MAXIMAL AMENABILITY OF THE RADIAL SUBALGEBRA IN FREE QUANTUM GROUP FACTORS

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Abstract. We show that the radial MASA in the orthogonal free quantum group algebra $\mathcal{L}(\mathbb{F}O_N)$ is maximal amenable if N is large enough, using the Asymptotic Orthogonality Property. This relies on a detailed study of the corresponding bimodule, for which we construct in particular a quantum analogue of Rădulescu's basis. As a byproduct we also obtain the value of the Pukánszky invariant for this MASA.

Contents

Introduction

The orthogonal free quantum groups $\mathbb{F}O_N$, for $N \in \mathbb{N}^*$, are discrete quantum groups which were introduced by Wang [Wan95] via their universal C^*-algebra defined by generators and relations:

$$C_u^*(\mathbb{F}O_N) = A_o(N) = C^*(u_{i,j}, 1 \le i,j \le N \mid u = \bar{u}, uu^* = u^*u = 1).$$

Here $u = (u_{i,j})_{i,j}$ is the matrix of generators, u^* is the usual adjoint in $M_N(C_u^*(\mathbb{F}O_N))$, and $\bar{u} = (u_{i,j}^*)_{i,j}$. There is a natural coproduct $\Delta : C_u^*(\mathbb{F}O_N) \to C_u^*(\mathbb{F}O_N) \otimes C_u^*(\mathbb{F}O_N)$ which encodes the quantum group structure, and which turns $C_u^*(\mathbb{F}O_N)$ into a Woronowicz C^*-algebra [Wor98]. In particular $C_u^*(\mathbb{F}O_N)$ is equipped with a canonical Δ-invariant tracial state h. In this article we are interested in the von Neumann algebra $\mathcal{L}(\mathbb{F}O_N) = \lambda(C_u^*(\mathbb{F}O_N))'' \subset B(H)$ generated by the image of $C_u^*(\mathbb{F}O_N)$ in the GNS representation λ associated with h. We still denote $u_{i,j} \in \mathcal{L}(\mathbb{F}O_N)$ the images of the generators.

The von Neumann algebras $\mathcal{L}(\mathbb{F}O_N)$, and their unitary variants $\mathcal{L}(\mathbb{F}U_N)$, can be seen as quantum, or matricial, analogues of the free group factors $\mathcal{L}(F_N)$. More precisely if we denote $FO_N = (\mathbb{Z}/2)^{*N}$, with canonical generators a_i, $1 \le i \le N$, we have a surjective $*$-homomorphism $\pi : C_u^*(\mathbb{F}O_N) \to C_u^*(FO_N)$, $u_{i,j} \mapsto \delta_{i,j} a_i$ compatible with coproducts. It turns out that this analogy is fruitful also at an analytical level: one can show that $\mathcal{L}(\mathbb{F}O_N)$ shares many properties with $\mathcal{L}(FO_N)$ and $\mathcal{L}(F_N)$, although the existence of π, which has a huge kernel, is useless to prove such properties. For instance, $\mathcal{L}(\mathbb{F}O_N)$ is non amenable for $N \ge 3$ [Ban97], and in fact it is a full and prime II_1 factor [VV07] without Cartan subalgebras [Iso15]. On the other hand it is not isomorphic to a free group factor [BV18].

The II_1 factor $M = \mathcal{L}(\mathbb{F}O_N)$ has a natural "radial" abelian subalgebra, $A = \chi_1'' \cap M$ where $\chi_1 = \chi_1^* = \sum_1^N u_{i,i}$ is the sum of the diagonal generators. It was shown, already in [Ban97], that $\chi_1/2$ is a semicircular variable with respect to h, in particular $\|\chi_1\| = 2$ in $\mathcal{L}(\mathbb{F}O_N)$. Since

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$\epsilon(\chi_1) = N$ in $C_u^*(\mathbb{F}O_N)$, this implies the non-amenability of $\mathbb{F}O_N$ for $N \geq 3$. The subalgebra $A \subset M$ is the quantum analogue of the radial subalgebra of $\mathcal{L}(F_N)$, generated by the sum $\chi_1 = \sum_1^N (a_i + a_i^*)$ of the generators $a_i \in F_N$ and their adjoints, which is known to be a maximal abelian subalgebra (MASA) since [Pyt81].

The position of A in M was already investigated in [FV16], where it was shown, for $N \geq 3$, to be a strongly mixing MASA. Note that $\mathbb{F}O_N$ admits deformations $\mathbb{F}O_Q$, where $Q \in M_N(\mathbb{C})$ is an invertible matrix such that $Q\bar{Q} = \pm I_N$. When Q is not unitary, the corresponding von Neumann algebra $M = \mathcal{L}(\mathbb{F}O_Q)$ is a type III factor, at least for small deformations [VV07]. One can still consider the abelian subalgebra $A = \chi_1'' \cap M$, but if Q is not unitary it is not maximal abelian anymore, as shown in [KW22]. More precisely, in this case the inclusion $A \subset M$ is quasi-split in the sense of [DL84].

The aim of the present article is to pursue the study of [FV16] in the non-deformed case. Our main result is the following theorem, proved at the end of Section 5. Here, and in the rest of the article, we fix a free ultrafilter ω on $\mathbb{N}$, but the result also holds for the Fréchet filter $\omega = \infty$.

Theorem A. *There exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$ the radial subalgebra $A \subset M = \mathcal{L}(\mathbb{F}O_N)$ satisfies the Asymptotic Orthogonality Property: for every $y \in A^\perp \cap M$ and for every bounded sequence of elements $z_r \in A^\perp \cap M$ such that $\forall a \in A$ $\|[a, z_r]\|_2 \to_\omega 0$, we have $(yz_r \mid z_r y) \to_\omega 0$.*

The Asymptotic Orthogonality Property (AOP) originates from Popa's seminal article [Pop83] where it was established for $A = a_1'' \subset \mathcal{L}(F_N)$, the generator MASA in free group factors, and proved to imply maximal amenability. It is often stated in a non-symmetric way, for scalar products of the form $(yz_r \mid z_r' y')$, but the version above is sufficient for our purposes. We can indeed formulate the following corollary, which is a quantum analogue of the result of [CFRW10] about the radial MASA in free group factors.

Corollary B. *There exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$ the radial subalgebra $A \subset M = \mathcal{L}(\mathbb{F}O_N)$ is maximal amenable: for any amenable subalgebra $P \subset M$ such that $A \subset P$, we have $A = P$.*

Proof. Since A is already known to be a singular MASA by [FV16, Corollary 5.8], this follows directly from [CFRW10, Corollary 2.3], whose proof uses only "symmetric" scalar products $(yz_r \mid z_r y)$. □

The proof of Theorem A follows a strategy which can also be traced back to Popa's work on the generator MASA of free group factors. One can identify the following ingredients:

(1) a good description of the A, A-bimodule $H = \ell^2(F_N)$;
(2) a decreasing sequence of subspaces $V_m \subset H$ such that, for $y \in A^\perp \cap M$ fixed and m big enough, $yV_m \perp V_m y$;
(3) the fact that elements $z \in A^\perp \cap M$ almost commuting to A are almost supported in V_m.

In the classical case the arguments for each of the above steps rely on the combinatorics of reduced words in the free group. In the quantum case the techniques are completely different and consist in performing analysis in the Temperley-Lieb category, which is naturally associated with $\mathbb{F}O_N$ as we recall in the preliminary Section 1. We give below more details about the strategy used for each of the three steps, in the classical and quantum cases, and present the organization of the article.

The more precise goal for (1) is to exhibit an orthonormal basis W of the A, A-bimodule $A^\perp \cap H$ with good combinatorial properties, which will allow to carry out computations. In the case of the generator MASA $a_1'' \subset \mathcal{L}(F_N)$, this basis is just given by the set of reduced words in F_N which do not start nor end with a_1 nor a_1^{-1}. In the case of the radial MASA in $\mathcal{L}(F_N)$, a convenient basis was constructed by Rădulescu [Răd91] to show that the radial MASA is singular with Pukánszky invariant $\{\infty\}$.

In Section 2 we construct an analogue $W = \bigsqcup_{k\geq 1} W_k$ of the Rădulescu basis for our free quantum groups. Surprisingly one has to take into account additional symmetries of H given by the rotation maps ρ_k which already played a (minor) role in [FV16]. Using this construction and a result from [FV16], we can already deduce (Corollary 2.13) that the Pukánszky invariant of the radial MASA in $\mathcal{L}(\mathbb{F}O_N)$ is $\{\infty\}$, a result that was missing in [FV16].

From $x \in W$ one can generate a natural $\mathbb{C}$-linear basis $(x_{i,j})_{i,j\in\mathbb{N}}$ of the cyclic submodule AxA. In Rădulescu's case, $(x_{i,j})$ is orthogonal as soon as $x \in W_k$ with $k \geq 2$, and for $k = 1$ it is nevertheless a Riesz basis. In our case, $(x_{i,j})$ is never orthogonal and we have to show that it is a Riesz basis, uniformly over $x \in W$. This is accomplished in Section 3, which is the most challenging technically, and we manage to reach this conclusion only if N is large enough.

The core of the strategy then lies in ingredient (2). In the case of the generator MASA in the free group factor $\mathcal{L}(F_N)$, V_m is simply the subspace of H generated by the reduced words of F_N that begin and end with a "large" power a_1^k of the generator, $|k| \geq m$, without being themselves a power of a_1. We have then clearly $V_m y \perp y V_m$ if $y \in A^\perp \cap M$ is supported on reduced words of length at most m.

In the case of the radial MASA in $\mathcal{L}(F_N)$, V_m is defined in terms of the Rădulescu basis as the subspace generated by the elements $x_{i,j}$, $x \in W$, i, $j \geq m$. We adopt the same definition in the quantum case, using our analogue of the Rădulescu basis, and we show in Section 4 that the orthogonality property $V_m y \perp y V_m$ holds in an approximate sense as $m \to \infty$. Note that we use one of the two main technical tools from [FV16], in an improved version (Lemma 1.6).

In the case of the classical generator MASA, the step (3) follows by observing that if $z \in M$ almost commutes to the powers a_1^k of the generator, then its components supported on a subset $S \subset F_N$ and on the subset $a_1^k S a_1^{-k}$ have approximately the same norm. If $S = S_m$ is the set of words starting with a power at most m (in absolute value) of a_1, for many values of k the subsets $a_1^k S_m a_1^{-k}$ will be pairwise disjoint, so that the norms of the corresponding components of z will be small. One can then show that z is "almost" contained in V_m, in a quantitative way.

In our case, we similarly relate various components of z using the commutator $[\chi_1, z]$, see Proposition 5.5 in Section 5. This requires to determine the structure constants for the left and right action of χ_1 on the basis $(x_{i,j})$ for a given $x \in W$. Then the components of z that we are able to relate in this way are not as simply "localized" as in the classical cases, and moreover the coefficients in these relations are only recursively specified and require a quite delicate analysis to reach the conclusion. For all this it is naturally necessary to know that the families $(x_{i,j})$ are Riesz bases, uniformly with respect to $x \in W$.

Assembling the results obtained in Sections 4 and 5 it is then easy to prove Theorem A.

1. Preliminaries

We denote by $\mathbb{N}$ the set of non-negative integers. Unless otherwise stated, all indices used in the statements belong to $\mathbb{N}$.

In this article, a discrete quantum group $\mathbb{\Gamma}$ is given by a Woronowicz C^*-algebra $C^*(\mathbb{\Gamma})$ [Wor98], i.e. a unital C^*-algebra equipped with a unital $*$-homomorphism $\Delta : C^*(\mathbb{\Gamma}) \to C^*(\mathbb{\Gamma}) \otimes C^*(\mathbb{\Gamma})$ satisfying the following two axioms: i) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ (co-associativity); ii) $\Delta(C^*(\mathbb{\Gamma}))(1 \otimes C^*(\mathbb{\Gamma}))$ and $\Delta(C^*(\mathbb{\Gamma}))(C^*(\mathbb{\Gamma}) \otimes 1)$ span dense subspaces of $C^*(\mathbb{\Gamma}) \otimes C^*(\mathbb{\Gamma})$ (bi-cancellation). This encompasses classical discrete groups, as well as duals of classical compacts groups G, given by $C^*(\mathbb{\Gamma}) = C(G)$.

In this setting, the existence and uniqueness of a bi-invariant state $h \in C^*(\mathbb{\Gamma})^*$, i.e. satisfying the relations $(h \otimes \mathrm{id})\Delta = 1h = (\mathrm{id} \otimes h)\Delta$, were proved by Woronowicz [Wor98] when $C^*(\mathbb{\Gamma})$ is separable, and by Van Daele [VD95] in general. We can consider the GNS representation λ associated with h and we shall mainly work with the corresponding von Neumann algebra $M = \mathcal{L}(\mathbb{\Gamma}) = \lambda(C^*(\mathbb{\Gamma}))''$ represented on the Hilbert space $H = \ell^2(\mathbb{\Gamma})$. We still denote h the factorization of the invariant state to M. As the notation suggests, in the classical case $\mathcal{L}(\Gamma)$ is the usual group von Neumann algebra with its canonical trace, whereas for the dual of a compact group G we have $\mathcal{L}(\mathbb{\Gamma}) = L^\infty(G)$ with the Haar integral.

A corepresentation of $\mathbb{\Gamma}$ is an element $u \in M \bar{\otimes} B(H_u)$ such that $u_{13}u_{23} = (\Delta \otimes \mathrm{id})(u)$. We will work exclusively with unitary and finite-dimensional corepresentations. We denote $\mathrm{Hom}(u, v) \subset B(H_u, H_v)$ the space of intertwiners from u to v, i.e. maps T such that $(1 \otimes T)u = v(1 \otimes T)$. A corepresentation u is irreducible if $\mathrm{Hom}(u,u) = \mathbb{C}\mathrm{id}$; two corepresentations u, v are equivalent if $\mathrm{Hom}(u,v)$ contains a bijection. The tensor product of u and v is $u \otimes v := u_{12}v_{13}$, with $H_{u\otimes v} = H_u \otimes H_v$. We have defined in this way a tensor C^*-category denoted $\mathrm{Corep}(\mathbb{F})$ with a fiber functor to Hilbert spaces.

Let $u \in M \otimes B(H_u)$ be a corepresentation of $\mathbb{F}$. For ζ, $\xi \in H_u$ we can consider the corresponding coefficient $u_{\zeta,\xi} = (\mathrm{id} \otimes \zeta^*)u(\mathrm{id} \otimes \xi) = (\mathrm{id} \otimes \mathrm{Tr})(u(1 \otimes \xi\zeta^*)) \in M$. More generally for $X \in B(H_u)$ we denote $u(X) = (\mathrm{id} \otimes \mathrm{Tr})(u(1 \otimes X))$ — although it would perhaps be more natural to denote this element $u(\varphi)$ where $\varphi = \mathrm{Tr}(\,\cdot\, X) \in B(H_u)^*$.

In the present article we will work only with unimodular discrete quantum groups, equivalently, the canonical state h will be a trace. In this case the Peter-Weyl-Woronowicz orthogonality relations read, for u irreducible:

$$(u(X) \mid u(Y)) = (\dim u)^{-1}(X \mid Y), \tag{1.1}$$

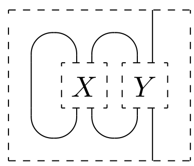
where we use on the left the scalar product associated with h, $(x \mid y) = h(x^*y)$, and on the right the Hilbert-Schmidt scalar product $(X \mid Y) = \mathrm{Tr}(X^*Y)$. On the other hand we have $(u(X) \mid v(Y)) = 0$ if u, v are irreducible and not equivalent.

The product in M can be computed according to the evident formula $u(X)v(Y) = (u \otimes v)(X \otimes Y)$. We have moreover $u(TX) = v(XT)$ for $X \in B(H_u, H_v)$ and $T \in \mathrm{Hom}(v,u)$. As a result, if we choose intertwiners $T_i \in \mathrm{Hom}(w_i, u \otimes v)$ such that $T_i^*T_i = \mathrm{id}$ and $\sum_i T_iT_i^* = \mathrm{id}$, we obtain the formula $u(X)v(Y) = \sum_i w_i(T_i^*(X \otimes Y)T_i)$, which we can use to compute the product of coefficients of irreducible corepresentations as a linear combination of coefficients of irreducible corepresentations.

In this article we consider the orthogonal free quantum groups $\mathbb{F} = \mathbb{F}O_N$ defined in the Introduction, and assume $N \geq 3$. Associated to N is the parameter $q \in]0,1[$ such that $q+q^{-1} = N$, which plays an important role in the computations. We have $q \to 0$ as $N \to \infty$. Banica [Ban96] showed that the C^*-tensor category $\mathrm{Corep}(\mathbb{F}O_N)$ is equivalent, as an abstract tensor category, to the Temperley-Lieb category TL_δ at parameter $\delta = N$, and that $\mathbb{F}O_N$ is realized via Tannaka-Krein duality by the fiber functor $F : TL_N \to \mathrm{Hilb}$ which sends the generating object to $H_1 := \mathbb{C}^N$, with corepresentation $u = (u_{i,j})_{i,j}$ given by the canonical generators of $\mathcal{L}(\mathbb{F}O_N)$, and the generating morphism to $F(\cap) = t := \sum_i e_i \otimes e_i \in H_1 \otimes H_1$, where $(e_i)_i$ is the canonical basis of $\mathbb{C}^N$. See [NT13, Section 2.5] for details about this category.

