be an orthonormal basis of H_α. The conjugate object $\bar{\alpha}$ of $\alpha \in \mathcal{C}$ is characterized, up to equivalence, by the existence of a conjugation map $j_\alpha : H_\alpha \to H_{\bar{\alpha}}$, $\zeta \mapsto \bar{\zeta}$ which is an anti-isomorphism such that $t_\alpha : 1 \mapsto \sum e_i \otimes \bar{e}_i$ and $t'_\alpha : \bar{\zeta} \otimes \xi \mapsto (\zeta|\xi)$ are resp. elements of Mor $(1, \alpha\otimes\bar{\alpha})$ and Mor $(\bar{\alpha}\otimes\alpha, 1)$. If α is irreducible, the conjugation map j_α is unique up to a scalar and one can renormalize it in such a way that Tr $j_\alpha^* j_\alpha =$ Tr $(j_\alpha^* j_\alpha)^{-1}$ and $j_{\bar{\alpha}} j_\alpha = \pm 1$.

The positive invertible maps $j_\alpha^* j_\alpha$ do not depend on the normalized j_α and describe the non-trivial interplay between the involution of $\mathcal{C}$ and its hilbertian structure. Putting them together, we obtain the modular element, a positive unbounded multiplier $F \in S^\eta$ such that $p_\alpha F = j_\alpha^* j_\alpha$ for every $\alpha \in$ Irr $\mathcal{C}$. One can show that $\delta(F) = F \otimes F$, this is equivalent to the fact that the restrictions of $j_{\alpha\otimes\beta} = \Sigma \circ (j_\alpha \otimes j_\beta)$ to the irreducible subspaces of $H_\alpha \otimes H_\beta$ are normalized conjugation maps if j_α, j_β are so.

The Haar weights of (S, δ) admit the following simple expressions in terms of F:

$$\forall a \in p_\alpha S \simeq L(H_\alpha) \quad h_L(a) = m_\alpha \mathrm{Tr}\,(F^{-1}a) \text{ and } h_R(a) = m_\alpha \mathrm{Tr}\,(Fa), \tag{2.1}$$

where the positive number $m_\alpha =$ Tr $p_\alpha F =$ Tr $p_\alpha F^{-1}$ is called the quantum dimension of α. We say that (S, δ) is unimodular if $F = 1$.

The antipode $\kappa : \mathcal{S} \to \mathcal{S}$ is a linear and antimultiplicative map such that $\kappa(p_\alpha) = p_{\bar{\alpha}}$ for all $\alpha \in \mathcal{C}$, and the co-unit $\varepsilon \in S'$ equals the trivial representation $1_{\mathcal{C}} \in$ Irr $\mathcal{C}$. Let us recall from [16] the following classical identities:

$$m \circ (\kappa\otimes\mathrm{id}) \circ \delta = m \circ (\mathrm{id}\otimes\kappa) \circ \delta = 1_S \varepsilon, \tag{2.2}$$

$$(\varepsilon\otimes\mathrm{id}) \circ \delta = (\mathrm{id}\otimes\varepsilon) \circ \delta = \mathrm{id} \text{ and } \delta \circ \kappa = \sigma \circ (\kappa\otimes\kappa) \circ \delta, \tag{2.3}$$

where $m : \mathcal{S}\otimes\mathcal{S} \to \mathcal{S}$ is the product of $\mathcal{S}$ and $\sigma : \mathcal{S}\otimes\mathcal{S} \to \mathcal{S}\otimes\mathcal{S}$ is the flip map.

The reduced dual Hopf C^*-algebra of (S, δ) will play an important role in this paper. It is defined from the left leg of the multiplicative unitary V under consideration:

$$\hat{S} = \overline{\mathrm{Lin}}\{(\mathrm{id}\otimes\omega)(V) \mid \omega \in B(H)_*\} \text{ and}$$
$$\forall \hat{a} \in \hat{S}\ \hat{\delta}(\hat{a}) = V^*(1\otimes\hat{a})V.$$

The pair $(\hat{S}, \hat{\delta})$ is then the (unital) Hopf C^*-algebra of a compact quantum group [27]. In particular it admits a two-sided Haar state $\hat{h}$ which is in our setting the vector state associated to any co-fixed unit vector $e \in H$. We say that (S, δ) is amenable if $(\hat{S}, \hat{\delta})$ admits a continuous co-unit.

We will consider the Fourier transform defined on the dense subspace $\mathcal{S} \subset S$ in the following way:

$$\forall a \in \mathcal{S}\ \mathcal{F}(a) = (\mathrm{id}\otimes h_R)(V^*(1\otimes a)) \in \hat{S},$$

and we denote by $\hat{\mathcal{S}}$ the image of $\mathcal{F}$, a dense subspace of $\hat{S}$. Observe that other choices are possible for the definition of $\mathcal{F}$. One can check that our choice makes

$\mathcal{F}$ isometric with respect to the GNS norms associated to h_R and $\hat{h}$. More precisely, let e be a co-fixed unit vector for V and put $\Lambda(a) = \mathcal{F}(a)e$ for $a \in \mathcal{S}$, this defines a GNS map Λ for h_R such that $\Lambda(ab) = a\Lambda(b)$ and $V(\Lambda\otimes\Lambda)(a\otimes b) = (\Lambda\otimes\Lambda)(\delta(a)(1\otimes b))$ for any $a, b \in \mathcal{S}$.

All the unbounded operators we have used in this introduction are of an almost trivial kind: they are self-adjoint and affiliated to S or $S\otimes S$. Due to the very special structure of S, operators affiliated to S, which are closed by definition, admit $\mathcal{H}$ as a core and they form a $*$-algebra S^η which identifies with the algebra $\prod_{\alpha\in\mathrm{Irr}\,\mathcal{C}} L(H_\alpha)$ of (algebraic) multipliers of the Pedersen ideal $\mathcal{S}$. Moreover in this case symmetry implies self-adjointness. Notice that one can apply $*$-homomorphisms such as δ and ε to elements of S^η, but also proper maps such as the antipode κ [16].

3. THE PROPERTY OF RAPID DECAY

3.1. LENGTHS. Before introducing the Property of Rapid Decay it is necessary to discuss the notion of length for discrete quantum groups. In particular we establish at Lemmas 3.3 and 3.4 some elementary properties of these lengths.

DEFINITION 3.1. Let (S, δ) be the Hopf C^*-algebra of a discrete quantum group. A length on (S, δ) is an unbounded multiplier $L \in S^\eta$ such that $L \geq 0$, $\varepsilon(L) = 0$, $\kappa(L)_{|\mathcal{H}} = L_{|\mathcal{H}}$ and $\delta(L) \leq 1\otimes L + L\otimes 1$. Given such a length, we denote by $p_n \in M(S)$ the spectral projection of L associated to the interval $[n, n+1[$, for $n \in \mathbb{N}$.

EXAMPLE 3.2.

1. A complex-valued function l on $\mathrm{Irr}\,\mathcal{C}$ will be called a length function if we have for any $\alpha, \beta, \beta' \in \mathrm{Irr}\,\mathcal{C}$
$$l(\alpha) \geq 0, l(1_{\mathcal{C}}) = 0, l(\bar{\alpha}) = l(\alpha) \text{ and}$$
$$\alpha \subset \beta\otimes\beta' \implies l(\alpha) \leq l(\beta) + l(\beta').$$
It is easy to observe that $L = \sum l(\alpha)p_\alpha$ is then a central length on (S, δ), and that all of these are obtained in this way. Notice that $\alpha \subset \beta\otimes\beta' \iff (p_\beta\otimes p_{\beta'})\delta(p_\alpha) \neq 0$, by definition of the representation $\beta\otimes\beta'$.
2. We say that (S, δ) is finitely generated if there exists a finite subset $\mathcal{D} = \bar{\mathcal{D}} \subset \mathrm{Irr}\,\mathcal{C}$, not containing $1_{\mathcal{C}}$, such that any element of $\mathrm{Irr}\,\mathcal{C}$ is contained in a multiple tensor product of elements of $\mathcal{D}$. The distance to the origin in the classical Cayley graph associated to $(\hat{S}, \mathcal{D})$, in the sense of [23], defines then a length function on $\mathrm{Irr}\,\mathcal{C}$ with values in $\mathbb{N}$ given by
$$l(\alpha) = \min\{k \in \mathbb{N} \mid \exists \beta_1, \ldots, \beta_k \in \mathcal{D}\ \alpha \subset \beta_1\otimes\cdots\otimes\beta_k\}.$$
The corresponding central length L_0 on (S, δ) is called the word length of (S, δ) with respect to $\mathcal{D}$.

LEMMA 3.3. *Let L_0 and L be lengths on (S,δ), and assume that L_0 is a word length. Then there exists $\epsilon > 0$ such that $L_0 \geq \epsilon L$.*

Proof. Using the spectral projections of L_0, put $p = p_0 + p_1 \in Z(\mathcal{S})$ and $C = ||Lp||$. For any $n \in \mathbb{N}^*$ we have then

$$p^{\otimes n}\delta^{n-1}(L) \leq p^{\otimes n}(L\otimes 1\otimes \cdots + 1\otimes L\otimes 1\otimes \cdots + \cdots + 1\otimes 1\otimes \cdots \otimes L) \leq Cn.$$

Take $\alpha \in \operatorname{Irr}\mathcal{C}$ such that $l(\alpha) = n$: we have $p^{\otimes n}\delta^{n-1}(p_\alpha) \neq 0$. The C^*-algebra $p_\alpha S$ being simple, this implies that the restriction of $p^{\otimes n}\delta^{n-1}$ to $p_\alpha S$ is an isometry. In particular we can write

$$\begin{aligned} Lp_\alpha \leq ||Lp_\alpha||p_\alpha &= ||p^{\otimes n}\delta^{n-1}(Lp_\alpha)||p_\alpha \\ &\leq ||p^{\otimes n}\delta^{n-1}(L)||p_\alpha \leq Cnp_\alpha = CL_0p_\alpha. \end{aligned}$$

Since this holds for any minimal central projection p_α, the desired result is proved with $\epsilon = 1/C$. ∎

LEMMA 3.4. *Let L be a central length on (S,δ). Denote by $\mathcal{T} \subset \mathbb{N}^3$ the set of triples (k,l,n) such that $\delta(p_n)(p_k\otimes p_l) \neq 0$.*

1. *$\delta(L) \geq 1\otimes L - L\otimes 1$ and $\delta(L) \geq L\otimes 1 - 1\otimes L$.*
2. *$\mathcal{T}$ is stable under permutation and contains $(n,n,0)$ for any $n \in \mathbb{N}$.*
3. *$(k,l,n) \in \mathcal{T} \Longrightarrow |k-l| - 2 \leq n \leq k+l+2$.*

Proof. 1. Let l be the length function on $\operatorname{Irr}\mathcal{C}$ corresponding to L like in Example 3.2.1. For the first inequality it is enough to prove that, for any inclusion $\alpha \subset \beta\otimes\beta'$:

$$\begin{aligned} (p_\beta\otimes p_{\beta'})\delta(p_\alpha)\delta(L) &\geq (p_\beta\otimes p_{\beta'})\delta(p_\alpha)(1\otimes L - L\otimes 1) \\ \Longleftrightarrow\ l(\alpha) &\geq l(\beta') - l(\beta). \end{aligned}$$

This results from the equivalence $\alpha \subset \beta\otimes\beta' \Longleftrightarrow \beta' \subset \bar{\beta}\otimes\alpha$ and the fact that l is a length function. Similarly the second inequality results from the other equivalent inclusion $\beta \subset \alpha\otimes\bar{\beta}'$.