This means that we have a pictorial representation of elements $A \in \mathrm{Hom}(H_1^{\otimes k}, H_1^{\otimes l})$. More precisely, denote $NC_2(k,l)$ the set of non-crossing pair partitions of $k+l$ points. For each partition $\pi \in NC_2(k,l)$ there is a morphism $T_\pi \in \mathrm{Hom}(H_1^{\otimes k}, H_1^{\otimes l})$ whose matrix coefficients $(e_{i_1} \otimes \cdots \otimes e_{i_l} \mid T_\pi(e_{j_1} \otimes \cdots \otimes e_{j_k}))$ are equal to 1 if "the indices i_s, j_t agree in each block of π", and to 0 otherwise. Then, for $N \geq 3$ the maps T_π with $\pi \in NC_2(k,l)$ form a linear basis of $\mathrm{Hom}(H_1^{\otimes k}, H_1^{\otimes l})$. Elements $\pi \in NC_2(k,l)$, and the corresponding morphisms T_π, are usually depicted inside a rectangle with k numbered points on the upper edge and l numbered points on the bottom edge by drawing non-crossing strings joining the two elements in each block of π.

More generally, the collection of spaces $B(H_1^{\otimes k}, H_1^{\otimes l})$ is an (even) planar algebra, meaning that linear maps obtained by composing and tensoring given maps $X_i \in B(H_1^{\otimes k_i}, H_1^{\otimes l_i})$ with maps T_π can be represented by means of a rectangular Temperley-Lieb diagram as above with p internal boxes representing the maps X_i. For instance, if X, $Y \in B(H_1^{\otimes 2})$ we have, drawing dashed internal and external boxes, and solid Temperley-Lieb strings:

$(t^* \otimes t^* \otimes \mathrm{id})(\mathrm{id} \otimes X \otimes Y)(t \otimes t \otimes \mathrm{id}) =$  $\in B(H_1)$.

The irreducible objects of the Temperley-Lieb category, and hence the irreducible corepresentations of $\mathbb{F}O_N$, can be labeled by integers $k \in \mathbb{N}$ up to equivalence, in such a way that

$u_0 = 1 \otimes \mathrm{id}_{\mathbb{C}}$ is the trivial corepresentation, $u_1 = u$ is the generating object, and the following fusion rules are satisfied:

$$u_k \otimes u_l \simeq u_{|k-l|} \oplus u_{|k-l|+2} \oplus \cdots \oplus u_{k+l}.$$

We denote H_k the Hilbert space associated with u_k and $d_k = \dim H_k$. We write Tr_k, tr_k the standard and normalized traces on $B(H_k)$. Note that $d_0 = 1$ and $d_1 = N$. The remaining dimensions can be computed using the fusion rules and are given by q-numbers:

$$d_k = [k+1]_q := \frac{q^{k+1} - q^{-(k+1)}}{q - q^{-1}}. \tag{1.2}$$

The irreducible characters are $\chi_k = (\mathrm{id} \otimes \mathrm{Tr}_k)(u_k) \in M$. It follows from the fusion rules and the Peter-Weyl-Woronowicz formula that they form an orthonormal basis of the $*$-subalgebra $\mathcal{A}$ generated by $\chi_1 = \sum u_{i,i}$, which is weakly-$*$ dense in $A = \chi_1''$.

According to the fusion rules, u_k appears with multiplicity 1 as a subobject of $u_1^{\otimes k}$. We agree to take for H_k the corresponding subspace of $H_1^{\otimes k}$, and we denote $P_k \in B(H_1^{\otimes k})$ the orthogonal projection onto H_k: this is the kth Jones-Wenzl projection. We have $P_k(P_a \otimes P_b) = P_k$, i.e. H_k is a subspace of $H_a \otimes H_b$, as soon as $k = a + b$. We shall use the notation id_k for the identity map *both on* H_k *or on* $H_1^{\otimes k}$; the space it is acting on should be clear from the context. We will also use the embeddings $H_k \subset H_a \otimes H_b \subset H_1^{\otimes k}$, when $a + b = k$, to identify an element $X \in B(H_k)$ with the corresponding elements of $B(H_a \otimes H_b)$ and $B(H_1^{\otimes k})$. This is especially used to take partial traces of X such as $(\mathrm{Tr}_a \otimes \mathrm{id})(X)$ or $(\mathrm{Tr}_1 \otimes \mathrm{id})(X)$, where Tr_k always stands for the trace of $B(H_k)$ as indicated above.

As another consequence of the fusion rules, there is a unique line of fixed vectors in $H_k \otimes H_k$. We already know the generator $t = t_1$ of $\mathrm{Hom}(H_0, H_1 \otimes H_1)$. This map satisfies the *conjugate equations* $(\mathrm{id}_1 \otimes t^*)(t \otimes \mathrm{id}_1) = \mathrm{id}_1 = (t^* \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes t)$. We slightly abuse notation by defining recursively $t_1^1 = t_1$, $t_1^k = (\mathrm{id}_1^{\otimes k-1} \otimes t_1 \otimes \mathrm{id}_1^{\otimes k-1}) t_1^{k-1} \in \mathrm{Hom}(H_0, H_1^{\otimes 2k})$, so that $\mathrm{Hom}(H_0, H_k \otimes H_k)$ is generated by $t_k := (P_k \otimes P_k) t_1^k = (\mathrm{id}_k \otimes P_k) t_1^k = (P_k \otimes \mathrm{id}_k) t_1^k$. Note that we have then $t_k^*(X \otimes \mathrm{id}_k) t_k = \mathrm{Tr}_k(X)$ for $X \in B(H_k)$, in particular $\|t_k\| = \sqrt{d_k}$.

Using the intertwiner t one can also investigate more precisely the position of H_n in $H_{n-1} \otimes H_1$, and this gives rise for instance to the Wenzl recursion relation [Wen87, Prop. 1], see also [FK97, Equation (3.8)] and [VV07, Notation 7.7]:

$$P_n = (P_{n-1} \otimes \mathrm{id}_1) + \sum_{l=1}^{n-1} (-1)^{n-l} \frac{d_{l-1}}{d_{n-1}} \left(\mathrm{id}_1^{\otimes (l-1)} \otimes t \otimes \mathrm{id}_1^{\otimes (n-l-1)} \otimes t^* \right) (P_{n-1} \otimes \mathrm{id}_1). \tag{1.3}$$

One can go further and define the *basic intertwiner* $V_m^{k,l} = (P_k \otimes P_l)(\mathrm{id}_{k-a} \otimes t_a \otimes \mathrm{id}_{l-a}) P_m$ which spans $\mathrm{Hom}(H_m, H_k \otimes H_l)$, where $m = k + l - 2a$. It is not isometric but its norm can be computed explicitly, see [Ver07, Lemma 4.8]. Following [FV16], we denote $\kappa_m^{k,l} = \|V_m^{k,l}\|^{-1}$. This yields the following explicit formula to compute the product of coefficients of irreducible corepresentations:

$$u_k(X) u_l(Y) = \sum_{a=0}^{\min(k,l)} \left(\kappa_m^{k,l} \right)^2 u_m \left(V_m^{k,l*} (X \otimes Y) V_m^{k,l} \right), \tag{1.4}$$

where we still agree to write $m = k + l - 2a$. This motivates the following notation (which is indeed connected with the convolution product in $c_c(\mathbb{F}O_N)$ up to constants).

Notation 1.1. For $X \in B(H_k)$, $Y \in B(H_l)$, $m = k + l - 2a$ we consider the following element of $B(H_m)$:

$$X *_m Y = V_m^{k,l*} (X \otimes Y) V_m^{k,l} = P_m (\mathrm{id}_{k-a} \otimes t_a^* \otimes \mathrm{id}_{l-a})(X \otimes Y)(\mathrm{id}_{k-a} \otimes t_a \otimes \mathrm{id}_{l-a}) P_m.$$

One can perform analysis in the tensor category $\mathrm{Corep}(\mathbb{F}O_N)$. Recall for instance Lemma 1.3 from [VV07] below, with some more precise information about constants.

Lemma 1.2. *For any* $k \in \mathbb{N}$ *we have* $q^{-k} \leq d_k \leq q^{-k}/(1 - q^2)$.

Proof. Clear from (1.2). □

Lemma 1.3. *Fix $q_0 \in]0,1[$ and assume that $q \in]0,q_0]$. Then there exists a constant C depending only on q_0 such that $\|(P_{a+b}\otimes \mathrm{id}_c)(\mathrm{id}_a\otimes P_{b+c}) - P_{a+b+c}\| \le Cq^b$ for all a, b, $c\in\mathbb{N}$.*

Proof. This is [VV07, Lemma A.4], we only have to check that the constant C remains bounded as $q\to 0$. The proof of [VV07] explicitly gives the following upper bound:

$$\|(P_{a+b}\otimes \mathrm{id}_c)(\mathrm{id}_a\otimes P_{b+c}) - P_{a+b+c}\| \le q^b\Big(\prod_0^\infty(1+Dq^k)\Big)\Big(\sum_0^\infty Cq^k\Big),$$

where C and D a priori depend on q. Let us show that one can choose C and D uniformly over $]0,q_0]$. Using Lemma 1.2 we have

$$q^{-b-c}\frac{[2]_q[a]_q}{[a+b+c+1]_q} \le q^{-b-c}\frac{q^{-1}q^{-a+1}}{q^{-a-b-c}(1-q^2)^2} \le \frac{1}{(1-q_0^2)^2}.$$

Similarly:

$$\begin{aligned}q^{-b-c}\Big|\frac{[2]_q[a+b]_q}{[a+b+c+1]_q} - \frac{[2]_q[b]_q}{[b+c+1]_q}\Big| &= q^{-b-c}\frac{[2]_q[a]_q[c+1]_q}{[a+b+c+1]_q[b+c+1]_q}\\ &\le q^{-b-c}\frac{q^{-1}q^{-a+1}q^{-c}}{q^{-a-b-c}q^{-b-c}(1-q^2)^3} \le \frac{q_0^b}{(1-q_0^2)^3} \le \frac{1}{(1-q_0^2)^3}.\end{aligned}$$

In [VV07], the only constraint on C is to be an upper bound for these two quantities, hence it can indeed be chosen to depend only on q_0. On the other hand, D should be an upper bound for

$$q^{-c}\frac{[2]_q[b]_q}{[b+c+1]_q} \le q^{-c}\frac{q^{-1}q^{-b+1}}{q^{-b-c}(1-q^2)^2} \le \frac{1}{(1-q_0^2)^2},$$

hence it can also be chosen to depend only on q_0. □

We also have estimates on the constants κ, already proved in [Ver07]. The formulae for $\kappa_m^{k,l}$ show that, again, the constant C is uniform for q varying in an interval $]0,q_0]$ with $q_0<1$, but we will not need this fact.

Lemma 1.4. *There exists a constant C, depending only on q, such that we have $1\le\sqrt{d_a}\kappa_m^{k,l}\le C$ for all k, l and $m=k+l-2a$.*

Proof. See the proof of [Ver07, Lemma 4.8], [BVY21, p. 1583], [BC18, Equation (6) and Proposition 3.1]. □

The following estimate appeared also in connection with Property RD [Ver07]. Recall that $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm on matrix spaces.

Lemma 1.5. *Consider integers such that $m=k+l-2a$. Then for any $X\in B(H_1^{\otimes k})$, $Y\in B(H_1^{\otimes l})$ we have $\|(\mathrm{id}_{k-a}\otimes t_a^*\otimes \mathrm{id}_{l-a})(X\otimes Y)(\mathrm{id}_{k-a}\otimes t_a\otimes\mathrm{id}_{l-a})\|_2 \le \|X\|_2\|Y\|_2$ and $\|(\mathrm{id}\otimes\mathrm{Tr}_a)(X)\|_2\le\sqrt{d_a}\|X\|_2$.*

Proof. The proof of [Ver07, Theorem 4.9] applies, although it was there used only for $X\in B(H_k)$, $Y\in B(H_l)$. Let us repeat it. Consider an orthonormal basis $(f_i)_i$ of H_a, then the basis $(\bar f_i)_i$ defined by putting $t_a=\sum_i f_i\otimes\bar f_i$ is orthonormal as well: indeed its Gram matrix is Woronowicz modular matrix F_a, which is equal to the identity in our unimodular case. Put $E_I=f_if_j^*$ and $\bar E_I=\bar f_i\bar f_j^*\in B(H_a)$ for $I=(i,j)$, these are orthonormal bases of $B(H_a)$ for the Hilbert-Schmidt structure and we have $t_a^*(E_I\otimes\bar E_J)t_a=\delta_{I,J}$. Decompose $(\mathrm{id}_{k-a}\otimes P_a)X(\mathrm{id}_{k-a}\otimes P_a)=\sum X_I\otimes E_I$ with $X_I\in B(H_1^{\otimes k-a})$ and similarly $(P_a\otimes\mathrm{id}_{l-a})Y(P_a\otimes\mathrm{id}_{l-a})=\sum\bar E_J\otimes Y_J$. We have then $\sum\|X_I\|_2^2=\|(\mathrm{id}_{k-a}\otimes P_a)X(\mathrm{id}_{k-a}\otimes P_a)\|_2^2\le\|X\|_2^2$ and similarly $\sum\|Y_J\|_2^2\le\|Y\|_2^2$. Finally we have by the triangle inequality and Cauchy-Schwarz :

$$\begin{aligned}\|(\mathrm{id}\otimes t_a^*\otimes\mathrm{id})(X\otimes Y)(\mathrm{id}\otimes t_a\otimes\mathrm{id})\|_2^2 &= \|\textstyle\sum_{I,J}t_a^*(E_I\otimes\bar E_J)t_a\times(X_I\otimes Y_J)\|_2^2\\ &\le \big(\textstyle\sum_I\|X_I\|_2\|Y_I\|_2\big)^2\\ &\le (\textstyle\sum_I\|X_I\|_2^2)(\sum_I\|Y_I\|_2^2)\le\|X\|_2^2\|Y\|_2^2.\end{aligned}$$

The second inequality of this lemma follows by taking $l = a$ and $Y = \mathrm{id}_a$, but can also be proved more directly by noticing that in the canonical isometric isomorphism $B(K\otimes L) \simeq K\otimes L\otimes\bar L\otimes\bar K$, the partial trace $\mathrm{id}\otimes\mathrm{Tr}_L$ corresponds to the map $\mathrm{id}\otimes t_L^*\otimes\mathrm{id}$, where $t_L : \mathbb{C}\to L\otimes\bar L$ is the canonical duality vector whose norm is $\sqrt{\dim L}$. □

We will use again one of the two main estimates from [FV16] about $\mathrm{Corep}(\mathbb{F}O_N)$. For a, b, $c\in\mathbb{N}$ consider $\Pi_{a,b,c} = (\mathrm{id}_a\otimes\mathrm{tr}_b\otimes\mathrm{id}_c)(P_{a+b+c})\in B(H_a\otimes H_c)$ — this time the analysis deals with $\mathrm{Corep}(\mathbb{F}O_N)$ together with its canonical fiber functor. Proposition 3.2 of [FV16] shows that $\Pi_{a,b,c}$ is almost scalar as $b\to\infty$. We give below an improvement of the corresponding constants.

Lemma 1.6. *For every $q_0\in\,]0,1[$ there exist constants $C>0$, $\alpha\in\,]0,1[$ such that, for all a, b, $c\in\mathbb{N}$ and $q\in\,]0,q_0]$ we have $\|\Pi_{a,b,c}-\lambda(\mathrm{id}_a\otimes\mathrm{id}_c)\|\le Cq^{\lfloor\alpha b\rfloor}$ for some scalar $\lambda\in\mathbb{C}$.*

Proof. Let us note first that in the case $c=0$ the map $d_b\Pi_{a,b,c} = (\mathrm{id}_a\otimes t_b^*)(P_{a+b}\otimes\mathrm{id}_b)(\mathrm{id}_a\otimes t_b)$ is an intertwiner of the irreducible space H_a, hence it is a multiple of the identity. On the other hand, for $c\ge 1$ Proposition 3.2 of [FV16] uses the scalar $\lambda=\lambda_{a,c}$ explicitly given by $\lambda_{a,c} = q^{-a-c}/d_ad_c$. Consider $\Pi'_{a,b,c} = d_b\Pi_{a,b,c} - d_b\lambda_{a,c}(\mathrm{id}_a\otimes\mathrm{id}_c)$. A direct computation shows that

$$\begin{aligned}\mathrm{Tr}(\Pi'_{a,b,c}) = d_{a+b+c} - q^{-a-c}d_b = q^{b+2}\frac{q^{-a-c}-q^{a+c}}{1-q^2}\\ \le \frac{1}{1-q^2}\sqrt{d_{a+c}d_ad_c} = \frac{\sqrt{d_{a+c}}}{1-q^2}(\mathrm{Tr}\,\mathrm{id}_a\otimes\mathrm{id}_c)^{1/2}.\end{aligned}$$

Now, [FV16] shows the existence of constants $D_{a,c}$ such that $|\mathrm{Tr}(\Pi'_{a,b,c}f)|\le D_{a,c}(\mathrm{Tr}\, f^*f)^{1/2}$ for $f\in B(H_a)\otimes B(H_c)$ with $\mathrm{Tr}(f)=0$. This implies

$$|\mathrm{Tr}(\Pi'_{a,b,c}f)|\le (d_{a+c}/(1-q^2)^2 + D_{a,c}^2)^{1/2}(\mathrm{Tr}\, f^*f)^{1/2}$$

for any $f\in B(H_a)\otimes B(H_c)$, hence $\|\Pi_{a,b,c}-\lambda_{a,c}\mathrm{id}\|_2\le (d_{a+c}/(1-q^2)^2 + D_{a,c}^2)^{1/2}d_b^{-1}$. Here we use the Hilbert-Schmidt norm in $B(H_a\otimes H_c)$, which is bigger than the operator norm.

Moreover, it is explicitly stated in the proof of [FV16, Prop. 3.2] that one can take the constants $D_{a,c}$ defined by induction over c as follows: $D_{a,0}=0$ and, for $c\ge 1$:

$$D_{a,c} = K_c\max(d_1^{1/2}D_{a,c-1} + d_1^{3/2}d_{a-1}, d_{a+c}^{1/2}),$$

where $1\le K_c = 1/(1-q^c)\le K := 1/(1-q)$. In particular $d_{a+c}\le D_{a,c}^2$ if $c\ge 1$. Putting $C_1 = \sqrt{2}/(1-q_0^2)$ we have thus, for all a, b, $c\in\mathbb{N}$, the existence of $\lambda\in\mathbb{C}$ such that $\|\Pi_{a,b,c}-\lambda\,\mathrm{id}\|\le C_1D_{a,c}q^b$.

One can then show by induction that the constants $D_{a,c}$ satisfy the estimate $D_{a,c}\le(2NK)^{a+c}$, where $N=d_1=q+q^{-1}$. Indeed $K_cd_{a+c}^{1/2}\le KN^{(a+c)/2}\le(2KN)^{a+c}$, and for $c\in\mathbb{N}^*$ we have by induction

$$K_c(d_1^{1/2}D_{a,c-1}+d_1^{3/2}d_{a-1})\le KN^{1/2}(2NK)^{a+c-1} + KN^{3/2}N^{a-1}\le(2NK)^{a+c}.$$

Of course this estimate is quite bad, but one can improve it using [VV07, Lemma A.4].

More precisely, let $\alpha>0$ be such that $(2KN)^{2\alpha}q = q^\alpha$. Take a, $c\ge\alpha b$. Denote C_0 the constant given by Lemma 1.3. Then we have

$$P_{a+b+c}\simeq(P_a\otimes\mathrm{id}_b\otimes P_c)(\mathrm{id}_{a-\lfloor\alpha b\rfloor}\otimes P_{b+2\lfloor\alpha b\rfloor}\otimes\mathrm{id}_{c-\lfloor\alpha b\rfloor})$$

up to $2C_0q^{\lfloor\alpha b\rfloor}$ in operator norm. Applying $\mathrm{id}\otimes\mathrm{tr}_b\otimes\mathrm{id}$, which is contracting, to this estimate we obtain

$$\Pi_{a,b,c}\simeq(P_a\otimes P_c)(\mathrm{id}_{a-\lfloor\alpha b\rfloor}\otimes\Pi_{\lfloor\alpha b\rfloor,b,\lfloor\alpha b\rfloor}\otimes\mathrm{id}_{c-\lfloor\alpha b\rfloor})\simeq\lambda(\mathrm{id}_a\otimes\mathrm{id}_c)$$

up to $2C_0q^{\lfloor\alpha b\rfloor} + C_1D_{\lfloor\alpha b\rfloor,\lfloor\alpha b\rfloor}q^b\le 2C_0q^{\lfloor\alpha b\rfloor}+C_1(2NK)^{2\lfloor\alpha b\rfloor}q^b$ in operator norm, for some $\lambda\in\mathbb{C}$. Since $q\le 1\le 2NK$ we have moreover $(2NK)^{2\lfloor\alpha b\rfloor}q^b\le(2NK)^{2\alpha b}q^b = q^{\alpha b}\le q^{\lfloor\alpha b\rfloor}$ by definition of α. This yields $\|\Pi_{a,b,c}-\lambda(\mathrm{id}_a\otimes\mathrm{id}_c)\|\le(2C_0+C_1)q^{\lfloor\alpha b\rfloor}$. This estimate is also valid if a, $c<\alpha b$

because in this case $D_{a,c}q^b \leq (2KN)^{2\alpha b}q^b = q^{\alpha b}$. It holds also in the remaining cases by using Lemma 1.3 only on one side.

Finally we have shown the existence of $D_0 > 0$, depending only on q_0, and $\alpha > 0$ such that for all a, b, c there exists a constant λ such that $\|\Pi_{a,b,c} - \lambda(\mathrm{id}_a \otimes \mathrm{id}_c)\| \leq D_0 q^{\lfloor \alpha b \rfloor}$. One should be careful that α depends on q. In fact it can be computed explicitly from the defining relation $(2KN)^{2\alpha}q = q^{\alpha}$, with $K = 1/(1-q)$ and $N = q + q^{-1}$: one gets

$$\alpha = \frac{1}{3}\left[1 - \frac{2\ln 2}{3\ln q} - \frac{2}{3\ln q}\ln\left(\frac{1+q^2}{1-q}\right)\right]^{-1}.$$

From this it follows that α is decreasing from $1/3$ to 0 as q varies from 0 to 1, and the result follows. ☐

Remark 1.7. For instance one can take $\alpha = 1/4$ for $q_0 \approx 0.15$ (or $N_0 = 7$). We also have $q^\alpha \sim Lq^{1/3}$ as $q \to 0$, where $L = \exp(2\ln(2)/9)$.

We will need in the next section one last tool about the representation category of $\mathbb{F}O_N$. The Wenzl recursion relation (1.3), applied twice, yields the following bilateral version.

Lemma 1.8. *For $n \geq 4$ we have the bilateral Wenzl recursion relation:*

$$\begin{aligned} P_n = {} & (\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) + \\ & - \frac{d_{n-2}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes(n-2)})(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ & - \frac{d_{n-2}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(\mathrm{id}_1^{\otimes(n-2)} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ & + \frac{(-1)^{n-1}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_1^{\otimes(n-2)} \otimes t^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ & + \frac{(-1)^{n-1}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t^* \otimes \mathrm{id}_1^{\otimes(n-2)} \otimes t)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ & + \frac{d_1 + d_{n-3}d_{n-2}}{d_{n-1}d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes(n-4)} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1). \end{aligned}$$

For $n = 3$ the formula still holds, without the last term.

Proof. We assume for this proof that $n \geq 4$. A similar calculation gives the result for $n = 3$.

We first multiply the relation (1.3) on the left by $(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)$. All terms except $l = 1$ and $l = n-1$ vanish because they involve $P_{n-2}(\mathrm{id}_i \otimes t \otimes \mathrm{id}_j)$, and we are left with :

$$\begin{aligned} P_n = {} & (P_{n-1} \otimes \mathrm{id}_1) + \frac{(-1)^{n-1}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_1^{\otimes(n-2)} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1) \\ & - \frac{d_{n-2}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(\mathrm{id}_1^{\otimes(n-2)} \otimes tt^*)(P_{n-1} \otimes \mathrm{id}_1). \end{aligned}$$

Let us denote A, B, C the three terms on the right-hand side above, without the numeric coefficients. We apply the left version Wenzl's recursion to the projections P_{n-1}:

$$P_{n-1} = (\mathrm{id}_1 \otimes P_{n-2}) + \sum_{k=1}^{n-2}(-1)^{n-1-k}\frac{d_{k-1}}{d_{n-2}}\left(t^* \otimes \mathrm{id}_1^{\otimes(n-k-2)} \otimes t \otimes \mathrm{id}_1^{\otimes(k-1)}\right)(\mathrm{id}_1 \otimes P_{n-2}).$$

Multiplying on the left by $(\mathrm{id}_1 \otimes P_{n-2})$ this yields

$$A = (\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) - \frac{d_{n-3}}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes(n-2)})(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1).$$

We proceed similarly with B: only the terms $k=1$, $k=2$ have a non-vanishing contribution and we obtain, applying the conjugate equation:

$$\begin{aligned} B &= (\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_1^{\otimes(n-2)} \otimes t^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ &\quad + \frac{(-1)^n}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes(n-2)})(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ &\quad + \frac{(-1)^{n-1} d_1}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes(n-4)} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1). \end{aligned}$$

Finally for C only the terms $k=1$, $k=n-2$ survive, yielding:

$$\begin{aligned} C &= (\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(\mathrm{id}_1^{\otimes(n-2)} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ &\quad + \frac{(-1)^n}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t^* \otimes \mathrm{id}_1^{\otimes(n-2)} \otimes t)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ &\quad - \frac{d_{n-3}}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes(n-4)} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1). \end{aligned}$$

The result follows by gathering A, B and C with their coefficients and using the relation $d_{n-3}d_{n-1} + 1 = d_{n-2}^2$. □

2. Decomposition of the Bimodule

In this section we consider the GNS space $H = \ell^2(\mathbb{F})$ of $M = \mathcal{L}(\mathbb{F})$ with respect to the Haar trace h. We identify M with a dense subspace of H. We shall study H as an A,A-bimodule for $A = \chi_1'' \cap M$. We will more specifically consider the orthogonal $H^\circ \subset H$ of the trivial bimodule $A \subset H$, and we shall decompose it into simpler, pairwise orthogonal submodules generated by natural elements, see Proposition 2.12. Moreover we will exhibit for each of these cyclic submodules $\mathcal{A}x\mathcal{A}$ a linear basis $(x_{i,j})_{i,j}$, see Proposition 2.9 and Corollary 2.14. Recall that $\mathcal{A}$ is the unital canonical dense sub-$*$-algebra of A generated by χ_1.

We denote $p_k \in B(H)$ the orthogonal projection onto the subspace $p_kH = u_k(B(H_k))$ spanned by coefficients of u_k. Note that p_k belongs in fact to the dual algebra $\ell^\infty(\mathbb{F})$, and that the projection $P_k \in B(H_1^{\otimes k})$ introduced in the preceding section is the image of p_k under the natural representation of $\ell^\infty(\mathbb{F})$ on the corepresentation space $H_1^{\otimes k}$.

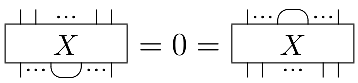
The space H° is spanned by its subspaces p_kH° and we have $p_kH^\circ = H^\circ \cap p_kH = u_k(B(H_k)^\circ)$ where $B(H_k)^\circ = \{X \in B(H_k) \mid \mathrm{Tr}(X) = 0\}$. In the case of the classical generator MASA $a_1'' \subset \mathcal{L}(F_N)$, the subspace analogous to p_kH° is spanned by reduced words of length k, different from $a_1^{\pm k}$. We introduce below a subspace $H^{\circ\circ} \subset H^\circ$ which is the quantum replacement for the set of words $g \in F_N$ that do not start nor end with a_1.

Notation 2.1. For $n \geq 1$ we denote

$$B(H_n)^{\circ\circ} = \{X \in B(H_n) \mid (\mathrm{Tr}_1 \otimes \mathrm{id})(X) = 0 = (\mathrm{id} \otimes \mathrm{Tr}_1)(X)\}.$$

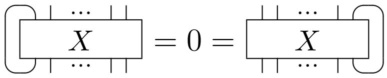
We denote $H^{\circ\circ}$ the closed linear span of the subspaces $u_n(B(H_n)^{\circ\circ})$ in H°.

Remark 2.2. It is well-known that $H_n \subset H_1^{\otimes n}$ is the subspace of vectors $\zeta \in H_1^{\otimes n}$ such that $(\mathrm{id}_i \otimes t^* \otimes \mathrm{id}_{n-i-2})(\zeta) = 0$, for all $i = 0, \dots, n-2$. This follows by induction from the fact that H_n is the kernel of $t^* \otimes \mathrm{id}_{n-2} : H_1 \otimes H_{n-1} \to H_{n-2}$, according to the fusion rules. As a consequence, an element $X \in B(H_1^{\otimes n})$ arises from an element of $B(H_n)$ **iff** we have $(\mathrm{id}_i \otimes t^* \otimes \mathrm{id}_{n-i-2})X = 0$ and $X(\mathrm{id}_i \otimes t \otimes \mathrm{id}_{n-i-2}) = 0$ for all i. Graphically this means we have $X \in B(H_n)$ **iff** we obtain 0 by applying to X any planar tangle which connects two consecutive points on the lower or upper edge of the internal box corresponding to X:



Since $(\mathrm{Tr}_1 \otimes \mathrm{id})(X) \in B(H_1^{\otimes n-1})$ (resp $(\mathrm{id} \otimes \mathrm{Tr}_1)(X)$) is obtained from X by applying the planar tangle connecting the upper left and lower left (resp. upper right and lower right) points of the internal box, we conclude that $X \in B(H_1^{\otimes n})$ belongs to $B(H_n)^{\circ\circ}$ **iff** we obtain 0 by applying to

X any planar tangle which connects any two consecutive points of the internal box corresponding to X. Diagrammatically this is represented by the additional constraints:



Now we compute the dimension of $B(H_n)^{\circ\circ}$, see Proposition 2.5. This will be useful to prove that the families $(x_{i,j})_{i,j}$ are linearly independent at Corollary 2.14. The latter also follows from the stronger results of Section 3, but there we will have to assume that N is large enough and the proofs are much more involved. Note however that the proof below is not optimal either, in the sense that the underlying technical result established at Lemma 2.4 does not hold if $q + q^{-1} \in]2, 2.41[$, which can occur for the non unimodular groups $\mathbb{F}O_Q$. We believe that Lemma 2.3 and Proposition 2.5 hold true for any group $\mathbb{F}O_Q$ with $q + q^{-1} > 2$, i.e. excluding the duals of $SU(2)$ and $SU_{-1}(2)$.

In the statement below we use the leg numbering notation: $t^*_{1,n} = \sum_i e^*_i \otimes \mathrm{id}_1 \otimes \cdots \otimes \mathrm{id}_1 \otimes e^*_i$, for $n \geq 2$. This application maps H_n to H_{n-2}, as can be seen when $n \geq 4$ by checking the condition $(\mathrm{id}_i \otimes t^* \otimes \mathrm{id}_{n-i-4})t^*_{1,n}(\zeta) = 0$, for any $\zeta \in H_n$.

Lemma 2.3. *Assume $N \geq 3$. For $n \geq 3$ the map $t^*_{1,n} : H_n \to H_{n-2}$ is surjective.*

Proof. We apply $t^*_{1,n} \cdot t_{1,n}$ to the bilateral Wenzl recursion formula from Lemma 1.8. Using the conjugate equations we have in $B(H_1^{\otimes(n-2)})$:

$$\begin{aligned}
&t^*_{1,n}(tt^* \otimes \mathrm{id}_1^{\otimes(n-2)})t_{1,n} = \mathrm{id}_1^{\otimes(n-2)} = t^*_{1,n}(\mathrm{id}_1^{\otimes(n-2)} \otimes tt^*)t_{1,n},\\
&t^*_{1,n}(t \otimes \mathrm{id}_1^{\otimes(n-2)} \otimes t^*)t_{1,n} = C^{-2}_{n-2},\\
&t^*_{1,n}(t^* \otimes \mathrm{id}_1^{\otimes(n-2)} \otimes t)t_{1,n} = C^{2}_{n-2},\\
&t^*_{1,n}(tt^* \otimes \mathrm{id}_1^{\otimes(n-4)} \otimes tt^*)t_{1,n} = t_{1,n-2}t^*_{1,n-2} \qquad (n \geq 4),
\end{aligned}$$

where $C_{n-2} : \xi \otimes \zeta \mapsto \zeta \otimes \xi$ for $\xi \in H_1$, $\zeta \in H_1^{\otimes(n-3)}$. Thus for $n \geq 4$ we obtain

$$\begin{aligned}
t^*_{1,n}P_n t_{1,n} = &\left(d_1 - \tfrac{2d_{n-2}}{d_{n-1}}\right) P_{n-2} + \\
&+ \frac{(-1)^{n-1}}{d_{n-1}} P_{n-2}(C^2_{n-2} + C^{-2}_{n-2})P_{n-2} + \frac{d_1 + d_{n-3}d_{n-2}}{d_{n-1}d_{n-2}} P_{n-2}t_{1,n-2}t^*_{1,n-2}P_{n-2}.
\end{aligned}$$

This formula also holds for $n = 3$, without the last term. Observe moreover that $\|P_{n-2}C^{\pm 2}_{n-2}P_{n-2}\| \leq 1$ and $\|P_{n-2}t_{1,n-2}t^*_{1,n-2}P_{n-2}\| \leq d_1$ by composition. Now, the inequality established in the next Lemma shows that $t^*_{1,n}P_n t_{1,n} \geq \epsilon P_{n-2}$ for some $\epsilon > 0$. As a result $t^*_{1,n}P_n t_{1,n} \in B(H_{n-2})$ is invertible and the result follows. ☐

Lemma 2.4. *Still assuming $N \geq 3$, we have for any $n \geq 3$:*

$$d_1 - \frac{2d_{n-2}}{d_{n-1}} > \frac{2}{d_{n-1}} + d_1 \frac{d_1 + d_{n-3}d_{n-2}}{d_{n-1}d_{n-2}}.$$

Proof. Denote $e_n = d_{n-1} - d_{n-2}$, $f_n = d_{n-5} + 1 + d_1$, with the convention $d_k = 0$ if $k < 0$. For $n = 3$ we have $e_3 = N^2 - N - 1$, $f_3 = 1 + N$ and since $N \geq 3 > 1 + \sqrt{3}$ we have $e_3 > f_3$.