2. First notice that the projections p_n are central. Using the identity (2.3) and the fact that $\kappa(p_n) = p_n$, we can write

$$\begin{aligned} \delta(p_n)(p_k\otimes p_l) &= \delta(\kappa(p_n))(p_k\otimes p_l) = \sigma(\kappa\otimes\kappa)\delta(p_n)\cdot(p_k\otimes p_l) \\ &= \sigma(\kappa\otimes\kappa)\left((p_l\otimes p_k)\delta(p_n)\right), \end{aligned}$$

hence $(k,l,n) \in \mathcal{T} \Longrightarrow (l,k,n) \in \mathcal{T}$. On the other hand, using the identities (2.2) and (2.3) we have

$$\begin{aligned} &(m\otimes\mathrm{id})(\kappa\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\delta)\ (\delta(p_n)(p_k\otimes p_l)) = \\ &\qquad = (m\otimes\mathrm{id})(\kappa\otimes\mathrm{id}\otimes\mathrm{id})\ (\delta^2(p_n)(p_k\otimes\delta(p_l))) \\ &\qquad = (p_k\otimes 1)\ [(m\otimes\mathrm{id})(\kappa\otimes\mathrm{id}\otimes\mathrm{id})\delta^2(p_n)]\ \delta(p_l) \\ &\qquad = (p_k\otimes 1)(1\otimes p_n)\delta(p_l) \end{aligned}$$

hence $(k,l,n) \in \mathcal{T} \Longrightarrow (k,n,l) \in \mathcal{T}$.

3. By definition one has $np_n \leq Lp_n \leq (n+1)p_n$, so that

$$\begin{aligned} \delta(L)\delta(p_n)(p_k \otimes p_l) &\geq n\delta(p_n)(p_k \otimes p_l) \text{ and} \\ \delta(L)\delta(p_n)(p_k \otimes p_l) &\leq (L \otimes 1 + 1 \otimes L)\delta(p_n)(p_k \otimes p_l) \\ &\leq (k+l+2)\delta(p_n)(p_k \otimes p_l). \end{aligned}$$

Now if $(k,l,n) \in \mathcal{T}$, the projection $\delta(p_n)(p_k \otimes p_l)$ is non-zero and the above inequalities show that $n \leq k+l+2$. Applying this result to (k,n,l) and (n,l,k) yields the two other inequalities of the statement. ∎

3.2. DEFINITION. We now introduce the Property of Rapid Decay for discrete quantum groups with respect to a central length. Like in the classical case, it is mainly about controlling the norm of the reduced dual C^*-algebra by Sobolev norms, which are much simpler. If L is a length on (S,δ) and $s \in \mathbb{R}_+$ we will use the following notation, where $e \in H$ is a co-fixed unit vector:

$$\begin{aligned} &\forall a \in \mathcal{S} \qquad ||a||_2^2 := h_R(a^*a),\ ||a||_{2,s} := ||(1+L)^s a||_2 \text{ and} \\ &\forall \hat{a} \in \hat{\mathcal{S}} \qquad ||\hat{a}||_2^2 := \hat{h}(\hat{a}^*\hat{a}) = ||\hat{a}e||^2,\ ||\hat{a}||_{2,s} := ||(1+L)^s \hat{a}e||. \end{aligned}$$

We will also denote by H_L^s (resp. $\hat{H}_L^s$) the completion of $\mathcal{S}$ (resp. $\hat{\mathcal{S}}$) with respect to the $2,s$-norm. We have the norm inequalities $||a||_2 \leq ||a||_{2,s}$ and $||a||_2 = ||\mathcal{F}(a)||_2 \leq ||\mathcal{F}(a)||$ and hence the following continuous embeddings:

$$\begin{array}{ccccc} H_L^s & \hookrightarrow & H & \hookrightarrow & S \\ \|\mathcal{F} & & \|\mathcal{F} & & \\ \hat{H}_L^s & \hookrightarrow & \hat{H} & \hookleftarrow & \hat{S} \end{array}$$

PROPOSITION AND DEFINITION 3.5. Let L be a central length on the Hopf C^*-algebra (S,δ) of a discrete quantum group. We say that (S,δ,L) has the Property of Rapid Decay (Property RD) if one of the following equivalent conditions is satisfied:

(i) $\exists\, C, s \in \mathbb{R}_+\ \forall a \in \mathcal{S}\ ||\mathcal{F}(a)|| \leq C||a||_{2,s}$
(ii) $\exists\, C, s \in \mathbb{R}_+\ \forall \hat{a} \in \hat{\mathcal{S}}\ ||\hat{a}|| \leq C||\hat{a}||_{2,s}$
(iii) $\hat{H}_L^\infty := \bigcap_{s \geq 0} \hat{H}_L^s \subset \hat{S}$ (as subspaces of H)
(iv) $\exists\, P \in \mathbb{R}[X]\ \forall n \in \mathbb{N},\ a \in p_n\mathcal{S}\ ||\mathcal{F}(a)|| \leq P(n)||a||_2$
(v) $\exists\, P \in \mathbb{R}[X]\ \forall n \in \mathbb{N},\ a \in p_n\mathcal{S}\ \forall k,l \in \mathbb{N}\ ||p_l\mathcal{F}(a)p_k|| \leq P(n)||a||_2$

We say that (S,δ) has Property RD if there exists a central length on it satisfying one of these conditions.

Proof. (i) $\Longleftrightarrow$ (ii) is clear because $\mathcal{F}$ is a bijection between $\mathcal{S}$ and $\hat{\mathcal{S}}$ such that $\Lambda(a) = \mathcal{F}(a)e$. (ii) $\Longrightarrow$ (iii) is immediate whereas (iii) $\Longrightarrow$ (ii) follows, like in the classical case, from the closed graph Theorem [6, §3, cor. 5] for the inclusion of the Fréchet space $\hat{H}_L^\infty$ into $\hat{S}$. Note that the family of closed balls centered in 0 with respect to all the norms $||\cdot||_{2,s}$ forms a fundamental system of neighbourhoods of 0 in $\hat{H}_L^\infty$.

(i) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) are evident. For (iv) $\Longrightarrow$ (i), choose $C, s \in \mathbb{R}_+$ such that $P(n) \leq C(n+1)^s$ for all $n \in \mathbb{N}$. We have then $||\mathcal{F}(a)|| \leq C\sum(n+1)^s||p_n a||_2$ and, like in the classical case [9, lemma 1.5], we use the Cauchy-Schwartz inequality and the fact that $(n+1)^{s+1}||p_n a||_2 \leq ||(L+1)^{s+1}p_n a||_2$ to conclude that $||\mathcal{F}(a)|| \leq C'||a||_{2,s+1}$.

For (v) $\Longrightarrow$ (iv) we will follow the proof of [9, lemma 1.4], using the results of Lemma 3.4. Let a be an element of $p_n\mathcal{S}$. According to the following computation, we have $p_l\mathcal{F}(a)p_k \neq 0 \Longrightarrow (k,l,n) \in \mathcal{T}$:

$$\begin{aligned} p_l\mathcal{F}(a)p_k &= (\mathrm{id}\otimes h_R)((p_l\otimes 1)V^*(p_k\otimes a)) \\ &= (\mathrm{id}\otimes h_R)(V^*\delta(p_l)(p_k\otimes p_n a)). \end{aligned}$$

As a result we can write, for any $\xi \in H$:

$$\begin{aligned} ||\mathcal{F}(a)\xi||^2 &= \sum_l ||p_l\mathcal{F}(a)\xi||^2 \leq \sum_l (\textstyle\sum_k ||p_l\mathcal{F}(a)p_k\xi||)^2 \\ &\leq P(n)^2||a||_2^2 \sum_l (\textstyle\sum_{(k,l,n)\in\mathcal{T}} ||p_k\xi||)^2. \end{aligned}$$

Moreover by Lemma 3.4 the cardinal of $\{p \mid (n,q,p) \in \mathcal{T}\}$ is bounded above by $2\min(q,n)+5 \leq 2n+5$. Using this estimate twice and the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} \sum_l (\textstyle\sum_{(k,l,n)\in\mathcal{T}} ||p_k\xi||)^2 &\leq \sum_l (2n+5)(\textstyle\sum_{(k,l,n)\in\mathcal{T}} ||p_k\xi||^2) \\ &\leq (2n+5)^2\textstyle\sum_k ||p_k\xi||^2 = (2n+5)^2||\xi||^2. \end{aligned}$$

Finally $||\mathcal{F}(a)|| \leq (2n+5)P(n)||a||_2$ and Condition (iv) is satisfied. ∎

REMARK 3.6. Conditions (i)–(iv) are still equivalent if L is a non-central length on (S,δ) and hence they can be used to define a Property RD with respect to non-central lengths. Besides, if L' is a central length on (S,δ) such that $L' \geq \epsilon L$ for some $\epsilon > 0$, it is easy to check that the Property RD for L implies the Property RD for L'. Now, assume that (S,δ) is finitely generated and choose a word length L_0 on it. The preceeding observations and Lemma 3.3 show that (S,δ) has Property RD, possibly with respect to a non-central length, if and only if (S,δ,L_0) has Property RD. As a result, the use of central lengths for the study of Property RD is not a restriction in the finitely generated case.