On the other hand we have, using the identity $Nd_{n-1} = d_n + d_{n-2}$ valid for $n \in \mathbb{Z}^*$:

$$e_{n+1} - e_n = d_n - 2d_{n-1} + d_{n-2} = (N-2)d_{n-1} \geq d_{n-1} \geq d_{n-4} - d_{n-5} = f_{n+1} - f_n.$$

An easy induction then shows that we have $e_n > f_n$ for every $n \geq 3$.

Multiplying this inequality by d_1 we find

$$\begin{aligned}
&d_1 d_{n-1} - d_1 d_{n-2} > d_1 d_{n-5} + d_1 + d_1^2\\
\iff\ &d_1 d_{n-1} - (d_1 - 1)d_{n-2} > d_1 d_{n-5} + d_{n-2} + d_1 + d_1^2\\
\implies\ &d_1 d_{n-1} - 2d_{n-2} > d_1 d_{n-3} + 2 + d_1^2/d_{n-2},
\end{aligned}$$

using the facts $d_1 \geq 3$, $d_1 d_{n-5} + d_{n-2} \geq d_{n-4} + d_{n-2} \geq d_1 d_{n-3} - 1$, and $d_{n-2} \geq 1$. Note that the inequality $d_1 d_{n-5} \geq d_{n-4}$, resulting from the fusion rules, does not hold for $n = 4$, but one can check directly that in this case $d_1 d_{n-5} + d_{n-2} = d_1 d_{n-3} - 1$. □

Proposition 2.5. *Still assume $N \geq 3$. For $n \geq 2$ we have $\dim B(H_n)^{\circ\circ} = \dim p_n H^{\circ\circ} = d_{2n} - d_{2n-2}$. For $n = 1$ we have $\dim B(H_1)^{\circ\circ} = \dim p_1 H^{\circ\circ} = d_2$.*

Proof. Recall the identification $B(H_n) \simeq H_n \otimes H_n$ via $X \mapsto x = (X \otimes \mathrm{id}) t_n$. In this identification the condition $(\mathrm{id} \otimes \mathrm{Tr}_1)(X) = 0$ reads $(\mathrm{id}_{n-1} \otimes t^* \otimes \mathrm{id}_{n-1})(x) = 0$ and the corresponding kernel is $H_{2n} \subset H_n \otimes H_n$. This holds as well if $N = 2$. Then the condition $(\mathrm{Tr}_1 \otimes \mathrm{id})(X) = 0$ reads $t^*_{1,2n}(x) = 0$, so that the result follows from the rank theorem and Lemma 2.3. For $n = 1$ both conditions coincide and we have $B(H_1)^{\circ\circ} = B(H_1)^\circ \simeq H_2$. □

On the other hand in the case $N = 2$ one can check that $t^*_{1,2n}$ vanishes on H_{2n} for all n, and thus $\dim B(H_n)^{\circ\circ} = \dim p_n H^{\circ\circ} = d_{2n}$ for all $n \geq 1$.

Recall then the "rotation operators" $\rho : B(H_k) \to B(H_k)$ already considered in [FV16] and defined as follows: $\rho(X) = (P_k \otimes t^*)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(t \otimes P_k)$. It follows from [FV16, Lemma 3.1] that ρ stabilizes the subspace $B(H_n)^\circ$ and contracts the Hilbert-Schmidt norm. On $B(H_n)^{\circ\circ}$ it behaves even better: as the next lemma shows, it is a finite order unitary — in particular, it is diagonalizable.

Lemma 2.6. *The map ρ is a bijection from $B(H_n)^{\circ\circ}$ to itself. Moreover we have $\rho^{2n} = \mathrm{id}$ and $\rho^* = \rho^{-1}$ on $B(H_n)^{\circ\circ}$.*

Proof. We first note that for $X \in B(H_n)^{\circ\circ}$ the element $Y = (\mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-1} \otimes \mathrm{id}_1)$ of $B(H_{n-1} \otimes H_1, H_1 \otimes H_{n-1})$ is directly equal to $\rho(X)$. This is clear if $n = 1$ since then $\mathrm{id}_1 \otimes \mathrm{id}_{n-1} = \mathrm{id}_1 = P_1$. Assume $n \geq 2$. Since $t^*(\mathrm{id} \otimes A)t = \mathrm{Tr}_1(A)$ for any $A \in B(H_1)$ we have $(t^* \otimes \mathrm{id}_{n-2})Y = (\mathrm{id} \otimes t^*)[(\mathrm{Tr}_1 \otimes \mathrm{id})(X) \otimes \mathrm{id}_1)] = 0$, and similarly $Y(\mathrm{id}_{n-2} \otimes t) = 0$, so that using Remark 2.2 we have $Y = P_n Y P_n = \rho(X)$. Then we compute $(\mathrm{Tr}_1 \otimes \mathrm{id})(Y)$ using again the morphism t. Thank to the conjugate equation we have

$$\begin{aligned}(\mathrm{Tr}_1 \otimes \mathrm{id})(Y) &= (t^* \otimes \mathrm{id}_{n-1})(\mathrm{id}_1 \otimes \mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes \mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)\\ &\qquad (\mathrm{id}_1 \otimes t \otimes \mathrm{id}_{n-1} \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-2} \otimes \mathrm{id}_1)\\ &= (\mathrm{id}_{n-1} \otimes t^*)(X \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-2} \otimes \mathrm{id}_1) = 0,\end{aligned}$$

since $X \in B(H_n)$. Similarly $(\mathrm{id} \otimes \mathrm{Tr}_1)(Y) = 0$ and this proves $\rho(B(H_n)^{\circ\circ}) \subset B(H_n)^{\circ\circ}$. The conjugate equation also implies that $(t^* \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes Y \otimes \mathrm{id}_1)(\mathrm{id}_n \otimes t) = X$ so that ρ is a bijection with $\rho^{-1}(X) = (t^* \otimes P_n)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(P_n \otimes t)$. This holds as well for $n = 1$.

Let us check that ρ^{-1} is the adjoint of ρ with respect to the Hilbert-Schmidt scalar product. Using twice the conjugate equation we have, for $X, Y \in B(H_n)^{\circ\circ}$:

$$\begin{aligned}\mathrm{Tr}_n(\rho^{-1}(X)^* Y) &= (\mathrm{Tr}_1 \otimes \mathrm{Tr}_{n-1})[(\mathrm{id}_n \otimes t^*)(\mathrm{id}_1 \otimes X^* \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_n) Y]\\ &= \mathrm{Tr}_{n-1}[(t^* \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes \mathrm{id}_1 \otimes X^* \otimes \mathrm{id}_1)\\ &\qquad (\mathrm{id}_1 \otimes t \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes Y)(t \otimes \mathrm{id}_{n-1})]\\ &= \mathrm{Tr}_{n-1}[(\mathrm{id}_{n-1} \otimes t^*)(X^* \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes Y)(t \otimes \mathrm{id}_{n-1})]\\ &= \mathrm{Tr}_{n-1}[(\mathrm{id}_{n-1} \otimes t^*)(X^* \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^* \otimes \mathrm{id}_1)\\ &\qquad (\mathrm{id}_1 \otimes Y \otimes \mathrm{id}_1 \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-1} \otimes t)]\\ &= (\mathrm{Tr}_{n-1} \otimes \mathrm{Tr}_1)[X^*(\mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes Y \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-1} \otimes \mathrm{id}_1)]\\ &= \mathrm{Tr}_n(X^* \rho(Y)).\end{aligned}$$

Recall the notation $t_1^n \in \mathrm{Hom}(\mathbb{C}, H_1^{\otimes n} \otimes H_1^{\otimes n})$ from the Preliminaries and consider the associated antilinear map $j^n : H_1^{\otimes n} \to H_1^{\otimes n}$ given by $j^n(\zeta) = (\zeta^* \otimes \mathrm{id}_n) t_1^n$. If $(e_i)_i$ is the canonical basis of $H_1 = \mathbb{C}^N$ we have $j^n(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_n} \otimes \cdots \otimes e_{i_1}$ so that $j^n \circ j^n = \mathrm{id}$ and $j^n(\zeta) = (\mathrm{id}_n \otimes \zeta^*) t^n$. Using the fact that $\rho(X) = (\mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-1} \otimes \mathrm{id}_1)$ for $X \in B(H_n)^{\circ\circ}$ we have easily $\rho^n(X) = (\mathrm{id}_n \otimes t^{n*})(\mathrm{id}_n \otimes X \otimes \mathrm{id})(t^n \otimes \mathrm{id}_n)$, which yields $(\zeta \mid \rho^n(X)\xi) = (j^n \xi \mid X j^n \zeta)$ for all $\zeta, \xi \in H_n$. Applying this identity a second time we get $\rho^{2n}(X) = X$. □

We shall now analyze the submodule AxA when x belongs to $H^{\circ\circ}$. In the analogy with the generator MASA $a_1'' \subset \mathcal{L}(F_N)$ in a free group factor, the vectors $x_{i,j}$ below play the role of the words $a_1^i g a_1^j \in F_N$, where $g \in F_N$ does not start nor end with a_1.

Notation 2.7. For $x \in H$ and $i, j \in \mathbb{N}$ we denote $x_{i,j} = \sum_n p_{i+n+j}(\chi_i p_n(x) \chi_j)$. For $X \in B(H_n)$ we denote $X_{i,j} = P_{i+n+j}(\mathrm{id}_i \otimes X \otimes \mathrm{id}_j) P_{i+n+j} \in B(H_{i+n+j})$.

Remark 2.8. The sum in the definition of $x_{i,j}$ indeed converges in H, since its terms are pairwise orthogonal an satisfy the inequality $\|\chi_i p_n(x) \chi_j\| \leq \|\chi_i\| \|\chi_j\| \|p_n(x)\|$. This yields a map $(x \mapsto x_{i,j})$ which is linear and bounded from H to H. We will mostly use the notation $x_{i,j}$ in the case when x belongs to one of the subspaces $p_n H$.

Note also that we have by construction $u_n(X)_{i,j} = u_{i+n+j}(X_{i,j})$ for $X \in B(H_n)$. Indeed, denote $x = u_n(X)$ and recall that $\chi_i = u_i(\mathrm{id}_i)$, $\chi_j = u_j(\mathrm{id}_j)$. To compute the component $p_{i+n+j}(\chi_i x \chi_j)$ one has to use an orthonormal basis of isometric intertwiners $T : H_{i+n+j} \to H_i \otimes H_n \otimes H_j$. But according to the fusion rules there is only one such intertwiner up to a phase, and by construction of the spaces H_k we can take for it the canonical inclusion of H_{i+n+j} into $H_i \otimes H_n \otimes H_j \subset H_1^{\otimes(i+n+j)}$, whose adjoint is given by P_{i+n+j}.

Finally, we record the fact that $X_{i,j}$ is the orthogonal projection of $\mathrm{id}_i \otimes X \otimes \mathrm{id}_j \in B(H_1^{\otimes(i+n+j)})$ onto $B(H_{i+n+j})$, with respect to the Hilbert-Schmidt scalar product — indeed for any Y, $Z \in B(H_1^{\otimes(i+n+j)})$ we have $\mathrm{Tr}(Y^* P_{i+n+j} Z P_{i+n+j}) = \mathrm{Tr}((P_{i+n+j} Y P_{i+n+j})^* Z)$.

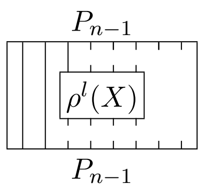
Proposition 2.9. *Fix $k \in \mathbb{N}^*$, $X \in B(H_k)^{\circ\circ}$ an eigenvector of ρ and $x = u_k(X) \in H^{\circ\circ}$. Then we have $\mathcal{A}x\mathcal{A} = \mathrm{Span}\{x_{i,j} \mid i, j \in \mathbb{N}\}$.*

Proof. Let us prove by induction over $i + j = n - k$ that $x_{i,j} \in \mathcal{A}x\mathcal{A}$. Assume that $x_{p,q} \in \mathcal{A}x\mathcal{A}$ if $p + q \leq i + j$ and compute $\chi_1 x_{i,j}$. We have $p_{n-1}(\chi_1 x_{i,j}) = (\kappa_{n-1}^{1,n})^2 u_{n-1}(\mathrm{id}_1 *_{n-1} X_{i,j})$ and

$$\begin{aligned} (2.1) \quad \mathrm{id}_1 *_{n-1} X_{i,j} &= (t^* \otimes P_{n-1})(\mathrm{id}_1 \otimes X_{i,j})(t \otimes P_{n-1}) \\ &= P_{n-1}(t^* \otimes \mathrm{id}_{n-1})(\mathrm{id}_1 \otimes P_n)(\mathrm{id}_{i+1} \otimes X \otimes \mathrm{id}_j)(\mathrm{id}_1 \otimes P_n)(t \otimes \mathrm{id}_{n-1}) P_{n-1}. \end{aligned}$$

Since the Jones-Wenzl projections P_n are intertwiners, we can expand them into linear combinations of Temperley-Lieb diagrams, so that $\mathrm{id}_1 *_{n-1} X_{i,j}$ is a linear combination of maps of the form $P_{n-1} T_\pi(X) P_{n-1}$, where π is a Temperley-Lieb diagram with $n - 1$ upper and lower points and an internal box with $2k$ points, and $T_\pi : B(H_1^{\otimes k}) \to B(H_1^{\otimes(n-1)})$ is the associated map. Since we multiply on the left and on the right by P_{n-1} and $X = P_k X P_k$ belongs to $B(H_k)^{\circ\circ}$, the term associated with π vanishes as soon as a string of π connects two upper points, or two lower points, or two internal points.

Now consider the string originating from the first top left external point in a diagram π such that $P_{n-1} T_\pi(X) P_{n-1} \neq 0$. If it is not connected to the internal box, it has to connect the top left point to the first bottom left external point, otherwise some other string would have to connect two upper or two lower external point, because of the non-crossing constraint. We can re-apply this reasoning to the following top left external points, until we find an external point connected to X, say with index $p + 1$ on the top external edge. Moreover up to replacing X by its image $\rho^l(X)$ under some iterated rotation we can assume that this external point is connected to the first top left point of the internal box by a vertical edge. Thus our diagram has the following form:

$P_{n-1} T_\pi(X) P_{n-1} =$  .

By the same reasoning we see that the $(p+2)^{\text{th}}$ external point on the top edge connects to the second top left point of the internal box $\rho^l(X)$ by a vertical string (if $k \geq 2$). Indeed if it is connected to another point of the internal box, there will be a string joining two internal points, and if it is connected to a bottom external point, there will be either a string connecting two upper external points, or a string connecting two internal points. Continuing like this, we see that the only possibility for a non-vanishing diagram is one composed entirely of vertical lines, i.e. $P_{n-1}(\mathrm{id}_p \otimes \rho^l(X) \otimes \mathrm{id}_q)P_{n-1}$, with $l \in \mathbb{Z}$ and $p+q = n-k-1$. Since X is an eigenvector of ρ, this shows that $p_{n-1}(\chi_1 x_{i,j})$ is a linear combination of vectors $x_{p,q}$ with $p+q < i+j$, which belong to $\mathcal{A}x\mathcal{A}$ by the induction hypothesis. Note that if $i = j = 0$ we have $p_{n-1}(\chi_1 x_{i,j}) = 0$; this can also be checked directly because (2.1) then equals $(\mathrm{Tr}_1 \otimes \mathrm{id})(X)$.

We have $p_{n+1}(\chi_1 x_{i,j}) = p_{n+1}(\chi_1 \chi_i x \chi_j) = x_{i+1,j}$ because $p_{n+1}(\chi_1 y) = 0$ if $y \in p_{k'}H$ with $k' < n$. We have thus $x_{i+1,j} = \chi_1 x_{i,j} - p_{n-1}(\chi_1 x_{i,j})$ and it follows that $x_{i+1,j}$ belongs to $\mathcal{A}x\mathcal{A}$. One can proceed in the same way on the right to show that $x_{i,j+1}$ belongs to $\mathcal{A}x\mathcal{A}$. By induction we have proved $x_{i,j} \in \mathcal{A}x\mathcal{A}$ for all i, j. Moreover from the identities $\chi_1 x_{i,j} = x_{i+1,j} + p_{n-1}(\chi_1 x_{i,j})$, $x_{i,j}\chi_1 = x_{i,j+1} + p_{n-1}(x_{i,j}\chi_1)$ and the fact that $p_{n-1}(\chi_1 x_{i,j})$, $p_{n-1}(x_{i,j}\chi_1)$ are linear combinations of vectors $x_{p,q}$ it also follows that $\mathrm{Span}\{x_{i,j}\}$ is stable under the left and right actions of $\mathcal{A}$. □

Remark 2.10. Using the Jones-Wenzl recursion relations, one can prove more precisely that $p_{n-1}(x_{i,j}\chi_1)$ is a linear combination of $x_{i-1,j}$, $\rho^{\pm 1}(x)_{i,j-1}$ and $x_{i+1,j-2}$, where we abusively write $\rho(u_k(X)) := u_k(\rho(X))$.

Notation 2.11. Choose for all $k \geq 1$ a basis $(X_r)_r \subset B(H_k)^{\circ\circ}$ of eigenvectors of ρ, normalized in such a way that $\|u_k(X_r)\|_2 = 1$. Denote $W_k = (u_k(X_r))_r$ its image in $p_k H^{\circ\circ}$. Put as well $W = \bigcup_k W_k$, which is a linearly independent family consisting of unital vectors in $H^{\circ\circ}$. For $x \in W$ we denote $H(x) = \overline{AxA}$, and for $k \in \mathbb{N}^*$, $H(k) = \overline{AW_kA}$, using the left and right actions of A. The previous lemma shows that the vectors $x_{i,j}$ span a dense subspace of $H(x)$.

Proposition 2.12. *The family W spans H° as a closed A,A-bimodule. Moreover, for $x \neq y \in W$ we have $H(x) \perp H(y)$.*

Proof. Denote $L_n = \mathrm{Span}\{X_{i,j} \mid X \in W_k, k \leq n, i+j+k = n\} \subset B(H_n)^\circ$, and let us show by induction over $n \geq 1$ that $L_n = B(H_n)^\circ$. For $n = 1$ we have by definition $L_1 = \mathrm{Span}\, W_1 = B(H_1)^{\circ\circ} = B(H_1)^\circ$. Assume that $L_n = B(H_n)^\circ$ and take $Y \in L_{n+1}^\perp \cap B(H_{n+1})^\circ$. We want to show that $Y = 0$. We consider first $(\mathrm{Tr}_1 \otimes \mathrm{id})(Y)$. For any generator $X_{i,j}$ of L_n we have

$$\mathrm{Tr}_n(X_{i,j}^*(\mathrm{Tr}_1 \otimes \mathrm{id})(Y)) = (\mathrm{Tr}_1 \otimes \mathrm{Tr}_n)(P_{n+1}(\mathrm{id}_1 \otimes X_{i,j}^*)P_{n+1}Y) = \mathrm{Tr}_{n+1}(X_{i+1,j}^* Y) = 0,$$

by assumption on Y. Since $L_n = B(H_n)^\circ$, this implies $(\mathrm{Tr}_1 \otimes \mathrm{id})(Y) = 0$. Similarly, $(\mathrm{id} \otimes \mathrm{Tr}_1)(Y) = 0$. As a result, $Y \in B(H_{n+1})^{\circ\circ}$. But $B(H_{n+1})^{\circ\circ} \subset L_{n+1}$, and $Y \perp L_{n+1}$, so that we have indeed proved $Y = 0$. Taking into account Proposition 2.9, this proves that $p_n H^\circ \subset \mathrm{Span}\, \mathcal{A}W\mathcal{A}$ for every n and the first result follows.

For the second part of the statement, take $x \in W_k$, $y \in W_l$ distinct, with $k \leq l$. The subspaces $H(x)$, resp. $H(y)$ are spanned by vectors $\chi_1^p x \chi_1^q$, resp. $\chi_1^r y \chi_1^s$. We have

$$(\chi_1^p x \chi_1^q \mid \chi_1^r y \chi_1^s) = (x \mid \chi_1^{p+r} y \chi_1^{q+s}) = (x \mid p_k(\chi_1^{p+r} y \chi_1^{q+s})).$$

But $\chi_1^{p+r} y \chi_1^{q+s} \in \mathrm{Span}\{y_{i,j}\}$ and $y_{i,j} \in p_{i+l+j}H$. Since $k \leq l$ this implies that $p_k(\chi_1^{p+r} y \chi_1^{q+s}) \in \mathbb{C}y$ and the result follows since $x \perp y$. □

Denote $B \subset B(H^\circ)$ the commutant of the left and right actions of A. Being the commutant of an abelian algebra, it is a type I von Neumann algebra, which can be decomposed into type I_n algebras. The numbers $n \in \mathbb{N}^* \cup \{\infty\}$ appearing in this way form the Pukánszky invariant of the maximal abelian subalgebra A.

Corollary 2.13. *The bimodule H° is isomorphic to $L^2(A) \otimes \ell^2(W) \otimes L^2(A)$. In particular the Pukánszky invariant of $A \subset M$ is $\{\infty\}$.*

Proof. Indeed the proof of in [FV16, Theorem 5.10] shows that the measure on $[-2,2] \times [-2,2]$ induced by a given $\zeta \in H^\circ \cap \mathcal{A}$ and the action of $A \otimes A$ on H° is equivalent to the Lebesgue measure — in fact it has a non-zero analytic density. As a result, the corresponding cyclic bimodule $H(\zeta)$ is isomorphic to the coarse bimodule $L^2(A) \otimes L^2(A)$. This applies to $\zeta = x \in W$. Now, Proposition 2.12 shows that we have an isomorphism of A, A-bimodules

$$H^\circ \simeq \bigoplus_{x \in W} L^2(A) \otimes L^2(A) \simeq L^2(A) \otimes \ell^2(W) \otimes L^2(A).$$

As a result $(A \otimes A)' \cap B(H^\circ) \simeq A \bar{\otimes} B(\ell^2(W)) \bar{\otimes} A$ and the value of the Pukánszky invariant follows since W is infinite. □

Corollary 2.14. *For $x \in W$ the vectors $x_{i,j}$ are linearly independent.*

Proof. Since the subspaces $p_n H^\circ$ are pairwise orthogonal, it suffices to consider a subfamily $(x_{i,j})$ with $i + k + j = n$ fixed. Note that $\#\{x_{i,j} \mid i + k + j = n\} = n - k + 1$. According to Proposition 2.12 we have

$$p_n H^\circ = \bigoplus_{k \leq n} \bigoplus_{x \in W_k} \operatorname{Span}\{x_{i,j} \mid i + k + j = n\},$$

so that $\dim p_n H^\circ \leq \sum_{k=1}^{n} (n - k + 1)\#W_k$. We will prove that this estimate is an equality, so that $\dim \operatorname{Span}\{x_{i,j} \mid i + k + j = n\} = n - k + 1$ for all $x \in W_k$, $k \leq n$, which implies the linear independence.

Recall from Proposition 2.5 that $\#W_k = \dim B(H_k)^{\circ\circ} = d_{2k} - d_{2k-2}$ for $k \geq 2$, and $\#W_1 = d_2$. We have then

$$\begin{aligned}\textstyle\sum_{k=1}^{n} (n - k + 1)\#W_k &= \textstyle\sum_{k=1}^{n} (n - k + 1) d_{2k} - \sum_{k=2}^{n} (n - k + 1) d_{2k-2} \\ &= \textstyle\sum_{k=1}^{n} (n - k + 1) d_{2k} - \sum_{k=1}^{n-1} (n - k) d_{2k} \\ &= \textstyle\sum_{k=1}^{n} d_{2k} = d_n^2 - 1 = \dim B(H_n)^\circ = \dim p_n H^\circ.\end{aligned}$$

The computation of the sum in the last line follows from the decomposition of $u_n \otimes u_n$ given by the fusion rules. □

Remark 2.15. As a result, the map $\Phi : c_c(\mathbb{N}) \otimes H^{\circ\circ} \otimes c_c(\mathbb{N}) \to H^\circ$, $\delta_i \otimes x \otimes \delta_j \mapsto x_{i,j}$ is injective with dense image. It is however not an isometry. We will see in the next section that, at least for "large N", it extends to an isomorphism from $\ell^2(\mathbb{N}) \otimes H^{\circ\circ} \otimes \ell^2(\mathbb{N})$ to H°.

We end this section with one further property of elements of $H^{\circ\circ}$ which is established using the action of planar tangles and will be used in Section 4.

Proposition 2.16. *For any $\zeta \in H(k)$, $\zeta' \in H(k')$ and $y \in p_n H$ with $n < |k - k'|$ we have $y\zeta \perp \zeta'$. If $k > n$ we have $y\zeta \in H^\circ$.*

Proof. By bilinearity one can assume $\zeta = x_{i,j} = u_{i+k+j}(X_{i,j})$, $\zeta' = x'_{i',j'} = u_{i'+k'+j'}(X'^{*}_{i',j'})$, with $X \in B(H_k)^{\circ\circ}$, $X' \in B(H_{k'})^{\circ\circ}$. Denote also $y = u_n(Y) \in M$ with $Y \in B(H_n)$. Then the product $y\zeta$ is a linear combination of elements $u_m(Y *_m X_{i,j})$ with $m = n + i + k + j - 2a$. Using the Peter-Weyl relations (1.1) it thus suffices to prove that $\operatorname{Tr}(X'_{i',j'}(Y *_m X_{i,j})) = 0$, with $m = i' + k' + j' = i + k + j - 2a$.

By definition, the element in the trace is computed by the following formula:

$$\begin{aligned}P_m(\mathrm{id}_{i'} \otimes X' \otimes \mathrm{id}_{j'}) P_m(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+j+k-a})(\mathrm{id}_n \otimes P_{i+k+j}) \\ (Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_j)(\mathrm{id}_n \otimes P_{i+k+j})(\mathrm{id}_{n-a} \otimes t_a \otimes \mathrm{id}_{i+j+k-a}) P_m.\end{aligned}$$

Since P_m, P_{i+k+j}, t_a are morphisms, this element is a linear combination of planar tangles on m lower and upper points, with 3 inside boxes, applied to X, X', Y. Since $\operatorname{Tr}(Z) = t_m^*(Z \otimes \mathrm{id}_m) t_m$, the scalar $\operatorname{Tr}(X'_{i',j'}(Y *_m X_{i,j}))$ is itself a linear combination of such planar tangles T, without external points, applied to X, X', Y.

Fix one of these tangles and consider the strings starting at one of the $2k$ points on the internal box corresponding to X. These strings can have their second ends on X, X' or Y. If $2k > 2k' + 2n$, the first possibility must happen at least once, i.e. there is a string connecting two points of X. Since the strings are non crossing, this implies that there is even a string connecting two consecutive points of the internal box corresponding to X. But then the value of the tangle applied to X, X', Y is 0 since $X \in B(H_k)^{\circ\circ}$: see Remark 2.2.

If $k < k' - n$ we proceed in the same way by considering strings starting on the internal box corresponding to X'. The last assertion of the statement amounts to considering the trace $\operatorname{Tr}(Y *_m X_{i,j})$ which is again a linear combination of planar tangles without external points applied to Y and X, and if $k > n$ the same argument as above applies. □

3. Invertibility of the Gram Matrix

In this section we fix $k \in \mathbb{N}^*$, $x = u_k(X) \in p_k H^{\circ\circ}$ with $X \in B(H_k)^{\circ\circ}$ an eigenvector of ρ with associated eigenvalue μ, $|\mu| = 1$. Recall the notation $x_{i,j} = p_{i+k+j}(\chi_i x \chi_j)$, $X_{i,j} = P_{i+k+j}(\mathrm{id}_i \otimes X \otimes \mathrm{id}_j)P_{i+k+j}$. We know from the previous section that $(x_{i,j})$ spans a dense subspace of the bimodule AxA. Our aim is now to show that it is a Riesz basis, i.e. it implements an isomorphism between $H(x) = \overline{AxA}$ in H and $\ell^2(\mathbb{N} \times \mathbb{N})$. We will only achieve this for small q, i.e. large N. We thus consider the associated Gram matrix, which is block diagonal since $p_m H \perp p_n H$ for $m \neq n$. Let us formalize this as follows:

Notation 3.1. We fix $k \in \mathbb{N}^*$ and a unital vector $x = u_k(X) \in W_k \subset p_k H^{\circ\circ}$. We denote $G = G(x)$ the Gram matrix of the family $(x_{i,j})_{i,j} \subset H$, and $G_n = G_n(x)$ its diagonal block corresponding to indices (i,j) such that $x_{i,j} \in p_n H$, i.e. $i + k + j = n$. Since k is fixed we drop the second index j and denote $x_{n;i} = x_{i,j}$, $X_{n;i} = X_{i,j}$. For i, $p \in \{0, \dots, n-k\}$ we denote accordingly

$$G_{n;i,p} = (x_{n;i} \mid x_{n;p}) = d_n^{-1}(X_{n;i} \mid X_{n;p}).$$

The second equality follows from the Peter-Weyl-Woronowicz orthogonality relations, using the Hilbert-Schmidt scalar product in $B(H_n)$. Let us record the following symmetry properties of G:

Lemma 3.2. *For any $n = i + k + j = p + k + q$ we have*

$$G_{n;i,p}(x) = \overline{G_{n;p,i}(x)} = G_{n;q,j}(x^*) = G_{n;j,q}(S(x)).$$

Proof. As a Gram matrix, G_n is self-adjoint, which corresponds to the first equality. Define maps J, $U : H \to H$ by $J(x) = x^*$, $U(x) = S(x)$ where S is the antipode. The maps are surjective isometries because we are in the Kac case, and since u_n is orthogonal they stabilize $p_n H$ and send χ_n to itself. We have then

$$\begin{aligned} G_{n;i,p}(x) &= (p_n(\chi_i x \chi_j) \mid p_n(\chi_p x \chi_q)) = (J p_n(\chi_p x \chi_q) \mid J p_n(\chi_i x \chi_j)) \\ &= (p_n(\chi_q x^* \chi_p) \mid p_n(\chi_j x^* \chi_i)) = G_{n;q,j}(x^*) \\ &= (U p_n(\chi_i x \chi_j) \mid U p_n(\chi_p x \chi_q)) = (p_n(\chi_j S(x) \chi_i) \mid p_n(\chi_q S(x) \chi_p)) = G_{n;j,q}(S(x)). \quad \square \end{aligned}$$

Our main aim is then to show the existence of a constant C such that $\|G_n\|$, $\|G_n^{-1}\| \leq C$ for all n. In fact we even want the constant C to be uniform over k and $x \in W_k$, so that the map Φ from Remark 2.15 will indeed be an isomorphism.

We shall first show that the Gram matrix $G = G(x)$ is bounded as an operator on $\ell^2(\mathbb{N} \times \mathbb{N})$. We start with an easy estimate, which is not sufficient for this purpose but will be useful later. We then prove an off-diagonal decay estimate for the coefficients of the Gram matrix, see Lemma 3.4, using the improvement of the main estimate of [FV16] established at Lemma 1.6. These two results easily imply the boundedness of $G(x)$ on $\ell^2(\mathbb{N} \times \mathbb{N})$, which we record at Proposition 3.5. Note however that the constant C obtained in this way depends on k, so that one cannot deduce the boundedness of the whole Gram matrix. This will be improved later.