EXAMPLE 3.7. When the C^*-algebra S is commutative, it is of the form $c_0(\Gamma)$ with Γ a discrete group, and the coproduct δ is given by the formula $\delta(f)(\alpha,\beta) = f(\alpha\beta)$. Then the notions of length and of Property RD studied in this paper coincide with the classical notions introduced by Jolissaint [12] after the founding paper of Haagerup [9] about the convolution algebras of the free groups. For a recent account on Property RD for discrete groups, including examples, counter-examples and more references, we refer the reader to [17, chapter 7].

4. SPECIAL CASES AND QUANTUM EXAMPLES

4.1. THE AMENABLE CASE. For amenable discrete groups, Property RD is equivalent to polynomial growth [12, cor. 3.1.8]. In this section we extend this result to the case of discrete quantum groups. The main motivation is the study of duals of compact Lie groups and their q-deformations: as a matter of fact these discrete quantum groups are all amenable [4, cor. 6.2].

DEFINITION 4.1. Let L be a central length on the Hopf C^*-algebra (S, δ) of a discrete quantum group. Denote by (p_n) the corresponding sequence of spectral projections. We say that (S, δ, L) has polynomial growth if

$$\exists\, P \in \mathbb{R}[X]\ \forall n \in \mathbb{N}\ h_R(p_n) \leq P(n).$$

REMARK 4.2. Let l be the length function on Irr $\mathcal{C}$ associated to L, and recall that we denote by m_α the quantum dimension of $\alpha \in$ Irr $\mathcal{C}$. From the expression of h_R given in Section 2 we see that

$$h_R(p_n) = \sum\{m_\alpha^2 \mid l(\alpha) \in [n, n+1[\} \in [0, +\infty].$$

In particular this implies that $h_R(p_n) = h_L(p_n)$. Moreover, put

$$s_n := \mathrm{Card}\{\alpha \in \mathrm{Irr}\,\mathcal{C} \mid l(\alpha) \in [n, n+1[\} \text{ and}$$
$$M_n := \sup\{m_\alpha \mid l(\alpha) \in [n, n+1[\}.$$

The previous expression shows that (S, δ, L) has polynomial growth **iff** the sequences (s_n) and (M_n) have polynomial growth. In the case of a discrete group, $M_n = 1$ for each n and one recovers the classical notion of polynomial growth with respect to l.

LEMMA 4.3. *Let L be a central length on the Hopf C^*-algebra (S, δ) of a non-unimodular discrete quantum group. Then (S, δ, L) does not have polynomial growth. Moreover if L is a word length the sequences $(||p_n F||)_n$ and $(||p_n F^{-1}||)_n$ grow geometrically.*

Proof. Let us first notice that L can be assumed to be a word length, even for the first assertion. As a matter of fact, the non-unimodularity implies the existence of an irreducible representation $\alpha \in$ Irr $\mathcal{C}$ such that $p_\alpha F \neq p_\alpha$. By restricting to the "subgroup generated by α", see [22, section 2], one can assume that $\mathcal{D} = \{\alpha, \bar{\alpha}\}$ generates $\mathcal{C}$. By Lemma 3.3 one can moreover assume that L is the word length associated to $\mathcal{D}$: as a matter of fact if $\epsilon L' \leq L$ with L' having polynomial growth, then L has polynomial growth.

Now let $\mathcal{D}$ be the generating subset defining the word length L. The equality Tr $p_\alpha F =$ Tr $p_\alpha F^{-1}$ shows that the greatest eigenvalue of $p_\alpha F$ is greater than or equal to 1, with equality **iff** $p_\alpha F = p_\alpha$. Let α be the element of $\mathcal{D}$ such that $p_\alpha F$ has the greatest eigenvalue λ. Because $\delta(F) = F \otimes F$, the n-th power λ^n is an eigenvalue of

$$p_\alpha^{\otimes n} \delta^{n-1}(F) = \sum\{p_\alpha^{\otimes n} \delta^{n-1}(p_\beta F) \mid \beta \subset \alpha^{\otimes n}\}.$$

Since the maps $p_\alpha^{\otimes n}\delta^{n-1}(p_\beta\,\cdot\,)$ are injective $*$-homomorphisms which have pairwise orthogonal ranges when β varies, we conclude that one of the $p_\beta F$ admits λ^n as an eigenvalue. In the same way one sees that the eigenvalues of $p_\beta F$, with $\beta \subset \alpha^{\otimes n-1}$, are less than λ^{n-1}. As a result $||p_n F|| = \lambda^n$ and $\lambda > 1$ by non-unimodularity. In particular (S, δ, L) does not have polynomial growth. One proceeds in the same way for $||p_n F^{-1}||$, using a minimal eigenvalue of the $p_\alpha F$, $\alpha \in \mathcal{D}$. ∎

PROPOSITION 4.4. *Let L be a central length on the Hopf C^*-algebra (S, δ) of a discrete quantum group.*

1. *If (S, δ) is amenable and (S, δ, L) has Property RD, then (S, δ, L) has polynomial growth.*
2. *If (S, δ, L) has polynomial growth, then (S, δ, L) has Property RD.*

Proof. 1. By hypothesis there exists a continuous co-unit $\hat{\varepsilon}$ on $(\hat{S}, \hat{\delta})$. Let us show that $x := (\hat{\varepsilon}\otimes\mathrm{id})(V)$ equals the identity: one has

$$x^2 = (\hat{\varepsilon}\otimes\hat{\varepsilon}\otimes\mathrm{id})(V_{13}V_{23}) = (\hat{\varepsilon}\otimes\hat{\varepsilon}\otimes\mathrm{id})(\hat{\delta}\otimes\mathrm{id})(V) = (\hat{\varepsilon}\otimes\mathrm{id})(V) = x,$$

but on the other hand x is unitary because $\hat{\varepsilon}$ is a $*$-character. Hence $(\hat{\varepsilon}\otimes\mathrm{id})(V) = \mathrm{id}$ and, by definition of $\mathcal{F}$, $\hat{\varepsilon} \circ \mathcal{F} = h_R$. As a result we can write, like in the classical case:

$$\forall a \in \mathcal{S}\ |h_R(a)| = |\hat{\varepsilon}\mathcal{F}(a)| \leq ||\mathcal{F}(a)|| \leq C||a||_{2,s},$$

for the constants $C, s \in \mathbb{R}_+$ given by Property RD. Applying this to the projections p_n gives the desired result:

$$h_R(p_n) \leq C||(1+L)^s p_n||_2 \leq C(2+n)^s\sqrt{h_R(p_n)},$$

hence $h_R(p_n) \leq C^2(2+n)^{2s}$ for all $n \in \mathbb{N}$.

2. Let $a \in p_n\mathcal{S}$, in particular we have $a = pa$ for some central projection $p \in S$. We consider the linear functional $ah_R := h_R(\,\cdot\, a)$ defined on the C^*-algebra S and we first assume that it is positive. One can then write

$$||\mathcal{F}(a)|| = ||(\mathrm{id}\otimes ah_R)(V^*)|| \leq ||ah_R||.$$

Let (u_i) be an approximate unit of S, we have $ah_R(u_i) = ah_R(u_i p)$ and, by taking limits, $||ah_R|| = ah_R(p) = h_R(a)$. As a result

$$||\mathcal{F}(a)|| \leq h_R(a) = h_R(p_n a) \leq (h_R(p_n)h_R(a^*a))^{1/2} \leq \sqrt{P(n)}\ ||a||_2.$$

The general case follows by decomposing $a \in p_n\mathcal{S}$ into a linear combination $\sum_{k=0}^3 i^k a_k$ of 4 positive elements: the negative and positive parts of Im a and Re a. Since (S, δ) is necessarily unimodular according to Lemma 4.3, h_R is a trace and one can check that $||a||_2 = (\sum ||a_k||_2^2)^{1/2}$, which is greater than $(\sum ||a_k||)/2$. ∎

EXAMPLE 4.5. Let G be a compact group and take $S = C^*(G)$, $\delta(U_g) = U_g\otimes U_g$. Then (S, δ) is the Hopf C^*-algebra of a discrete quantum group which is called the dual of G. By definition, $\mathcal{C}$ identifies in this case with the category of the

finite-dimensional unitary representations of G. Besides, $\hat{S}$ identifies with $C(G)$ and $\mathcal{F}$ coincides with the Fourier transform of [7, §8, n°1]. The finitely generated case corresponds to the case of compact Lie groups.

Let G be a connected compact Lie group and choose a maximal torus $T \subset G$. Following [7], we denote by X the dual group of T, by $R \subset X$ the set of roots of G, by W the Weyl group of (G, T) acting on X and by w_0 the longest element of W. We moreover choose a subset of positive roots $R_+ \subset R$ and denote by X_{++} the associated set of dominant weights. Taking the highest weight of irreducible representations defines a canonical identification between Irr $\mathcal{C}$ and X_{++}.

Let $||\cdot||$ be a W-invariant definite-positive quadratic form on $X \otimes_{\mathbb{Z}} \mathbb{R}$. Its restriction l to Irr $\mathcal{C} \simeq X_{++}$ is in fact a length function: in our identification, we have $\alpha \subset \beta \otimes \beta' \Longrightarrow \alpha \leq \beta + \beta'$, and because such inequalities are conserved by scalar product against dominant weights we obtain

$$||\alpha||^2 \leq (\alpha|\beta + \beta') \leq ||\beta + \beta'||^2 \leq (||\beta|| + ||\beta'||)^2.$$

Moreover, $||\bar{\alpha}|| = ||-w_0(\alpha)|| = ||\alpha||$ and $||1_{\mathcal{C}}|| = ||0_X|| = 0$. We denote by L the length function on (S, δ) associated to l.

It is clear by definition of L that (s_n) has polynomial growth. Moreover M_n, which is the maximal dimension of the representations of length n, is also polynomialy growing by the dimension formula of H. Weyl:

$$\dim \beta = \prod_{\alpha \in R_+} \frac{\langle \beta + \rho, K_\alpha \rangle}{\langle \rho, \alpha \rangle},$$

where $2\rho = \sum_{\alpha \in R_+} \alpha$ and the $\langle \cdot, K_\alpha \rangle$ are linear forms on X. As a result, duals of connected compact Lie groups have Property RD. In fact with little more work [7, §8, thm. 1] one can see that $\hat{H}_L^\infty$ coincides in this case with $C^\infty(G)$, which is evidently included in $C(G)$.

On the other hand the q-deformations of simple compact Lie groups are usually non-unimodular, hence their duals do not have Property RD by Lemma 4.3 and Proposition 4.4. This is for instance the case of the duals of the compact quantum groups $SU_q(N)$ for $q \in]0, 1[$, $N \geq 3$. For $N = 2$, the quantum group $SU_q(2)$ is defined for $q \in [-1, 1] \setminus 0$ and is non-unimodular for $|q| < 1$. For $q = -1$ it has the same semi-ring of representations and the same dimension map as $SU(2)$, hence it has Property RD: this is a first non-commutative, non-cocommutative example.

4.2. THE NON-UNIMODULAR CASE. We have remarked in the previous section that Property RD is not conserved by non-unimodular deformations. In this section we will see that Property RD is in fact uncompatible with non-unimodularity. This will result from the geometric necessary condition (4.1) for Property RD which we will also use in the next section.

To move smoothly to the geometric point of view, let us use the notion of convolution of elements of S [16]. For a, $b \in \mathcal{S}$, we denote by $\mathrm{Conv}(a \otimes b)$ the unique element $c \in \mathcal{S}$ satisfying the relation $(ah_R \otimes bh_R) \circ \delta = ch_R$, where $xh_R :=$

$h_R(\,\cdot\, x)$. This clearly defines a linear map $\mathrm{Conv} : \mathcal{S}\otimes_{\mathrm{alg}}\mathcal{S} \to \mathcal{S}$, which is related to the Fourier transform in the following way:

$$\begin{aligned}\mathcal{F}(a)\mathcal{F}(b) &= (\mathrm{id}\otimes bh_R\otimes ah_R)(V_{13}^*V_{12}^*)\\ &= (\mathrm{id}\otimes bh_R\otimes ah_R)(\mathrm{id}\otimes\delta)(V^*) = \mathcal{F}(\mathrm{Conv}(b\otimes a)).\end{aligned}$$

In particular this yields the equality $\mathcal{F}(a)\Lambda(b) = \Lambda(\mathrm{Conv}(b\otimes a))$. Moreover it is easy to check from the definition that

$$\mathrm{Conv}(\delta(s)\, a\otimes b) = s\,\mathrm{Conv}(a\otimes b) \text{ and } \mathrm{Conv}(a\otimes b\,\delta(s)) = \mathrm{Conv}(a\otimes b)\, s,$$

using for the second equality the KMS property of h_R. In particular we have $p_\gamma\mathrm{Conv}(a\otimes b) = \mathrm{Conv}(\delta(p_\gamma)(a\otimes b)\delta(p_\gamma))$ and $p_\gamma\mathrm{Conv}(x\otimes y)$ is a multiple of p_γ for any $x, y \in Z(\mathcal{S})$, because it is then central.

LEMMA 4.6. *Let $\gamma \subset \beta\otimes\alpha$ be an inclusion without multiplicity, with α, β, $\gamma \in$ Irr $\mathcal{C}$. Then we have*

$$\forall x \in p_\beta S\otimes p_\alpha S\ ||p_\gamma\mathrm{Conv}(x)||_2 = \sqrt{\frac{m_\beta m_\alpha}{m_\gamma}}\ ||\delta(p_\gamma)x\delta(p_\gamma)||_2.$$

Proof. Let $\lambda \in \mathbb{C}$ be the scalar such that $p_\gamma\mathrm{Conv}(p_\beta\otimes p_\alpha) = \lambda p_\gamma$. By the hypothesis on the inclusion $\gamma \subset \beta\otimes\alpha$, for any $x \in p_\beta S\otimes p_\alpha S$ there exists $y \in p_\gamma S$ such that $\delta(p_\gamma)x\delta(p_\gamma) = \delta(y)(p_\beta\otimes p_\alpha)$. We have then the following equalities : $p_\gamma\mathrm{Conv}(x) = \mathrm{Conv}(\delta(p_\gamma)x\delta(p_\gamma)) = yp_\gamma\mathrm{Conv}(p_\beta\otimes p_\alpha) = \lambda y$. As a result

$$\begin{aligned}||p_\gamma\mathrm{Conv}(x)||_2^2 &= \bar\lambda\ h_R(y^*p_\gamma\mathrm{Conv}(x)) = \bar\lambda\ (h_R\otimes h_R)(\delta(y^*)\delta(p_\gamma)x)\\ &= \bar\lambda\ (h_R\otimes h_R)(\delta(p_\gamma)x^*\delta(p_\gamma)x) = \bar\lambda\ ||\delta(p_\gamma)x\delta(p_\gamma)||_2^2.\end{aligned}$$

We obtain the desired value of λ by the following computation:

$$\begin{aligned}\lambda\, m_\gamma^2 &= \lambda\, h_R(p_\gamma) = h_R(p_\gamma\mathrm{Conv}(p_\beta\otimes p_\alpha)) = (h_R\otimes h_R)(\delta(p_\gamma)(p_\beta\otimes p_\alpha))\\ &= m_\beta m_\alpha(\mathrm{Tr}\otimes\mathrm{Tr})((F\otimes F)\delta(p_\gamma)) = m_\beta m_\alpha m_\gamma. \quad \blacksquare\end{aligned}$$

Let us now fix a central length L on (S,δ) and denote by l the associated length function on Irr $\mathcal{C}$. In view of the link between convolution and Fourier transform, we have for any $a \in p_\alpha S$, $b \in p_\beta S$:

$$||p_\gamma\mathcal{F}(a)p_\beta\Lambda(b)|| = ||p_\gamma\mathrm{Conv}(b\otimes a)||_2.$$

As a result Lemma 4.6 gives a necessary condition for caracterization (v) of Property RD to be fulfilled: there should exist a polynomial $P \in \mathbb{R}[X]$ such that one has, for any inclusion $\gamma \subset \beta\otimes\alpha$ without multiplicity and $a \in p_\alpha S$, $b \in p_\beta S$:

$$||\delta(p_\gamma)(b\otimes a)\delta(p_\gamma)||_2 \le \sqrt{\frac{m_\gamma}{m_\beta m_\alpha}}\ P(\lfloor l(\alpha)\rfloor)\ ||b\otimes a||_2. \tag{4.1}$$

Let us observe that this last condition is of geometric nature: it concerns the relative position in $H_\beta\otimes H_\alpha$ of the cone of decomposable tensors and of the γ-homogeneous subspace. To emphasize this point of view we will now work in the identifications $p_\alpha S \simeq L(H_\alpha)$ and use the twisted Hilbert-Schmidt norms

$||x||_{HS}^2 = \mathrm{Tr}\ (Fx^*x)$ on $L(H_\alpha)$, which only differs from the 2-norm on $p_\alpha S$ by a coefficient m_α. In particular, Inequality (4.1) can equivalently be expressed with the twisted Hilbert-Schmidt norms.

Inequality (4.1) must in particular be satisfied for $\gamma = 1_{\mathcal{C}}$. In this case we have $\beta = \bar{\alpha}$, the inclusion is automatically multiplicity free and it is realized by $t_{\bar{\alpha}}$, up to a scalar. Since $||t_{\bar{\alpha}}|| = \sqrt{m_\alpha}$, we have $||\delta(p_\gamma)(b\otimes a)\delta(p_\gamma)||_{HS} = |t_{\bar{\alpha}}^*(b\otimes a)t_{\bar{\alpha}}|/m_\alpha$ and our necessary condition now reads: $\forall \alpha \in \mathrm{Irr}\ \mathcal{C}$, $a \in L(H_\alpha)$, $b \in L(H_{\bar{\alpha}})$

$$|t_{\bar{\alpha}}^*(b\otimes a)t_{\bar{\alpha}}| \leq P(\lfloor l(\alpha)\rfloor)||b\otimes a||_{HS}. \tag{4.2}$$

The left-hand side has an even simpler expression, which comes from the definition of the morphisms $t_{\bar{\alpha}}$ and holds in fact even if α is not irreducible. Let (e_i) be an orthonormal basis of $H_{\bar{\alpha}}$ and put $\bar{a} = j_{\bar{\alpha}}^* a j_{\bar{\alpha}}$ for $a \in L(H_\alpha)$, for any such a and $b \in L(H_{\bar{\alpha}})$ we have

$$t_{\bar{\alpha}}^*(b\otimes a)t_{\bar{\alpha}} = \sum (e_i|be_j)(\bar{e}_i|a\bar{e}_j) = \mathrm{Tr}\ \bar{a}^* b. \tag{4.3}$$

PROPOSITION 4.7. *Let L be a central length on the Hopf C^*-algebra (S,δ) of a discrete quantum group. If (S,δ,L) has Property RD, then (S,δ) is unimodular.*

Proof. Let $\lambda \in \mathbb{R}_+$ be an eigenvalue of $p_n F$, there exists a corresponding unit eigenvector $\xi \in H_{\bar{\alpha}}$ for some $\alpha \in \mathrm{Irr}\ \mathcal{C}$ with $l(\alpha) \in [n, n+1[$. We take $b = p_\xi$, the orthogonal projection onto $\mathbb{C}\xi$, and $a = \bar{p}_\xi$. According to (4.3), the left-hand side of (4.2) equals then $\mathrm{Tr}\ p_\xi = 1$. On the other hand we have

$$\begin{aligned} ||p_\xi||_{HS}^2 &= \mathrm{Tr}\ Fp_\xi = \lambda \text{ and} \\ ||\bar{p}_\xi||_{HS}^2 &= \mathrm{Tr}\ Fj^*p_\xi jj^*p_\xi j = \mathrm{Tr}\ F^{-2}p_\xi F^{-1}p_\xi = \lambda^{-3}. \end{aligned}$$