Lemma 3.3. *We have $\|x_{n;i}\|_2 \leq (1-q^2)^{-3/2}\|x\|_2$, hence $|G_{n;i,p}| \leq (1-q^2)^{-3}\|x\|_2^2$, for all n, $0 \leq i, p \leq n-k$.*

Proof. We have

$$\begin{aligned} \|X_{n;i}\|_2^2 &= \mathrm{Tr}(P_n(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)P_n(\mathrm{id}_i \otimes X \otimes \mathrm{id}_j)P_n) \\ &\leq \mathrm{Tr}(P_n(\mathrm{id}_i \otimes X^*X \otimes \mathrm{id}_j)P_n) \leq \mathrm{Tr}(P_i \otimes X^*X \otimes P_j) = d_i d_j \|X\|_2^2, \end{aligned}$$

hence $\|x_{n;i}\|_2^2 \leq (d_i d_j d_k / d_n)\|x\|_2^2$. The result then follows from Lemma 1.2. $\square$

Lemma 3.4. *For every $q_0 \in]0,1[$ there exists $\alpha \in]0,1[$ and $C > 0$ depending only on q_0 such that $|G_{n;i,p}| \leq C q^{\alpha(|p-i|-k)-2-k}\|x\|_2^2$ for all n, i, p such that $|p-i| \geq k$, as soon as $q \in]0,q_0]$.*

Proof. The reader will find after the proof a graphical "explanation" of the computations. Write $n = i + k + j = p + k + q$. We have $(x_{n;i} \mid x_{n;p}) = \mathrm{Tr}(X^*_{i,j} X_{p,q})/d_n$. We first assume $p - i \geq k$ and put $a = \lfloor (p - i - k)/2 \rfloor$. By Lemma 1.3 we have $\|P_n - (\mathrm{id}_{i+k+a} \otimes P_{j-a})(P_p \otimes \mathrm{id}_{k+q})\| \leq Dq^{p-(i+k+a)} \leq Dq^a$, where $D > 0$ is a constant depending only on q_0. This yields

$$\begin{aligned}(x_{n;i} \mid x_{n;p}) &= d_n^{-1} \mathrm{Tr}_n[P_n(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)P_n(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)P_n] \\ &\simeq d_n^{-1} \mathrm{Tr}_n[P_n(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_a \otimes P_{j-a})(P_p \otimes X \otimes \mathrm{id}_q)P_n] \\ &= d_n^{-1}(\mathrm{Tr}_{i+k} \otimes \mathrm{Tr}_a \otimes \mathrm{Tr}_{j-a})[(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_a \otimes P_{j-a})(P_p \otimes X \otimes \mathrm{id}_q)P_n]. \end{aligned} \tag{3.1}$$

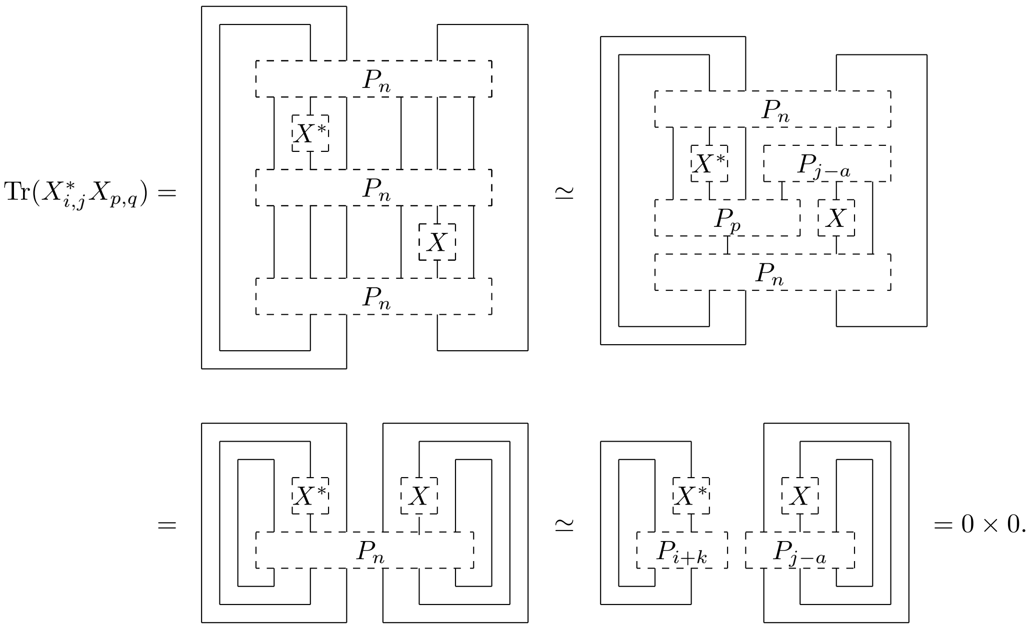
Since $d_n^{-1} \mathrm{Tr}_n(P_n \cdot P_n)$ is a state, the error is bounded by $Dq^a\|X\|^2$. In the last expression, the projection $P_p \otimes \mathrm{id}_k \otimes \mathrm{id}_q$ is absorbed in P_n, and since $j - a \geq k + q$ the partial trace $(\mathrm{id}_{i+k} \otimes \mathrm{Tr}_a \otimes \mathrm{id}_{j-a})(P_n)$ appears. We know from Lemma 1.6 that this partial trace is equal to a multiple λ of the identity up to $Ed_a q^{\lfloor \beta a \rfloor}$ if $q \in]0, q_0]$, for some $\beta \in]0, 1[$ and $E > 0$ depending only on q_0. Applying the remaining traces and dividing by d_n the total error is controlled by

$$\begin{aligned} Dq^a\|X\|^2 + Eq^{\lfloor \beta a \rfloor} \frac{d_{i+k} d_a d_{j-a}}{d_n} \|X\|^2 &\leq q^{\lfloor \beta a \rfloor}(D + E/(1 - q_0^2)^3)\|X\|_2^2 \\ &\leq Cq^{\alpha(p-i-k)-2-k}\|x\|_2^2, \end{aligned}$$

for $C = [D + E/(1 - q_0^2)^3]/(1 - q_0)$ and $\alpha = \beta/2$ — recall that $\|X\|_2^2 = d_k\|x\|_2^2 \leq q^{-k}\|x\|_2^2/(1 - q)$. But if we replace $(\mathrm{id}_{i+k} \otimes \mathrm{Tr}_a \otimes \mathrm{id}_{j-a})(P_n)$ by $\lambda(P_{i+k} \otimes P_{j-a})$ in (3.1) we can see the trace $\mathrm{Tr}((\mathrm{id}_i \otimes X^*)P_{i+k})$ which vanishes (as well as $\mathrm{Tr}((\mathrm{id}_{a'} \otimes X \otimes \mathrm{id}_q)P_{j-a})$, where $a' = \lceil (p - i - k)/2 \rceil$).

This proves the result if $p - i \geq k$. If $i - p \geq k$ we can proceed in the same way "on the other side" and the result follows because then $q - j = |p - i| \geq k$. □

We give below a graphical version of the above proof, for the convenience of the reader, in the case $p - i \geq k$. Of course it is still necessary to carry out the quantitative bookkeeping of approximations, as we did above. It is possible to draw similar graphical computations for many lemmata in this section and the following ones.



Proposition 3.5. *Fix $q \in]0, 1[$ and assume that $q \in]0, q_0]$. There exists a constant $C > 0$, depending on k and q_0, such that $\|G_n\| \leq C\|x\|_2^2$ for all n. In particular $G(x)$ is bounded.*

Proof. Take the constants α, C provided by Lemma 3.4. Put $l = k + \lceil (2 + k)/\alpha \rceil$, so that $\alpha k + 2 + k \leq \alpha l$, and decompose $G_n = \hat{G}_n + \check{G}_n$, where $\hat{G}_{n;i,p} = \delta_{|i-p| \leq l} G_{n;i,p}$. From Lemma 3.4

we have $|\check G_{n;i,p}| \le Cq^{\alpha(|p-i|-l)}\|x\|_2^2$ and it is then a standard fact that $\check G$ is bounded. More precisely for any $\lambda \in \ell^2(\mathbb{N})$ we have by Cauchy-Schwarz

$$\left|\sum_{i,p}\bar\lambda_i\lambda_p\check G_{n;i,p}\right| \le (\sum_{i,p}|\lambda_i|^2|\check G_{n;i,p}|)^{1/2}(\sum_{i,p}|\lambda_p|^2|\check G_{n;i,p}|)^{1/2}$$
$$\le C\|x\|_2^2 \sum_i|\lambda_i|^2\sum_{|p-i|>l}q^{\alpha(|p-i|-l)} \le \frac{2q^\alpha C\|x\|_2^2}{1-q^\alpha}\|\lambda\|^2.$$

This shows that $\|\check G_n\| \le 2q_0^\alpha C\|x\|_2^2/(1-q_0^\alpha)$ for all n and $q\in\,]0,q_0]$.

On the other hand by Lemma 3.3 we have $|G_{n;i,p}| \le (1-q_0^2)^{-3}\|x\|_2^2$ for all n, i, p and it follows easily $\|\hat G_n\| \le (2l+1)(1-q_0^2)^{-3}\|x\|_2^2$. □

Now we want to prove that G has a bounded inverse and obtain uniform estimates with respect to k. This requires a finer analysis of the band matrices $\hat G_n$ of the previous proof. We first show that for $m<n$ the diagonal blocks of size $m-k+1$ of G_n "resemble" G_m, with a better approximation order for blocks that are far away from the "borders" of G_n. This will allow to reduce the analysis of $\hat G_n$ to that of a "fixed size" matrix G_m (in fact m will depend on k, but not on n).

Lemma 3.6. *Fix $q_0 \in\,]0,1[$ and assume that $q\in\,]0,q_0]$. Assume that $n=m+a+b$ and $m=i+k+j=p+k+q$. Then there exists a constant C depending only on q_0 such that*

$$|G_{m;i,p}-G_{n;i+a,p+a}| \le \begin{cases} C\|x\|_2^2\, q^{\max(j,q)-k} & \text{if } a=0,\\ C\|x\|_2^2\, q^{\max(i,p)-k} & \text{if } b=0.\end{cases}$$

We also have $|G_{m;i,p}-G_{n;i+a,p+a}| \le C\|x\|_2^2\, q^{\min(i,j,p,q)-k}$ for a, b arbitrary.

Proof. By definition we have $n=(i+a)+k+(j+b)=(p+a)+k+(q+b)$. The case $b=0$ follows from the case $a=0$ by symmetry. The "general case" follows from the first two cases by going first from m to $n'=m+b$ and then from n' to $n=n'+a$, and observing that $Cq^{\max(j,q)}+Cq^{\max(i,p)} \le 2Cq^{\min(i,j,p,q)}$. So we assume $a=0$.

According to Lemma 1.3 we have $\|P_n-(\mathrm{id}_{i+k}\otimes P_{j+b})(P_m\otimes\mathrm{id}_b)\| \le Cq^j$, which yields

$$\begin{aligned}(x_{n;i}\mid x_{n;p}) &= d_n^{-1}\,\mathrm{Tr}_n[P_n(\mathrm{id}_i\otimes X^*\otimes\mathrm{id}_{j+b})P_n(\mathrm{id}_p\otimes X\otimes\mathrm{id}_{q+b})P_n]\\ &\simeq d_n^{-1}\,\mathrm{Tr}_n[P_n(\mathrm{id}_i\otimes X^*\otimes P_{j+b})(P_m\otimes\mathrm{id}_b)(\mathrm{id}_p\otimes X\otimes\mathrm{id}_{q+b})P_n]\\ &= d_n^{-1}(\mathrm{Tr}_m\otimes\mathrm{Tr}_b)[P_n(\mathrm{id}_i\otimes X^*\otimes\mathrm{id}_{j+b})(P_m\otimes\mathrm{id}_b)(\mathrm{id}_p\otimes X\otimes\mathrm{id}_{q+b})]\end{aligned}$$

up to $Cq^j\|X\|^2 \le Cq^jd_k\|x\|_2^2$, since P_{j+b} is absorbed in P_n. Since by [VV07, Proposition 1.13] we have $(\mathrm{id}\otimes\mathrm{Tr}_b)(P_n)=(d_n/d_m)P_m$, this reads

$$(x_{n;i}\mid x_{n;p}) \simeq d_m^{-1}\,\mathrm{Tr}_m[P_m(\mathrm{id}_i\otimes X^*\otimes\mathrm{id}_j)P_m(\mathrm{id}_p\otimes X\otimes\mathrm{id}_q)] = (x_{m;i}\mid x_{m;j})$$

up to $Cq^jd_k\|x\|_2^2 \le Cq^{j-k}\|x\|_2^2/(1-q_0)$. If $j\le q$ we proceed in the same way starting with the estimate $P_n \simeq (P_m\otimes\mathrm{id}_b)(\mathrm{id}_{p+k}\otimes P_{q+b})$ up to Cq^q. □

In the next Theorem we show that the blocks G_n of the Gram matrix G are related by a recursion formula, which allows at Lemma 3.9 to obtain estimates on G_m with a good behavior as $k\to\infty$, improving the "naive" Lemma 3.3.

Theorem 3.7. *Fix $n>k>0$ and $x=u_k(X)\in W_k$ with $\rho(X)=\mu X$. For $0\le i<n-k$ and $0\le p\le n-k$ we have:*

$$G_{n;i,p} = \delta_{p<n-k}(1-A_p^n)G_{n-1;i,p}+\delta_{p>0}B_p^nG_{n-1;i,p-1}+\delta_{p>1}C_p^nG_{n-1;i,p-2} \text{ where}$$
$$A_p^n = \frac{d_{p+k}d_{p+k-1}}{d_nd_{n-1}},\ B_p^n = 2(-1)^k\,\mathrm{Re}(\mu)\frac{d_{p+k-1}d_{p-1}}{d_nd_{n-1}},\ C_p^n = -\frac{d_{p-1}d_{p-2}}{d_nd_{n-1}}.$$

Note that $A_p^n=1$ if $p=n-k$, $B_p^n=0$ if $p=0$ and $C_p^n=0$ if $p=0$ or 1, if one puts $d_{-l}=0$ for $l>0$. Hence the corresponding terms vanish "naturally" from the recursion equation.

The proof of the theorem will easily follow from the following Lemma, that we will reuse in Section 5, and which relies on two applications of Wenzl's recursion relation (1.3).

Lemma 3.8. *For $X \in B(H_k)^{\circ\circ}$, $k \in \mathbb{N}^*$, such that $\rho(X) = \mu X$, we have for all p, $q \in \mathbb{N}$ and $n = p + k + q$:*

$$\frac{d_{n-1}}{d_n}(\mathrm{id} \otimes \mathrm{Tr}_1)(X_{p,q}) = \delta_{q>0}(1 - A_p^n)X_{p,q-1} + \delta_{p>0}B_p^n X_{p-1,q} + \delta_{p>1}C_p^n X_{p-2,q+1}.$$

Proof. **Step 1.** In this proof we denote $T = X_{p,q}$. We will use the Jones-Wenzl recursion formula for each projection P_n appearing in the definition $T = P_n(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)P_n$, starting with the left occurrence. By the adjoint of (1.3) we have $T = \sum_{l=1}^n (-1)^{n-l}(d_{l-1}/d_{n-1})T_l$ where

$$T_n = (P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)P_n \qquad \text{and}$$
$$T_l = (P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t)(\mathrm{id}_{l-1} \otimes t^* \otimes \mathrm{id}_{n-l-1})(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)P_n \qquad \text{for } l < n.$$

Step 2. Denote $M = T_n$. Recall that $(\mathrm{id} \otimes \mathrm{Tr}_1)(P_n) = (d_n/d_{n-1})P_{n-1}$, so that if $q \geq 1$ we have $(\mathrm{id} \otimes \mathrm{Tr}_1)(M) = (d_n/d_{n-1})X_{p,q-1}$. If $q = 0$ we have to apply (1.3) to the second occurrence of P_n. This yields $M = \sum_{l=1}^n (-1)^{n-l}\ (d_{l-1}/d_{n-1})M_l$ where

$$M_n = (P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_p \otimes X)(P_{n-1} \otimes \mathrm{id}_1) \qquad \text{and}$$
$$M_l = (P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_p \otimes X)(\mathrm{id}_{l-1} \otimes t \otimes \mathrm{id}_{n-l-1})(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1) \qquad \text{for } l < n.$$

In $(\mathrm{id} \otimes \mathrm{Tr}_1)(M_n)$ we can factor $(\mathrm{id} \otimes \mathrm{Tr}_1)(X) = 0$ so this term disappears. Moreover all terms M_l vanish because t hits $X = XP_k$ or P_{n-1}, except M_p. By the conjugate equation we have

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{Tr}_1)(M_p) &= (\mathrm{id}_{n-1} \otimes t^*)(M_p \otimes \mathrm{id}_1)(\mathrm{id}_{n-1} \otimes t)\\ &= P_{n-1}(\mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_p \otimes X \otimes \mathrm{id}_1)(\mathrm{id}_{p-1} \otimes t \otimes \mathrm{id}_{n-p})P_{n-1}\end{aligned}$$

and we recognize $(\mathrm{id} \otimes \mathrm{Tr}_1)(M_p) = P_{n-1}(\mathrm{id}_{p-1} \otimes \rho(X))P_{n-1}$. Altogether we can thus write $(\mathrm{id} \otimes \mathrm{Tr}_1)(M) = \delta_{p<n-k}(d_n/d_{n-1})X_{p,q-1} + (-1)^k\delta_{p=n-k}\mu(d_{p-1}/d_{n-1})X_{p-1,0}$.

Step 3. Now we come back to the terms T_l with $l < n$. Most of them vanish because t^* hits either $X = P_kX$ or P_n. The only remaining terms are $M' := T_{p+k}$, which appears only if $p < n - k$ (i.e. $q \geq 1$), and $M'' := T_p$, which appears if $p \geq 1$. For these terms we apply as well (1.3) to the second occurrence of P_n. This yields $M' = \sum_{l=1}^n (-1)^{n-l}(d_{l-1}/d_{n-1})M'_l$ where

$$M'_n = (P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t)(\mathrm{id}_{p+k-1} \otimes t^* \otimes \mathrm{id}_{q-1})(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)(P_{n-1} \otimes \mathrm{id}_1) \quad \text{and}$$
$$\begin{aligned}M'_l = (P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t)(\mathrm{id}_{p+k-1} \otimes t^* \otimes \mathrm{id}_{q-1})&\\ (\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)(\mathrm{id}_{l-1} \otimes t \otimes \mathrm{id}_{n-l-1})(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1) &\quad \text{for } l < n.\end{aligned}$$

One can simplify $(\mathrm{id}\otimes\mathrm{Tr}_1)(M'_n) = (\mathrm{id}_{n-1}\otimes t^*)(M'_n\otimes\mathrm{id}_1)(\mathrm{id}_{n-1}\otimes t)$ using the conjugate equation:

$$(\mathrm{id} \otimes \mathrm{Tr}_1)(M'_n) = P_{n-1}(\mathrm{id}_{p+k-1} \otimes t^* \otimes \mathrm{id}_q)(\mathrm{id}_p \otimes X \otimes \mathrm{id}_{q+1})(P_{n-1} \otimes t).$$

This vanishes if $q \geq 2$ because in this case t hits P_{n-1}. If $q = 1$ applying once again the conjugate equation we recognize $(\mathrm{id} \otimes \mathrm{Tr}_1)(M'_n) = P_{n-1}(\mathrm{id}_p \otimes X \otimes \mathrm{id}_{q-1})P_{n-1}$. Finally we have $(\mathrm{id} \otimes \mathrm{Tr}_1)(M'_n) = \delta_{p=n-k-1}X_{p,q-1}$.