Hence the condition (4.2) reads in this particular case $\lambda \leq P(n)$. Taking the supremum over the eigenvalues λ shows that $||p_n F|| \leq P(n)$ for all n. This is impossible for a non-unimodular discrete quantum group by Lemma 4.3. ∎

4.3. THE FREE QUANTUM GROUPS. We will mainly study the case of the duals of the orthogonal free quantum groups $\hat{S} = A_o(Q)$, with $Q \in GL(N,\mathbb{C})$. Recall that $A_o(Q)$ is the C^*-algebra generated by N^2 generators u_{ij} and the relations making $U = (u_{ij})$ unitary and $Q\bar{U}Q^{-1}$ equal to U [24, 21]. As usual, we will assume that $\bar{Q}Q$ is a scalar matrix, so that the fundamental corepresentation U is irreducible. When $N = 2$, the discrete quantum groups in consideration correspond in fact to the duals of the quantum groups $SU_q(2)$ [3, section 5], which we have already studied. Moreover the dual of $A_o(Q)$ is unimodular **iff** Q is a multiple of a unitary matrix, and up to an isomorphism one can then assume that $Q = I_N$ or $Q = \left(\begin{smallmatrix} 0 & I_k \\ -I_k & 0 \end{smallmatrix}\right)$ [5].

It is known that $\mathcal{C}$ identifies to the representation theory of $SU(2)$ [2]: the irreducible representations $\alpha_n = \bar{\alpha}_n$ are indexed by integers $n \in \mathbb{N}$ in such a way that $\alpha_0 = 1_{\mathcal{C}}$ and $\alpha_1 \otimes \alpha_n \simeq \alpha_{n-1} \oplus \alpha_{n+1}$. In particular we have $l(\alpha_n) = n$ with respect to $\mathcal{D} = \{\alpha_1\}$ and the sequence of quantum dimensions $(m_n)_n$ satisfies the

recursive equation $m_1 m_n = m_{n-1} + m_{n+1}$. Moreover $m_0 = 1$ and m_1 is strictly greater than 2 when $N \geq 3$, hence in this case $m_n = r^n(1-s^{n+1})/(1-s)$ for some $r > 1$ and $s = r^{-2} < 1$ [23, lemma 2.1].

In the case of the orthogonal free quantum groups, there is only one irreducible representation of a given length, and consequently (4.1) is equivalent to Property RD. More precisely we have by Lemma 4.6

$$||p_l \mathcal{F}(a) p_k \Lambda(b)|| = ||p_l \mathrm{Conv}(b \otimes a)||_2 = \sqrt{\frac{m_k m_n}{m_l}}\, ||\delta(p_l) x \delta(p_l)||_2,$$

for $a \in p_n S$ and $b \in p_k S$. As a result caracterization (v) of Property RD is satisfied **iff** we have, for any integers k, l, n and every $a \in p_n S$, $b \in p_k S$:

$$||\delta(p_l)(b \otimes a)\delta(p_l)||_2 \leq \sqrt{\frac{m_l}{m_n m_k}} P(n)\, ||b||_2\, ||a||_2. \tag{4.4}$$

Let us remark that the numerical coefficient in the right-hand side tends to zero as n goes to infinity. Hence the free quantum group under consideration has Property RD **iff** the cone of decomposable tensors in $H_k \otimes H_n$ is "asymptotically far" from the subspace equivalent to H_l. Like in Section 4.2, we will study this geometric condition in the identifications $p_n S \simeq L(H_n)$ and using the twisted Hilbert-Schmidt norms.

For any representation $\alpha \in \mathcal{C}$ we have $\bar{\alpha} = \alpha$, recall that we denote by $t_\alpha : \mathbb{C} = H_{1_\mathcal{C}} \to H_\alpha \otimes H_\alpha$ the morphism associated to a normalized conjugation map on H_α. For any Hilbert spaces H, H' we will also call t_α the map $\mathrm{id} \otimes t_\alpha \otimes \mathrm{id} : H \otimes H' \to H \otimes H_\alpha \otimes H_\alpha \otimes H'$. When $\alpha = \alpha_k^{\otimes n}$ we use the conjugation map $j_{\alpha_k^{\otimes n}} = \Sigma \circ (j_\alpha \otimes j_{\alpha_k^{\otimes n-1}})$ and the notation $t_k^n := t_\alpha$.

LEMMA 4.8. *Let L be the word length induced by $\mathcal{D} = \{\alpha_1\}$ on the dual of some $A_o(Q)$ with $N \geq 3$. Then (S, δ, L) has Property RD **iff** there exists $P \in \mathbb{R}[X]$ such that for all $k, l, n \in \mathbb{N}$, $a \in L(H_n)$, $b \in L(H_k)$:*

$$||t_1^{q*} \delta(p_l)(b \otimes a)\delta(p_l) t_1^q||_{HS} \leq P(n)\, ||b||_{HS}\, ||a||_{HS},$$

where we put $q = (n+k-l)/2$ and the norm in the left-hand side is the twisted Hilbert-Schmidt norm on $L(H_{k-q} \otimes H_{n-q})$.

Proof. We have $(b \otimes a)\delta(p_l) t_1^q = (b \otimes a)(p_k \otimes p_n) t_1^q \delta(p_l)$. Since $(p_k \otimes p_n) t_1^q : H_{k-q} \otimes H_{n-q} \to H_k \otimes H_n$ is a morphism, it is a multiple of an isometry on the highest homogeneous subspace $H_l \simeq \delta(p_l)(H_{k-q} \otimes H_{n-q})$. In view of the characterization of Property RD given by Inequality (4.4), the proof reduces to controlling the norm of $(p_k \otimes p_n) t_1^q \delta(p_l)$. To do so we notice that

$$(p_k \otimes p_n) t_1^q = (p_k \otimes p_n) t_1 \circ (p_{k-1} \otimes p_{n-1}) t_1 \circ \cdots \circ (p_{k-q+1} \otimes p_{n-q+1}) t_1.$$

Now, the norm of each morphism $T_{p,p'} := (p_{p+1}\otimes p_{p'+1})t_1$ on the subspace of $H_p\otimes H_{p'}$ equivalent to H_l is given by [23, prop. 2.3]:

$$||T_{p,p'}\delta(p_l)||^2 = \frac{m_{p+1}}{m_p}\left(1-\frac{m_{p-q}m_{p'-q-1}}{m_{p+1}m_{p'}}\right) =: \frac{m_{p+1}}{m_p}N^l_{p,p'},$$

with $q = \frac{p+p'-l}{2}$. So the numeric quantity we have to control is the following one:

$$\frac{m_k}{m_{k-q}}\sqrt{\frac{m_l}{m_n m_k}}\ \cdot\ (N^l_{k-q,n-q}\cdots N^l_{k-2,n-2}N^l_{k-1,n-1}).$$

Using the explicit expression of m_p given at the beginning of the section, it is a boring but easy exercise to check that this quantity is bounded from above and from below by two non-zero constants independant of k, l, n. ∎

THEOREM 4.9. *Let $Q \in M_N(\mathbb{C})$ be an invertible matrix with $\bar{Q}Q \in \mathbb{C}I_N$ and $N \geq 3$. Then the dual of $A_o(Q)$ has Property RD with respect to the natural word length* ***iff*** *Q is unitary up to a scalar.*

Proof. We have already seen that the dual of $A_o(Q)$ does not have Property RD when $Q \notin \mathbb{C}U(N)$, and hence we restrict to the case $Q \in U(N)$. Then $F = 1$ and in particular there is no twisting in the Hilbert-Schmidt structures. Let (e_I) be a orthonormal basis of $L(H^{\otimes q})$, ie a basis such that Tr $(e_I^* e_J) = \delta_{I,J}$ for all I, J. We put $\bar{e}_I = j^* e_I j \in L(H^{\otimes q})$, in our unimodular case $(\bar{e}_I)$ is again an orthonormal basis.

Take $a \in L(H_n)$ and $b \in L(H_k)$. We consider H_n (resp. H_k) as the highest homogeneous subspace of $H_1^{\otimes n}$ (resp. $H_1^{\otimes k}$) and we simply denote by p_n (resp. p_k) the corresponding orthogonal projection. Write $p_n a p_n = \sum \bar{e}_I\otimes a_I$ and $p_k b p_k = \sum b_I\otimes e_I$ with $a_I \in L(H_{n-q})$, $b_I \in L(H_{k-q})$. We have, using the identity (4.3):

$$\begin{aligned} t_1^{q*}(b\otimes a)t_1^q &= (\mathrm{id}\otimes t_1^{q*}\otimes id)(p_k b p_k\otimes p_n a p_n)(\mathrm{id}\otimes t_1^q\otimes id) \\ &= \sum b_I\otimes t_1^{q*}(e_I\otimes\bar{e}_J)t_1^q\otimes a_J = \sum b_I\otimes a_I. \end{aligned}$$

From this we get a first upper bound for the left-hand side of the inequality of Lemma 4.8: since $\delta(p_l) \in L(H_k\otimes H_n)$ is an orthogonal projection,

$$||\delta(p_l)t_1^{q*}(b\otimes a)t_1^q\delta(p_l)||_{HS} \leq ||t_1^{q*}(b\otimes a)t_1^q||_{HS} = ||\sum b_I\otimes a_I||_{HS}.$$

We then use the triangle and the Cauchy-Schwartz inequalities:

$$||\sum b_I\otimes a_I||^2_{HS} \leq \left(\sum ||b_I||_{HS}||a_I||_{HS}\right)^2 \leq \sum ||b_I||^2_{HS}\sum ||a_I||^2_{HS}.$$

Since (e_I), $(\bar{e}_I)$ are orthonormal bases, we have $||a||^2_{HS} = \sum ||a_I||^2_{HS}$ and $||b||^2_{HS} = \sum ||b_I||^2_{HS}$. Hence the above estimate shows that the condition of Lemma 4.8 for Property RD is fulfilled with $P = 1$. ∎

We will now address briefly the case of the unitary free quantum groups $A_u(Q)$ with $N \geq 3$. Its definition is similar to the one of $A_o(Q)$, using the relations that make U and $Q\bar{U}Q^{-1}$ unitary but not equal anymore. The corepresentation U can then be considered as a representation of S. It comes out that the result of Theorem 4.9 also holds for the duals of these quantum groups, the heuristic reason being that $A_u(Q)$ is a mixing of the geometry of $A_o(Q)$ and of the combinatorics of the free group F_2.

Let us recall the structure of $\mathcal{C}$ from [3]: Irr $\mathcal{C}$ can be identified with the free monoid on two generators U, $\bar{U}$ in such a way that the involutive semi-ring structure is given by $\overline{\alpha U} = \bar{U}\bar{\alpha}$, $\overline{U\alpha} = \bar{\alpha}\bar{U}$ and the recursive identities

$$\alpha U \otimes U\beta = \alpha UU\beta, \alpha U \otimes \bar{U}\beta = \alpha U\bar{U}\beta \oplus \alpha \otimes \beta.$$

In particular the word length of the free monoid coincides with the word length on Irr $\mathcal{C}$ associated to the generating subset $\mathcal{D} = \{U, \bar{U}\}$.

THEOREM 4.10. *The dual of $A_u(Q)$, with $Q \in GL_N(\mathbb{C})$ and $N \geq 3$, has Property RD with respect to the natural word length* ***iff*** *Q is unitary up to a scalar.*

Proof. Like in the orthogonal case m_U is the geometric mean of Tr Q^*Q and Tr $(Q^*Q)^{-1}$ and hence the dual of $A_u(Q)$ is unimodular **iff** $Q \in \mathbb{C}U(N)$. When this is not the case, we already know that Property RD is not satisfied. Moreover in the unimodular case Q can be replaced with I_N without changing the discrete quantum group under consideration.

One can then follow the arguments of the orthogonal case to check that the necessary condition (4.1) is still satisfied. As a matter of fact Lemma 4.8 relies on the technical result [23, prop. 2.3] which holds in the unitary case for representations $\bar{\beta}$, α of the form $U\bar{U}U\cdots$, with respective lengths k, n. The reduction to this case is straightforward because $\alpha' UU\bar{U}U\cdots \simeq \alpha' U \otimes U\bar{U}U\cdots$, compare [23, rem. 6.4.2]. In this way one obtains for $A_u(I_N)$ the existence of a positive constant C such that

$$\forall\, \alpha, \beta, \gamma \in \text{Irr}\, \mathcal{C},\ a, b \in \mathcal{S}\ ||p_\gamma \mathcal{F}(p_\alpha a)\Lambda(p_\beta b)||_2 \leq C\ ||p_\alpha a||_2 ||p_\beta b||_2. \tag{4.5}$$

Because the combinatorics of the free monoid Irr $\mathcal{C}$ is analogous (and in fact simpler for our purposes) to the one of the free group, one can show that this property is in fact sufficient, by adapting the ideas of [9, lemma 1.3] in the following way.

Let us fix n, k, $l \in \mathbb{N}$, $a \in p_n S$, $b \in p_k S$, and put $q = \frac{n-k-l}{2}$. For any α, β, $\gamma \in$ Irr $\mathcal{C}$ such that $p_\gamma \mathcal{F}(p_\alpha a)\Lambda(p_\beta b)$ is non zero, there is an inclusion $\gamma \subset \beta \otimes \alpha$, and hence we can write $\alpha = \tau\alpha'$, $\beta = \beta'\bar{\tau}$ and $\gamma = \beta'\alpha'$ with $l(\tau) = q$. Moreover, all triples (τ, α', β') with $l(\tau) = q$, $l(\alpha') = n - q$ and $l(\beta') = k - q$ are obtained exactly once in this way. Using this "change of indices" one can prove Property RD via the last characterization of Definition 3.5. We compute indeed, for $a \in p_n\mathcal{S}$ and $b \in p_k\mathcal{S}$:

$$\begin{aligned}||p_l\mathcal{F}(a)\Lambda(b)||^2 &= \Big|\Big|\sum_{l(\gamma)=l}\sum_{\substack{l(\alpha)=n\\ l(\beta)=k}} p_\gamma\mathcal{F}(p_\alpha a)\Lambda(p_\beta b)\Big|\Big|^2\\ &= \sum_{\substack{l(\alpha')=n-q\\ l(\beta')=k-q}}\Big|\Big|\sum_{l(\tau)=q} p_{\beta'\alpha'}\mathcal{F}(p_{\tau\alpha'}a)\Lambda(p_{\beta'\bar{\tau}}b)\Big|\Big|^2,\end{aligned}$$

because the projections $p_{\beta'\alpha'}$ are mutually orthogonal for different values of β', α'. We use then the triangle inequality, (4.5) and the Cauchy-Schwartz inequality:

$$\begin{aligned}||p_l\mathcal{F}(a)\Lambda(b)||^2 &\leq C^2\sum_{\alpha',\beta'}\Big(\sum_\tau ||p_{\tau\alpha'}a||_2||p_{\beta'\bar{\tau}}b||_2\Big)^2\\ &\leq C^2\sum_{\alpha',\beta'}\Big(\sum_\tau ||p_{\tau\alpha'}a||_2^2\Big)\Big(\sum_\tau ||p_{\beta'\bar{\tau}}b||_2^2\Big) = C^2||a||_2^2||b||_2^2. \quad \blacksquare\end{aligned}$$

5. *K*-THEORY

In this section we will check that the classical applications of Property RD to K-theory still hold in the quantum case. More precisely, if (S,δ,L) has Property RD we will prove that the subspaces $\hat{H}_L^\infty$ and $\hat{H}_L^s$, for s big enough, are subalgebras of $\hat{S}$ having the same K-theory as $\hat{S}$: compare [11, thm. A] and [14, prop. 1.2] respectively. Of course we restrict ourselves to unimodular discrete quantum groups, since we have seen in Section 4.2 that unimodularity is necessary for Property RD to hold.

In fact following the methods of [10] and [14] this goes down to establishing some norm inequalities, which we do at Proposition 5.2 and Proposition 5.5. In particular in this section the difficulties of the quantum generalization are only of technical nature. However, after having presented a definition and quantum examples, it is also important to know that the applications are still working.

5.1. THE FRÉCHET ALGEBRA $\hat{H}_L^\infty$.

Let L be a closed operator on H admitting $\mathcal{H}$ as a core. For any bounded $x \in B(H)$, the commutator $[L,x]$ is a priori an unbounded operator which needs not to be closable nor densely defined. Let Dom $D \subset B(H)$ be the subspace of operators x such that $x\mathcal{H} \subset$ Dom L and $[L,x]$ is bounded on $\mathcal{H}$, and let us denote by $D(x) \in B(H)$ the closure of $[L,x]$, for $x \in$ Dom D. This defines an unbounded linear map D : Dom $D \to B(H)$. Because L is closed, it is a standard fact that D is a closed derivation.

LEMMA 5.1. *Let L be a length on the Hopf C^*-algebra (S,δ) of a discrete quantum group. If p_0 has finite rank, we have $\hat{S} \cap$ Dom $D^k \subset \hat{H}_L^k$ (as subspaces of H).*

Proof. Let $e \in H$ be a co-fixed unit vector for V and $\Lambda : a \mapsto \mathcal{F}(a)e$ the associated GNS map for h_R. Since $\varepsilon(L) = 0$ we have $Le = 0$: one can indeed check that $\Lambda(p_\varepsilon)$ is a multiple of e, using the expression of V on the image of $\Lambda\otimes\Lambda$. In particular we have $D(x)e = Lxe$ for any $x \in$ Dom D.

It is easy to check by induction that for any $x \in \mathrm{Dom}\ D^k$ we have $xe \in \mathrm{Dom}\ L^k$ and $D^k(x)e = L^k(xe)$: if this holds, let x be in the domain of D^{k+1}, we have $D^k(x) \in \mathrm{Dom}\ D$ so that $D^k(x)e = L^k(xe) \in \mathrm{Dom}\ L$, and hence $xe \in \mathrm{Dom}\ L^{k+1}$. Moreover $D^{k+1}(x)e = D(D^k(x))e = LD^k(x)e = L^{k+1}xe$. In particular if $\hat{a} \in \hat{S} \cap \mathrm{Dom}\ D^k$, then $ae \in \mathrm{Dom}\ L^k$.

Now observe that $L(1-p_0) \leq (1+L)(1-p_0) \leq 2L(1-p_0)$. Hence if p_0 has finite rank we have $\mathrm{Dom}\ L^k = \mathrm{Dom}\ (1+L)^k$. This concludes the proof because $\hat{H}_L^k = \mathrm{Dom}\ (1+L)^k$ by definition. ∎

PROPOSITION 5.2. *Let L be a central length on the Hopf C^*-algebra (S, δ) of a unimodular discrete quantum group. For any $k \in \mathbb{N}$ we have $\hat{\mathcal{S}} \subset \mathrm{Dom}\ D^k$. Moreover if (S, δ, L) has Property RD with constants s, C we have*

$$\forall k \in \mathbb{N},\ \hat{a} \in \hat{\mathcal{S}}\ ||D^k(\hat{a})|| \leq 4C||\hat{a}||_{2,s+k}.$$

Proof. Proceeding by induction, we assume that the result holds for $k-1$. For $\hat{a} \in \hat{\mathcal{S}}$ it is clear that $D^{k-1}(\hat{a})$ stabilizes $\mathcal{H}$, and hence $[L, D^{k-1}(\hat{a})]$ is defined on $\mathcal{H}$ as well as its adjoint. We denote this operator by $D^k(\hat{a})$ and we want to show that it is bounded. For $a \in \mathcal{S}$ it is easy to check by induction that

$$D^k(\mathcal{F}a) = (\mathrm{id}\otimes h_R)(V^*(\delta(L) - L\otimes 1)^k(1\otimes a)).$$