Again most of the terms M'_l with $l < n$ vanish because the last t hits either $X = XP_k$ or P_{n-1}. The first non-vanishing term, if $p \geq 1$, is M'_p and we recognize $(\mathrm{id}_k \otimes t^*)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_k) = \rho(X) = \mu X$. By the conjugate equation we have $(\mathrm{id} \otimes \mathrm{Tr}_1)(L \otimes tt^*) = L \otimes \mathrm{id}_1$ so that

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{Tr}_1)(M'_p) &= \mu(\mathrm{id} \otimes \mathrm{Tr}_1)[(P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t)\\ &\qquad (\mathrm{id}_{p-1} \otimes X \otimes \mathrm{id}_{q-1})(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1)]\\ &= \mu P_{n-1}(\mathrm{id}_{p-1} \otimes X \otimes \mathrm{id}_q)P_{n-1} = \mu X_{p-1,q}.\end{aligned}$$

The second non-vanishing term is M'_{p+k}, but it contains the term $(\mathrm{id}_{k-1} \otimes t^*)(X \otimes \mathrm{id}_1)$ $(\mathrm{id}_{k-1} \otimes t) = (\mathrm{id} \otimes \mathrm{Tr}_1)(X)$ hence it vanishes as well. Finally we have M'_{p+k+1} which appears if $q \geq 2$ and by the conjugate equation can also be written

$$M'_{p+k+1} = (P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t)(\mathrm{id}_p \otimes X \otimes \mathrm{id}_{q-2})(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1).$$

As for M'_p we have the further simplification $(\mathrm{id}\otimes\mathrm{Tr}_1)(M'_{p+k+1}) = P_{n-1}(\mathrm{id}_p\otimes X\otimes\mathrm{id}_{q-1})P_{n-1} = X_{p,q-1}$.

Step 4. We proceed similarly for M'', writing $M'' = \sum_{l=1}^{n}(-1)^{n-l}(d_{l-1}/d_{n-1})M''_l$ with

$$M''_n = (P_{n-1}\otimes \mathrm{id}_1)(\mathrm{id}_{n-2}\otimes t)(\mathrm{id}_{p-1}\otimes t^*\otimes \mathrm{id}_{k+q-1})(\mathrm{id}_p\otimes X\otimes \mathrm{id}_q)(P_{n-1}\otimes \mathrm{id}_1) \quad \text{and}$$
$$M''_l = (P_{n-1}\otimes \mathrm{id}_1)(\mathrm{id}_{n-2}\otimes t)(\mathrm{id}_{p-1}\otimes t^*\otimes \mathrm{id}_{k+q-1})$$
$$(\mathrm{id}_p\otimes X\otimes \mathrm{id}_q)(\mathrm{id}_{l-1}\otimes t\otimes \mathrm{id}_{n-l-1})(\mathrm{id}_{n-2}\otimes t^*)(P_{n-1}\otimes \mathrm{id}_1) \quad \text{for } l<n.$$

As in the case of M'_n we find

$$(\mathrm{id}\otimes \mathrm{Tr}_1)(M''_n) = P_{n-1}(\mathrm{id}_{p-1}\otimes t^*\otimes \mathrm{id}_{q+k})(\mathrm{id}_p\otimes X\otimes \mathrm{id}_{q+1})(P_{n-1}\otimes t),$$

which vanishes as soon as $q\geq 1$ because t then hits P_{n-1}. If $q=0$ we recognize $(\mathrm{id}\otimes \mathrm{Tr}_1)(M''_n) = P_{n-1}(\mathrm{id}_{p-1}\otimes \rho^*(X))P_{n-1}$. Altogether we have $(\mathrm{id}\otimes \mathrm{Tr}_1)(M''_n) = \delta_{p=n-k}\bar\mu X_{p-1,q}$.

The first non-vanishing term M''_l is M''_{p-1}, if $p\geq 2$, which by the conjugate equation reads $M''_{p-1} = (P_{n-1}\otimes \mathrm{id}_1)(\mathrm{id}_{n-2}\otimes t)(\mathrm{id}_{p-2}\otimes X\otimes \mathrm{id}_q)(\mathrm{id}_{n-2}\otimes t^*)(P_{n-1}\otimes \mathrm{id}_1)$. As for M' it follows $(\mathrm{id}\otimes \mathrm{Tr}_1)(M''_{p-1}) = P_{n-1}(\mathrm{id}_{p-2}\otimes X\otimes \mathrm{id}_{q+1})P_{n-1} = X_{p-2,q+1}$. The second non-vanishing term would be M''_p but it contains $(\mathrm{Tr}_1\otimes \mathrm{id})(X)$ hence in fact it vanishes. The last term to consider is M''_{p+k} which appears if $q=n-p-k>0$ and we recognize

$$M''_{p+k} = (P_{n-1}\otimes \mathrm{id}_1)(\mathrm{id}_{n-2}\otimes t)(\mathrm{id}_{p-1}\otimes \rho^*(X)\otimes \mathrm{id}_{q-1})(\mathrm{id}_{n-2}\otimes t^*)(P_{n-1}\otimes \mathrm{id}_1)$$

which yields as before $(\mathrm{id}\otimes \mathrm{Tr}_1)(M''_{p+k}) = \bar\mu X_{p-1,q}$.

Step 5. Finally we can collect all terms as follows:

$$\begin{aligned}(\mathrm{id}\otimes \mathrm{Tr}_1)(T) = (\mathrm{id}\otimes \mathrm{Tr}_1)\Bigg[&M + (-1)^{n-p-k}\delta_{p<n-k}\frac{d_{p+k-1}}{d_{n-1}}M'_n + (-1)^k\delta_{n-k>p\geq 1}\frac{d_{p+k-1}d_{p-1}}{d_{n-1}^2}M'_p\\ &- \delta_{p<n-k-1}\frac{d_{p+k-1}d_{p+k}}{d_{n-1}^2}M'_{p+k+1} + (-1)^{n-p}\delta_{p\geq 1}\frac{d_{p-1}}{d_{n-1}}M''_n\\ &-\delta_{p\geq 2}\frac{d_{p-1}d_{p-2}}{d_{n-1}^2}M''_{p-1} + (-1)^k\delta_{n-k>p\geq 1}\frac{d_{p-1}d_{p+k-1}}{d_{n-1}^2}M''_{p+k}\Bigg].\end{aligned}$$

According to the computations carried above we obtain:

$$\begin{aligned}d_{n-1}^2(\mathrm{id}\otimes \mathrm{Tr}_1)(T) = {}&\delta_{p<n-k}d_nd_{n-1}X_{p,q-1} + (-1)^k\delta_{p=n-k}\mu d_{p+k-1}d_{p-1}X_{p-1,q}\\ &- \delta_{p=n-k-1}d_{p+k-1}d_{p+k}X_{p,q-1} + (-1)^k\delta_{n-k>p\geq 1}\mu d_{p+k-1}d_{p-1}X_{p-1,q}\\ &- \delta_{p<n-k-1}d_{p+k-1}d_{p+k}X_{p,q-1} + (-1)^k\delta_{p=n-k}\bar\mu d_{p-1}d_{p+k-1}X_{p-1,q}\\ &- \delta_{p\geq 2}d_{p-1}d_{p-2}X_{p-2,q+1} + (-1)^k\delta_{n-k>p\geq 1}\bar\mu d_{p-1}d_{p+k-1}X_{p-1,q}.\end{aligned}$$

Merging cases together as appropriate this yields the expression in the statement. □

Proof of Theorem 3.7. Recall that by assumption $j\geq 1$, but we allow $q=n-p-k=0$. Multiplying the outcome of Lemma 3.8 by $(\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_{j-1})$ on the left, we obtain

$$\begin{aligned}\frac{d_{n-1}}{d_n}(\mathrm{id}\otimes \mathrm{Tr}_1)((\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_j)X_{p,q}) = {}&\delta_{q>0}(1-A_p^n)(\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_{j-1})X_{p,q-1}+\\ &+ \delta_{p>0}B_p^n(\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_{j-1})X_{p-1,q} + \delta_{p>1}C_p^n(\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_{j-1})X_{p-2,q+1}.\end{aligned}$$

We apply $\mathrm{Tr}_1^{\otimes(n-1)}$ to this identity. Since $X_{p,q} = P_nX_{p,q}P_n$ we have e.g.

$$\mathrm{Tr}_1^{\otimes n}((\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_j)X_{p,q}) = \mathrm{Tr}_1^{\otimes n}(X_{i,j}^*X_{p,q}) = \mathrm{Tr}_n(X_{i,j}^*X_{p,q}),$$

hence we obtain

$$\begin{aligned}d_n^{-1}\,\mathrm{Tr}_n(X_{i,j}^*X_{p,q}) = {}&\delta_{q>0}(1-A_p^n)d_{n-1}^{-1}\,\mathrm{Tr}_{n-1}(X_{i,j-1}^*X_{p,q-1})+\\ &+ \delta_{p>0}B_p^nd_{n-1}^{-1}\,\mathrm{Tr}_{n-1}(X_{i,j-1}^*X_{p-1,q}) + \delta_{p>1}C_p^nd_{n-1}^{-1}\,\mathrm{Tr}_{n-1}(X_{i,j-1}^*X_{p-2,q+1}).\end{aligned}$$

This corresponds to the identity in the statement by definition of the Gram matrix G_n, using the Peter-Weyl expression of the scalar product. □

Lemma 3.9. *Fix $q_0 \in]0,1[$ and assume that $q \in]0,q_0]$. Then there exists a constant $C > 0$, depending only on q_0, such that*

$$|G_{m;i,p}| \le C(m-k+1)q^{k+1}\|x\|_2^2 \qquad \text{and}$$
$$G_{m;p,p} \ge (C^{-1} - C(m-k)q^{k+1})\|x\|_2^2$$

if $x \in W_k$ and $0 \le i \ne p \le m-k$.

Proof. Since G_m is symmetric we can assume $i < p \le m-k$. We have $|\operatorname{Re}(\mu)| \le 1$ and for $p \le m-k$ Lemma 1.2 shows that we have $|B_p^m| \le 2q^{k+1}/(1-q^2)^2$, $|C_p^m| \le q^{2(k+1)}/(1-q^2)^2$. Since moreover $A_p^m \in [0,1]$, the recursion formula of Theorem 3.7 and Lemma 3.3 imply

$$|G_{m;i,p}| \le \delta_{p<m-k}|G_{m-1;i,p}| + 3q^{k+1}(1-q^2)^{-5}\|x\|_2^2.$$

We iterate this inequality $m-p-k+1$ times, until we reach $G_{p+k-1;i,p}$, in which case the first term disappears. This yields the first estimate with $C = 3/(1-q_0^2)^5$.

For the second one, let us start with $G_{m;0,0}$. In the recursion relation of Theorem 3.7 only the first term is non zero when $i = p = 0$. By an easy induction we have thus

$$G_{m;0,0} = G_{k;0,0}\prod_{l=k+1}^{m}(1-A_0^l) = \|x\|_2^2\prod_{l=k+1}^{m}\left(1-\frac{d_kd_{k-1}}{d_ld_{l-1}}\right).$$

Using the explicit expression of the dimensions d_i and the fact that $1-q^{2k} \le 1-q^{2l}$ if $l \ge k$ we obtain the following lower bound, which depends only on q_0:

$$G_{m;0,0} = \|x\|_2^2\prod_{l=k+1}^{m}\left(1-q^{2(l-k)}\frac{(1-q^{2k+2})(1-q^{2k})}{(1-q^{2l+2})(1-q^{2l})}\right)$$
$$\ge \|x\|_2^2\prod_{l=k+1}^{\infty}\left(1-q^{2(l-k)}\right) \ge \|x\|_2^2\prod_{i=1}^{\infty}(1-q_0^{2i}) \ge C^{-1}\|x\|_2^2,$$

increasing C if necessary. Since $\|x\|_2 = \|x^*\|_2$, the same estimate is true for $G_{m;m-k,m-k}$ by Lemma 3.2.

For the other diagonal terms we use again the recursion equation, which yields for $p < m-k$:

$$G_{m;p,p} \ge (1-A_p^m)G_{m-1;p,p} - 3q^{k+1}(1-q^2)^{-5}\|x\|_2^2.$$

Again we iterate until $m = p+k+1$, obtaining

$$G_{m;p,p} \ge G_{p+k;p,p}\textstyle\prod_{l=p+k+1}^{m}(1-A_p^l) - 3(m-p-k)q^{k+1}(1-q^2)^{-5}\|x\|_2^2.$$

The coefficients $1-A_p^l$ do not appear in the second term since they are dominated by 1. We have already proved above that $\prod_{l=p+k+1}^{m}(1-A_p^l) \ge C^{-1}$ (replace k by $k+p$) and so we obtain $G_{m;p,p} \ge C^{-2}\|x\|_2^2 - C(m-k)q^{k+1}\|x\|_2^2$. □

The estimates we have obtained about the "fixed size" matrix G_m will be sufficient for our purposes only in the $q \to 0$ limit. This corresponds to letting $q + q^{-1} = N \to \infty$, and apparently we are thus varying the spaces H_1, H_k. However, let us note that the numbers

$$G_{n;i,p} = d_n^{-1}\operatorname{Tr}_1^{\otimes n}[P_n(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)P_n(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)]$$

do not really depend on the precise form of the matrix $X \in B(H_k)^{\circ\circ}$, but only on k, $\|x\|_2$ and on the eigenvalue μ of the rotation operator ρ corresponding to X. Indeed, we can expand the projections P_n into linear combination of Temperley-Lieb diagrams π, whose coefficients depend on n, π and the parameter q. Moreover, after this expansion the evaluation of

$$\operatorname{Tr}_1^{\otimes n}[T_\pi(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)T_{\pi'}(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)]$$

is given by the evaluation of a Temperley-Lieb tangle at X^* and X. Non-vanishing terms necessarily correspond to tangles where strings cannot start and end on the same internal box, and so they are of the form $\operatorname{Tr}_k[\rho^r(X)^*\rho^s(X)] = \mu^{s-k}\|X\|_2^2 = d_k\mu^{s-k}\|x\|_2^2$. As a result $G_{n;i,p}$

can as well be considered as a function of k, μ, $\|x\|_2$ and q. Then it makes sense to take a limit $q \to 0$, and this will allow to prove results "for large N".

Theorem 3.10.
(1) For all $q_0 \in]0,1[$ there exists $C > 0$ such that, assuming $q \le q_0$, we have $\|G_n(x)\| \le C$ for all $x \in W$ and all n.
(2) There exists $q_1 \in]0,1[$ and $D > 0$ such that, assuming $q \le q_1$, we have $\|G_n(x)^{-1}\| \le D$ for all $x \in W$ and all n.

This shows in particular that $\{x_{i,j} \mid x \in W, i, j \in \mathbb{N}\}$ is a Riesz basis of $H^{\circ\circ}$ if $q \le q_1$, and that the map $\Phi : \ell^2(\mathbb{N}) \otimes H^{\circ\circ} \otimes \ell^2(\mathbb{N}) \to H^\circ$, $\delta_i \otimes x \otimes \delta_j \mapsto x_{i,j}$ from Remark 2.15 is an isomorphism.

Proof. Fix $q_0 \in]0,1[$ and assume $q < q_0$. In this proof C denotes a "generic constant" depending on q_0, that we will only modify a finite number of times. We take the constants $C > 0$ and $\alpha > 0$ of Lemma 3.4 and we fix the "cut-off width" $l = k + \lceil (3+k)/\alpha \rceil$. We will distinguish three regimes for the coefficients of our Gram matrix: the diagonal entries, for which we have the trivial estimate of Lemma 3.3 and the lower bound of Lemma 3.9 ; the entries $G_{n;i,p}$ with $0 < |i-p| < 2l$ for which we have the uniform estimate of Lemma 3.9 with a good behavior as $k \to \infty$; and the entries such that $|i-p| \ge 2l$ for which we have the off-diagonal decay estimate of Lemma 3.4 with a bad behavior as $k \to \infty$.

Recall that Lemma 3.4 shows that $|G_{n;i,p}| \le Cq^{\alpha(|p-i|-k)-2-k}\|x\|_2^2$ if $|p-i| \ge k$, which by definition of l yields $|G_{n;i,p}| \le Cq^{1+\alpha(|p-i|-l)}\|x\|_2^2$. In particular for $|p-i| \ge 2l$ we obtain $|G_{n;i,p}| \le Cq^{1+\alpha|p-i|/2}\|x\|_2^2$.