Note that this is just the definition of $\mathcal{F}(a)$ for $k = 0$, and use the identity $(L\otimes 1)V^* = V^*\delta(L)$ to proceed to the induction. Using the expression of V on the image of $\Lambda\otimes\Lambda$ recalled in Section 2 we obtain

$$D^k(\mathcal{F}a)^*\Lambda(b) = (\Lambda\otimes h_R)((1\otimes a^*)(\delta(L) - L\otimes 1)^k\delta(b)).$$

We first assume that a, $b \in \mathcal{S}$ are positive. By the first point of Lemma 3.4 and since $\delta(b)$ commutes to $\delta(L) - L\otimes 1$ on $\mathcal{H}\otimes\mathcal{H}$ we have

$$-(1\otimes L^k)\delta(b) \leq (\delta(L) - L\otimes 1)^k\delta(b) \leq (1\otimes L^k)\delta(b).$$

Because h_R is central, this yields

$$(\mathrm{id}\otimes h_R)((1\otimes a)(\delta(L) - L\otimes 1)^k\delta(b)) \leq (\mathrm{id}\otimes h_R)((1\otimes aL^k)\delta(b))$$

and similarly with the left inequality. But one can check that, for a central weight φ, inequalities of the form $-s \leq t \leq s$ with $t = t^* \in \mathcal{S}$ and $s \in \mathcal{S}_+$ imply the inequality $||t||_\varphi \leq ||s||_\varphi$ of the GNS norms. As a result we obtain

$$||D^k(\mathcal{F}a)^*\Lambda(b)||_2 \leq ||\mathcal{F}(L^k a)^*\Lambda(b)||_2 \leq ||\mathcal{F}(L^k a)||\ ||b||_2.$$

This result is then easily generalized to any $b \in \mathcal{S}$, exactly like in the proof of Proposition 4.4. Hence we have shown that $D^k(\mathcal{F}a)$ is bounded. Moreover if Property RD is satisfied we have the following estimate on its norm:

$$||D^k(\mathcal{F}a)|| \leq 2\ ||\mathcal{F}(L^k a)|| \leq 2C\ ||L^k a||_{2,s} \leq 2C\ ||a||_{2,s+k}.$$

Again this can be generalized to any $a \in \mathcal{S}$ and we get the estimate of the statement, with $\hat{a} = \mathcal{F}(a)$. ∎

COROLLARY 5.3. *Let (S,δ) be the Hopf C^*-algebra of a unimodular, finitely generated discrete quantum group with Property RD, and L a word length on it. Then $\hat{H}_L^\infty$ is dense in $\hat{S}$ and coincides with $\bigcap \mathrm{Dom}\ D^k \cap \hat{S}$. In particular it is a dense subalgebra which is stable under holomorphic functional calculus in $\hat{S}$, and the inclusion $\hat{H}_L^\infty \subset \hat{S}$ induces isomorphisms in K-theory.*

Proof. Let s be an exponent realizing Property RD, and $k \in \mathbb{N}$. Let $\hat{a}$ be an element of $\hat{H}_L^{k+s}$: there exists a sequence $\hat{a}_n$ in $\hat{\mathcal{S}}$ converging to $\hat{a}$ in the $(2,k+s)$-norm. It is easy to check by induction that $\hat{a} \in \mathrm{Dom}\ D^l$ and $D^l\hat{a}_n \to D^l\hat{a}$ for $l = 0,\dots,k$. As a matter of fact, $(\hat{a}_n)_n$ converges in particular in the $(2,l+s)$ norm, hence by Proposition 5.2 the sequence $D^l\hat{a}_n = D(D^{l-1}\hat{a}_n)$ has a limit in $B(H)$. Since $D^{l-1}\hat{a}_n \to D^{l-1}\hat{a}$ and D is closed, this implies that $D^{l-1}\hat{a} \in \mathrm{Dom}\ D$ and $D^l\hat{a}_n \to D^l\hat{a}$.

Hence we have proved that $\hat{H}_L^{k+s} \subset \mathrm{Dom}\ D^k \cap \hat{S}$ for all k. Because we are using a word length, the hypothesis of Lemma 5.1 is satisfied and we also have $\mathrm{Dom}\ D^k \cap \hat{S} \subset \hat{H}_L^k$. This proves that $\hat{H}_L^\infty = \bigcap \mathrm{Dom}\ D^k \cap \hat{S}$. This subspace is dense because it contains $\hat{\mathcal{S}}$. It is then a general fact for closed derivations that $\hat{H}_L^\infty$ is a dense subalgebra stable under holomorphic functional calculus in $\hat{S}$, cf [10, thm. 1.2]. This implies in turn that the canonical inclusion induces isomorphisms in K-theory, cf e.g. [17, prop. 8.14] for a recent statement of this classical result. ∎

5.2. THE BANACH ALGEBRAS $\hat{H}_L^s$. We start with a Lemma which is proved using the same techniques as for Proposition 5.2:

LEMMA 5.4. *Let L be a central length on the Hopf C^*-algebra (S,δ) of a unimodular discrete quantum group. For any $(a_i) \in \mathcal{S}_+^n$ and $s,t \geq 0$ we have*

$$||\mathcal{F}(a_1)\cdots\mathcal{F}(a_n)||_{2,s+t} \leq n^t \textstyle\sum_i ||\mathcal{F}(a_1)\cdots\mathcal{F}((1+L)^t a_i)\cdots\mathcal{F}(a_n)||_{2,s}.$$

Proof. Using the identity $(\mathrm{id}\otimes\kappa)(V) = V^*$ and the fact that $h_R\kappa^{-1} = h_R$ in the unimodular case, one sees that $\mathcal{F}(a)^* = \mathcal{F}(\kappa(a^*))$. Since $\kappa(\mathcal{S}_+) = \mathcal{S}_+$ by unimodularity and $\kappa(L^*) = L$ by hypothesis, this allows to replace on both sides of the statement the first $n-1$ terms $\mathcal{F}(\cdot)$ by $\mathcal{F}(\cdot)^*$. In this way we avoid using κ in the rest of the proof. We have indeed, using the identity $V_{12}V_{13}\cdots V_{1n} = (\mathrm{id}\otimes\delta^{n-2})(V)$:

$$\begin{aligned}\mathcal{F}(a_1)^*\cdots\mathcal{F}(a_{n-1})^*\mathcal{F}(a_n)e &= \mathcal{F}(a_1)^*\cdots\mathcal{F}(a_{n-1})^*\Lambda(a_n)\\ &= (\Lambda\otimes h^{\otimes n-1})((1\otimes a_1\otimes\cdots\otimes a_{n-1})\delta^{n-1}(a_n)).\end{aligned}$$

Like in the classical case the proof relies on the following elementary inequality. For $t \geq 0$ the function $f_t : x \mapsto (1+x)^t$ is growing on $\mathbb{R}_+$ and hence we have, for any $(x_i) \in \mathbb{R}_+^n$:

$$\begin{aligned}(1+\textstyle\sum x_i)^t &\leq n^t(1+\textstyle\sum \frac{x_i}{n})^t\\ &\leq n^t(1+\max x_i)^t \leq n^t\textstyle\sum(1+x_i)^t.\end{aligned}$$

We apply this inequality to the following iterated version of the first point of Lemma 3.4, whose right-hand side is a sum of n commuting terms:

$$\begin{aligned} & L\otimes 1\otimes\cdots\otimes 1 \leq 1\otimes L\otimes 1\otimes\cdots+\ldots+1\otimes\cdots\otimes 1\otimes L+\delta^{n-1}(L) \\ \Longrightarrow \quad & f_t(L)\otimes 1^{\otimes n-1} \leq n^t\left(1\otimes f_t(L)\otimes 1^{\otimes n-2}+\ldots+\delta^{n-1}(f_t(L))\right). \end{aligned}$$

This inequality can be multiplied by $(f_s(L)\otimes 1^{\otimes n-1})\delta^{n-1}(a_n)$, which is positive and commutes to all the terms. Applying moreover the positive functional $\mathrm{id}\otimes h_R a_1\otimes\cdots\otimes h_R a_{n-1}$ we obtain

$$\begin{aligned} (f_{s+t}(L)\otimes h^{\otimes n-1})((1\otimes a_1\otimes\cdots\otimes a_{n-1})\delta^{n-1}(a_n)) \leq \\ \leq n^t(f_s(L)\otimes h^{\otimes n-1})((1\otimes f_t(L)a_1\otimes a_2\otimes\cdots\otimes a_{n-1})\delta^{n-1}(a_n))+ \\ +\ldots+n^t(f_s(L)\otimes h^{\otimes n-1})((1\otimes a_1\otimes\cdots\otimes a_{n-1})\delta^{n-1}(f_t(L)a_n)). \end{aligned}$$

Since h_R is tracial, this inequality between positive elements of $\mathcal{S}$ implies the inequality of their 2-norms and finally, by the triangle inequality, the inequality of the statement. ∎

PROPOSITION 5.5. *Let L be a central length on the Hopf C^*-algebra (S,δ) of a unimodular discrete quantum group. Assume that (S,δ,L) has Property RD with exponent s_0. For any $s\geq s_0$, $\hat{H}_L^s$ is a Banach subalgebra of $\hat{S}$. Moreover for any $t\geq 0$ there exists a constant $K_{s,t}>0$ such that*

$$\forall \hat{a}\in\hat{\mathcal{S}}\ \exists C>0\ \forall n\in\mathbb{N}\ ||\hat{a}^n||_{2,s+t}\leq CK_{s,t}^n||\hat{a}||_{2,s}^n.$$

Proof. We first apply Lemma 5.4 with $n=2$ and $s=0$. Taking into account the unimodular fact that $||\hat{a}^*||_2=||\hat{a}||_2$ for $\hat{a}\in\hat{S}$, we obtain

$$\begin{aligned} ||\mathcal{F}(a_1)\mathcal{F}(a_2)||_{2,t} \quad \leq \quad & 2^t||\mathcal{F}(a_2)||\ ||\mathcal{F}((1+L)^t a_1)||_2+ \\ & +2^t||\mathcal{F}(a_1)||\ ||\mathcal{F}((1+L)^t a_2)||_2 \end{aligned}$$

for $a_1, a_2\in\mathcal{S}_+$. By definition we have $||\mathcal{F}((1+L)^t a_i)||_2=||\mathcal{F}(a_i)||_{2,t}$ and using moreover Property RD, which holds for any exponent $t\geq s_0$, we get

$$||\mathcal{F}(a_1)\mathcal{F}(a_2)||_{2,t}\leq 2^{t+1}C||\mathcal{F}(a_1)||_{2,t}||\mathcal{F}(a_2)||_{2,t}.$$

Now this extends to any $a, a'\in\mathcal{S}$ like in the proof of Proposition 4.4. This proves that $||\hat{a}\hat{a}'||_{2,t}\leq 2^{t+3}C||\hat{a}||_{2,t}||\hat{a}'||_{2,t}$ for any $\hat{a}, \hat{a}'\in\hat{\mathcal{S}}$ and hence the first statement.

Fix $s\geq s_0$ and let $K_s\geq 1$ be such that $||\hat{a}\hat{a}'||_{2,s}\leq K_s||\hat{a}||_{2,s}||\hat{a}'||_{2,s}$ for any $\hat{a}$, $\hat{a}'\in\hat{\mathcal{S}}$. We get from Lemma 5.4, for any $(a_i)\in\mathcal{S}_+^n$ and $t\geq 0$:

$$\begin{aligned} & ||\mathcal{F}(a_1)\cdots\mathcal{F}(a_n)||_{2,s+t}\leq \\ & \leq n^t K_s^{n-1}\textstyle\sum_i ||\mathcal{F}(a_1)||_{2,s}\cdots||\mathcal{F}(a_i)||_{2,s+t}\cdots||\mathcal{F}(a_n)||_{2,s}. \end{aligned} \tag{5.1}$$

Now let b be an element of $\mathcal{S}$ and write again the "canonical decomposition" $b=\sum_{k=0}^3 i^k b_k$, where the elements $b_k\in\mathcal{S}_+$ are the positive and negative parts of

Re b and Im b. Using the triangle and Cauchy-Schwartz inequalities as well as (5.1) we write

$$\begin{aligned}||\mathcal{F}(b)^n||_{2,s+t}^2 &\leq (\textstyle\sum_{(k_j)_j} ||\mathcal{F}(b_{k_1})\cdots\mathcal{F}(b_{k_n})||_{2,s+t})^2\\ &\leq (\textstyle\sum_{i,(k_j)_j} n^t K_s^n ||\mathcal{F}(b_{k_1})||_{2,s}\cdots||\mathcal{F}(b_{k_i})||_{2,s+t}\cdots||\mathcal{F}(b_{k_n})||_{2,s})^2\\ &\leq n^{2t}K_s^{2n} n 4^n \textstyle\sum_{i,(k_j)_j} ||\mathcal{F}(b_{k_1})||_{2,s}^2\cdots||\mathcal{F}(b_{k_i})||_{2,s+t}^2\cdots||\mathcal{F}(b_{k_n})||_{2,s}^2.\end{aligned}$$

As already mentioned we have $||b||_{2,s}^2 = \sum ||b_k||_{2,s}^2$ because h_R is tracial. Hence the last upper bound has the right form:

$$\begin{aligned}n^{2t+1}(2K_s)^{2n}\textstyle\sum_i ||\mathcal{F}(b)||_{2,s}^2\cdots||\mathcal{F}(b)||_{2,s+t}^2\cdots||\mathcal{F}(b)||_{2,s}^2 &=\\ = n^{2t+2}(2K_s)^{2n}||\mathcal{F}(b)||_{2,s+t}^2\,||\mathcal{F}(b)||_{2,s}^{2n-2}.\quad\blacksquare\end{aligned}$$

COROLLARY 5.6. *Let L be a central length on the Hopf C^*-algebra (S,δ) of a unimodular discrete quantum group. Assume that (S,δ,L) has Property RD with exponent s_0. Then the canonical inclusion of Banach algebras $\hat{H}_L^s \subset \hat{S}$ induces isomorphisms in K-theory for any $s \geq s_0$.*

Proof. The proof goes exactly like in [14, prop. 1.2]. Denote by $\rho_s(\hat{a})$ the spectral radius of $\hat{a}$ in $\hat{H}_L^s$. Taking the nth-root and letting n go to infinity in the estimate of Proposition 5.5, we see that $\rho_{s+t}(\hat{a}) \leq K_{s,t}||\hat{a}||_{2,s}$ for any $\hat{a} \in \hat{\mathcal{S}}$. Applying this to $\hat{a}^n$ and repeating the same process yields $\rho_{s+t}(\hat{a}) \leq \rho_s(\hat{a})$, hence $\hat{a}$ has the same spectral radius in all the Banach algebras $\hat{H}_L^s$ with $s \geq s_0$.

We use then an interpolation inequality for our Sobolev spaces which results from Hölder's inequality for the series $\sum_\alpha (1+l(\alpha))^{2s}||p_\alpha a||_2^2$, as in the classical case. We obtain more precisely, for $s' > s \geq 0$:

$$\forall n \in \mathbb{N}\ ||\hat{a}^n|| \geq ||\hat{a}^n||_2 \geq ||\hat{a}^n||_{2,s}^{s'/(s'-s)}||\hat{a}^n||_{2,s'}^{-s/(s'-s)},$$

if $\hat{a} \in \hat{\mathcal{S}}$ is such that $\rho_{s'}(\hat{a}) \neq 0$. Again this yields an inequality between spectral radii, which reads $\rho_{\hat{S}}(\hat{a}) \geq \rho_s(\hat{a}) = \rho_{s'}(\hat{a})$ when $s, s' \geq s_0$. Since $\hat{H}_L^s$ is dense in $\hat{S}$, this proves that it is stable under holomorphical calculus. $\blacksquare$

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REFERENCES

[1] SAAD BAAJ AND GEORGES SKANDALIS, Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres, *Ann. Sci. École Norm. Sup. (4)* **26** (1993), 425–488.

[2] TEODOR BANICA, Théorie des représentations du groupe quantique compact libre $O(n)$, *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996), 241–244.

[3] TEODOR BANICA, Le groupe quantique compact libre $\mathrm{U}(n)$, *Comm. Math. Phys.* **190** (1997), 143–172.

[4] TEODOR BANICA, Representations of compact quantum groups and subfactors, *J. Reine Angew. Math.* **509** (1999), 167–198.

[5] JULIEN BICHON, AN DE RIJDT, AND STEFAAN VAES, Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups, *Comm. Math. Phys.* **262** (2006), 703–728.

[6] N. BOURBAKI, *Espaces vectoriels topologiques sur un corps valué*, volume 1 of *Espaces vectoriels topologiques*, In *Éléments de Mathématiques*, Masson, Paris 1981.

[7] N. BOURBAKI, *Groupes de Lie réels compacts*, volume 9 of *Groupes et Algèbres de Lie*, In *Éléments de Mathématiques*, Masson, Paris 1982.

[8] EDWARD G. EFFROS AND ZHONG-JIN RUAN, Discrete quantum groups. I. The Haar measure, *Internat. J. Math.* **5** (1994), 681–723.

[9] UFFE HAAGERUP, An example of a nonnuclear C^*-algebra, which has the metric approximation property, *Invent. Math.* **50** (1978/79), 279–293.

[10] RONGHUI JI, Smooth dense subalgebras of reduced group C^*-algebras, Schwartz cohomology of groups, and cyclic cohomology, *J. Funct. Anal.* **107** (1992), 1–33.

[11] PAUL JOLISSAINT, K-theory of reduced C^*-algebras and rapidly decreasing functions on groups, *K-Theory* **2** (1989), 723–735.

[12] PAUL JOLISSAINT, Rapidly decreasing functions in reduced C^*-algebras of groups, *Trans. Amer. Math. Soc.* **317** (1990), 167–196.

[13] JOHAN KUSTERMANS AND STEFAAN VAES, Locally compact quantum groups, *Ann. Sci. École Norm. Sup. (4)* **33** (2000), 837–934.

[14] VINCENT LAFFORGUE, A proof of property (RD) for cocompact lattices of $\mathrm{SL}(3,\mathbf{R})$ and $\mathrm{SL}(3,\mathbf{C})$, *J. Lie Theory* **10** (2000), 255–267.

[15] VINCENT LAFFORGUE, K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes, *Invent. Math.* **149** (2002), 1–95.

[16] PIOTR PODLEŚ AND STANISŁAW L. WORONOWICZ, Quantum deformation of Lorentz group, *Comm. Math. Phys.* **130** (1990), 381–431.

[17] ALAIN VALETTE, *Introduction to the Baum-Connes conjecture*, Lectures in Mathematics ETH Zürich, from notes taken by Indira Chatterji, with an appendix by Guido Mislin, Birkhäuser Verlag, Basel 2002.

[18] JEAN-MICHEL VALLIN, C^*-algèbres de Hopf et C^*-algèbres de Kac, *Proc. London Math. Soc. (3)* **50** (1985), 131–174.

[19] ALFONS VAN DAELE, Multiplier Hopf algebras, *Trans. Amer. Math. Soc.* **342** (1994), 917–932.

[20] ALFONS VAN DAELE, Discrete quantum groups, *J. Algebra* **180** (1996), 431–444.

[21] ALFONS VAN DAELE AND SHUZHOU WANG, Universal quantum groups, *Internat. J. Math.* **7** (1996), 255–263.

[22] ROLAND VERGNIOUX, K-amenability for amalgamated free products of amenable discrete quantum groups, *J. Funct. Anal.* **212** (2004), 206–221.

[23] ROLAND VERGNIOUX, Orientation of quantum Cayley trees and applications, *J. Reine Angew. Math.* **580** (2005), 101–138.

[24] SHUZHOU WANG, Free products of compact quantum groups, *Comm. Math. Phys.* **167** (1995), 671–692.

[25] STANISŁAW L. WORONOWICZ, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.

[26] STANISŁAW L. WORONOWICZ, Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted $\mathrm{SU}(N)$ groups, *Invent. Math.* **93** (1988), 35–76.

[27] STANISŁAW L. WORONOWICZ, Compact quantum groups, In *Symétries quantiques (Les Houches, 1995)*, pages 845–884, North-Holland, Amsterdam 1998.

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