We then deal with the entries such that $0 < |p-i| < 2l$. First assuming $n > 2l + 5k + 1$, we approximate each such entry $G_{n;i,p}$ by a corresponding entry $G_{m;i-a,p-a}$ of the smaller matrix G_m with $m - k = 2l + 4k + 1$, using Lemma 3.6. Write $n = i + k + j = p + k + q = m + a + b$. If i, j, p, $q > 2k$ we can choose a, b such that $i - a$, $p - a$, $j - b$, $q - b \ge 2k + 1$ — we can e.g. take $a = \min(i,p) - 2k - 1$, and since $|i-p| < 2l$ we have $i - a < 2l + 2k + 1$ hence $j - b = 2l + 4k - (i-a) + 1 > 2k$ and, similarly, $q - b > 2k$. We have then

$$|G_{n;i,p} - G_{m;i-a,p-a}| \le Cq^{1+k}\|x\|_2^2.$$

If $i \le 2k$ or $p \le 2k$ we use the case $a = 0$, we have then $j - b \ge 2k+1$ (resp. $q - b \ge 2k + 1$) and the estimate still holds. Similarly if $j \le 2k$ or $q \le 2k$ we use the case $b = 0$.

Now if $i \ne p$ Lemma 3.9 shows that $|G_{m;i-a,p-a}| \le C(m-k+1)q^{k+1}\|x\|_2^2$. Altogether we have obtained the estimate $|G_{n;i,p}| \le C(2l+4k+3)q^{1+k}\|x\|_2^2$ if $0 < |p-i| < 2l$. It holds also if $n \le 2l + 5k + 1$ by applying directly Lemma 3.9 with $m = n$. Observe moreover that $l \le (1+\alpha^{-1})k + 1 + 3\alpha^{-1} \le 6\alpha^{-1}k$. In particular the sequence $v_k = (2l+4k+3)q_0^{k/2}$ is bounded, hence we can modify C so that $|G_{n;i,p}| \le Cq^{1+k/2}\|x\|_2^2$ for $0 < |p-i| < 2l$. In that case we have $|p-i| < 12\alpha^{-1}k$, hence we have as well $|G_{n;i,p}| \le Cq^{1+\alpha|p-i|/24}\|x\|_2^2$. Merging this with the case $|p-i| \ge 2l$ we have finally $|G_{n;i,p}| \le Cq^{1+\alpha|p-i|/24}\|x\|_2^2$ for all $i \ne p$, where C and α depend only on q_0.

Decompose $G_n = \hat{G}_n + \check{G}_n$, where $\hat{G}_n$ is diagonal with the same diagonal entries as G_n. The previous estimate shows that $\check{G}_n$ is bounded, more precisely for any $\lambda \in \ell^2(\mathbb{N})$ we have by Cauchy-Schwarz

$$\begin{aligned}\Big|\textstyle\sum_{i,p}\bar{\lambda}_i\lambda_p\check{G}_{n;i,p}\Big| &\le (\textstyle\sum_{i,p}|\lambda_i|^2|\check{G}_{n;i,p}|)^{1/2}(\textstyle\sum_{i,p}|\lambda_p|^2|\check{G}_{n;i,p}|)^{1/2} \\ &\le Cq\|x\|_2^2 \textstyle\sum_i|\lambda_i|^2\textstyle\sum_{|p-i|\ge 1}q^{\alpha|p-i|/24} \le \dfrac{2q^{1+\alpha/24}C\|x\|_2^2}{1-q^{\alpha/24}}\|\lambda\|^2.\end{aligned}$$

This shows that $\|\check{G}_n\| \le Cq\|x\|_2^2$ for all n and x, after dividing C by $2/(1-q_0^{\alpha/24})$. On the other hand we also have $\|\hat{G}_n\| \le \|x\|_2^2/(1-q_0^2)^3$ by Lemma 3.3 and the first assertion is proved.

For the inverse of G, we need a lower bound on the diagonal entries. We proceed as above, approximating each coefficient $G_{n;p,p}$ by a diagonal coefficient $G_{m;p-a,p-a}$ of a smaller matrix G_m, with $m - k = 1 + 4k$, and either $a = 0$, $b = 0$, or $p - a = 2k + 1 = q - b$. This yields

$|G_{n;p,p} - G_{m;p-a,p-a}| \leq Cq^{k+1}\|x\|_2^2$. Then we use the lower bound of Lemma 3.9, obtaining

$$G_{n;p,p} \geq C^{-1}\|x\|_2^2 - C(m-k+1)q^{k+1}\|x\|_2^2 \geq C^{-1}\|x\|_2^2 - Cq(4k+2)q_0^k\|x\|_2^2.$$

Again if $m = 1+5k \geq n$ we obtain this estimate directly from Lemma 3.9, without using G_m. Since the sequence $v_k = (4k+2)q_0^k$ is bounded, we can modify C so as to obtain $G_{n;p,p} \geq (C^{-1} - Cq)\|x\|_2^2$. For q small enough, $C^{-1} - Cq > 0$ and this shows $\hat{G}_n^{-1} \leq (C^{-1} - Cq)^{-1}\|x\|_2^{-2}\,\mathrm{I}$.

Now we write $G_n = (\mathrm{I} + \check{G}_n\hat{G}_n^{-1})\hat{G}_n$. The estimates obtained above show that we have $\|\check{G}_n\hat{G}_n^{-1}\| \leq D(q) := Cq/(C^{-1} - Cq)$. For q_1 small enough and $q \leq q_1$ we have $D(q) \leq D(q_1) < 1$ so that G_n is invertible. Moreover we have

$$G_n^{-1} = \hat{G}_n^{-1}\textstyle\sum_{i=0}^{\infty}(-1)^i[\check{G}_n\hat{G}_n^{-1}]^i \tag{3.2}$$

so that $\|G_n^{-1}\| \leq (C^{-1} - Cq_1)^{-1}(1 - D(q_1))^{-1}\|x\|_2^{-2}$. □

Remark 3.11. Using the recursion relation of Theorem 3.7 and the symmetry properties of G_n, one can compute G_n by induction on n. Numerical experiments then show the existence, for all $q \in\,]0,1[$, of a constant $C > 0$ such that $\|G_n\| \leq C\|x\|_2^2$, $\|G_n^{-1}\| \leq C\|x\|_2^{-2}$ for all n, k, $x \in W_k$. Thus our proof is far from optimal and we strongly believe that the results of Theorem 3.10 hold for all $q \in\,]0,1[$ (with constants depending on q).

4. An Orthogonality Property

Recall from Sections 2 and 3 that we have an isomorphism of normed spaces $\Phi : \ell^2(\mathbb{N}) \otimes H^{\circ\circ} \otimes \ell^2(\mathbb{N}) \to H^\circ$. In this section we shall establish a crucial asymptotic orthogonality property of the following subspaces:

Notation 4.1. For every $m \in \mathbb{N}$ we consider the following subspace of H°:

$$V_m = \Phi\left(\ell^2(\mathbb{N}_{\geq m}) \otimes H^{\circ\circ} \otimes \ell^2(\mathbb{N}_{\geq m})\right) = \overline{\mathrm{Span}}\{x_{i,j} \mid x \in H^{\circ\circ}, i,j \geq m\}.$$

In the rest of this section we will prove that for $y \in p_n(H^\circ) \subset M$ the scalar product $(\zeta y \mid y\zeta)$ becomes small, uniformly on unit vectors $\zeta \in V_m$, as $m \to \infty$, cf. Theorem 4.9. We start by computations in $\mathrm{Corep}(\mathbb{F}O_N)$ which culminate in the "local estimate" of Theorem 4.5. In these computations $x = u_k(X)$ is a fixed element of $p_k H^{\circ\circ}$, which is not assumed to be an eigenvector of the rotation map ρ. We then assemble the pieces to come back to $H(k)$ and finally H°.

Recall from Section 1 that by Tannaka-Krein duality products $x_{i,j}y$, $yx_{i,j}$ can be computed from the elements $X_{i,j} *_m Y$, $Y *_m X_{i,j} \in B(H_m)$ if $x = u_k(X)$ and $y = u_n(Y)$. Recall also that we use the Hilbert-Schmidt norm $\|X\|_2 := \mathrm{Tr}(X^*X)^{1/2}$ on $B(H_k)$. We have $\|AXB\|_2 \leq \|A\|\|X\|_2\|B\|$, where $\|A\|$, $\|B\|$ are the operator norms of A, $B \in B(H_k)$. This yields for instance the inequality $\|X_{i,j} *_m Y\|_2 \leq d_a\|X_{i,j}\|_2\|Y\|_2$, where $m = i+k+j+n-2a$, and we recall moreover that $\|X_{i,j}\|_2 \leq \sqrt{d_id_j}\|X\|_2$, see e.g. the proof of Lemma 3.3. We still make repeated use of Lemma 1.2 which allows to replace d_l with q^{-l} up to multiplicative constants.

Lemma 4.2. *Fix k, $n \in \mathbb{N}$ and $X \in B(H_k)^{\circ\circ}$, $Y \in B(H_n)^\circ$. Then for all $i \geq n$, $r \in \mathbb{N}$, $j \geq 2r$, $m = n+i+k+j-2a$ with $0 \leq a \leq n$, there exists $Z \in B(H_{m-2r})$ such that*

$$\|Y *_m X_{i,j} - P_m(Z \otimes \mathrm{id}_{2r})P_m\|_2 \leq Cd_aq^{i+k+j-a-2r}\sqrt{d_id_j}\|X\|_2\|Y\|_2,$$

where C is a constant depending only on q, and $\|Z\|_2 \leq d_a\sqrt{d_id_{j-2r}}\|X\|_2\|Y\|_2$.

Proof. We have by definition

$$Y *_m X_{i,j} = P_m(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+k+j-a})(\mathrm{id}_n \otimes P_{i+k+j})(Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_j)$$
$$(\mathrm{id}_n \otimes P_{i+k+j})(\mathrm{id}_{n-a} \otimes t_a \otimes \mathrm{id}_{i+k+j-a})P_m.$$

We use the estimate from Lemma 1.3 as follows: $P_{i+k+j} \simeq (\mathrm{id}_a \otimes P_{i+k+j-a})\,(P_{i+k+j-2r} \otimes \mathrm{id}_{2r})$ up to $Cq^{i+k+j-a-2r}$ in operator norm. Since $(\mathrm{id}_{n-a} \otimes P_{i+k+j-a})$ is absorbed by P_m we can write

$$Y *_m X_{i,j} \simeq P_m(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+k+j-a})(\mathrm{id}_n \otimes P_{i+k+j-2r} \otimes \mathrm{id}_{2r})(Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_j)$$
$$(\mathrm{id}_n \otimes P_{i+k+j-2r} \otimes \mathrm{id}_{2r})(\mathrm{id}_{n-a} \otimes t_a \otimes \mathrm{id}_{i+k+j-a})P_m$$

up to $2C\|t_a\|^2q^{i+k+j-a-2r}\|Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_j\|_2 = 2Cd_aq^{i+k+j-a-2r}\sqrt{d_id_j}\|X\|_2\|Y\|_2$ in HS norm.

This yields the result with

$$Z = P_{m-2r}(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+k+j-a-2r})(\mathrm{id}_n \otimes P_{i+k+j-2r})(Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_{j-2r})$$
$$(\mathrm{id}_n \otimes P_{i+k+j-2r})(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+k+j-a-2r})P_{m-2r}$$

which satisfies the right norm estimate. Note that we have $Z = Y *_{m-2r} X_{i,j-2r}$. □

Lemma 4.3. *Fix k, $n \in \mathbb{N}$ and $X \in B(H_k)^{\circ\circ}$, $Y \in B(H_n)^\circ$. Then for all $i \geq n$, $p \in \mathbb{N}$, $j \geq 2n+3p$, $m = n+i+k+j-2a$ with $0 \leq a \leq n$, we have*

$$\|(\mathrm{id} \otimes \mathrm{Tr}_{n+2p})(X_{i,j} *_m Y)\|_2 \leq C q^{\alpha p} q^{-p} q^{-a} q^{-(i+j+n)/2} \|X\|_2 \|Y\|_2,$$

where $C > 0$, $\alpha \in]0,1[$ are constants depending only on q.

Proof. In this proof C denotes a generic constant, depending only on q and that we will modify only a finite number of times.

We write $\mathrm{Tr}_{n+2p} = (\mathrm{Tr}_p \otimes \mathrm{Tr}_{p+a} \otimes \mathrm{Tr}_{n-a})\ (P_{n+2p} \cdot P_{n+2p})$. Applying this to $X_{i,j} *_m Y$, the projections $\mathrm{id} \otimes P_{n+2p}$ are absorbed in P_m:

$$(\mathrm{id} \otimes \mathrm{Tr}_{n+2p})(X_{i,j} *_m Y) = (\mathrm{id}_{m-2p-n} \otimes \mathrm{Tr}_p \otimes \mathrm{Tr}_{p+a} \otimes \mathrm{Tr}_{n-a})[$$
$$P_m(\mathrm{id}_{m-n+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})(X_{i,j} \otimes Y)(\mathrm{id}_{m-n+a} \otimes t_a \otimes \mathrm{id}_{n-a})P_m]$$

We shall proceed to three successive approximations to show that this quantity is almost zero.

We first use the estimate $P_m \simeq (\mathrm{id}_{m-p-n} \otimes P_{p+n})(P_{m-n+a} \otimes \mathrm{id}_{n-a})$ up to Cq^{p+a} in operator norm, from Lemma 1.3. The projections $P_{m-n+a} \otimes \mathrm{id}$ are absorbed by $X_{i,j}$ so that

$$P_m(\mathrm{id}_{m-n+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})(X_{i,j} \otimes Y)(\mathrm{id}_{m-n+a} \otimes t_a \otimes \mathrm{id}_{n-a})P_m \simeq$$
$$\simeq (\mathrm{id}_{m-p-n} \otimes P_{p+n})(\mathrm{id}_{m-n+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})$$
$$(X_{i,j} \otimes Y)(\mathrm{id}_{m-n+a} \otimes t_a \otimes \mathrm{id}_{n-a})(\mathrm{id}_{m-p-n} \otimes P_{p+n}),$$

with an error controlled by $2Cq^{p+a}\|t_a\|^2\|X_{i,j} \otimes Y\|_2 \leq 2Cq^{p+a} d_a \sqrt{d_i d_j}\|X\|_2\|Y\|_2$ in Hilbert-Schmidt norm. Observing that Tr_p hits only $X_{i,j}$, we obtain

$$(\mathrm{id} \otimes \mathrm{Tr}_{2p+n})(X_{i,j} *_m Y) \simeq (\mathrm{id}_{m-2p-n} \otimes \mathrm{Tr}_{p+a} \otimes \mathrm{Tr}_{n-a})[(\mathrm{id}_{m-2p-n} \otimes P_{p+n})$$
$$(\mathrm{id}_{m-n-p+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})(Z \otimes Y)(\mathrm{id}_{m-n-p+a} \otimes t_a \otimes \mathrm{id}_{n-a})(\mathrm{id}_{m-2p-n} \otimes P_{p+n})],$$

where $Z = (\mathrm{id}_{m-2p-n} \otimes \mathrm{Tr}_p \otimes \mathrm{id}_{p+2a})(X_{i,j})$. We denote the right-hand side by $\Phi(Z)$, with $\Phi : B(H_{m-2p-n} \otimes H_{p+2a}) \to B(H_{m-2p-n})$. After applying the trace $\mathrm{Tr}_p \otimes \mathrm{Tr}_{p+a} \otimes \mathrm{Tr}_{n-a}$, see e.g. Lemma 1.5, the error is controlled as follows:

$$\|(\mathrm{id} \otimes \mathrm{Tr}_{2p+n})(X_{i,j} *_m Y) - \Phi(Z)\|_2 \leq 2Cq^{p+a} d_a \sqrt{d_p d_{p+a} d_{n-a} d_i d_j}\|X\|_2\|Y\|_2$$
$$\leq 2Cq^{-(i+j+n)/2}\|X\|_2\|Y\|_2,$$

up to dividing C by the appropriate power of $1/(1-q^2)$, cf. Lemma 1.2. This error is less that the upper bounded in the statement.

Then we use the estimate $P_{i+j+k} \simeq (P_{i+k+p} \otimes \mathrm{id}_{j-p})(\mathrm{id}_{i+k} \otimes P_j)$, up to Cq^p in operator norm, in the expression of Z. We have by assumption $j \geq 3p+2a$, and in particular we can write

$$Z \simeq (\mathrm{id}_{i+k+j-2p-2a} \otimes \mathrm{Tr}_p \otimes \mathrm{id}_{p+2a})[(P_{i+k+p} \otimes \mathrm{id}_{j-p})(\mathrm{id}_i \otimes X \otimes P_j)(P_{i+k+p} \otimes \mathrm{id}_{j-p})]$$
$$= (P_{i+k+p} \otimes \mathrm{id}_{j-2p})(\mathrm{id}_i \otimes X \otimes P_j')(P_{i+k+p} \otimes \mathrm{id}_{j-2p}) =: Z'$$

where $P_j' = (\mathrm{id}_{j-2p-2a} \otimes \mathrm{Tr}_p \otimes \mathrm{id}_{p+2a})(P_j) \in B(H_{j-2p-2a} \otimes H_{p+2a})$. The error in Z is controlled in HS norm by $2Cq^p\sqrt{d_p}\|P_i \otimes X \otimes P_j\|_2 = 2Cq^p\sqrt{d_p d_i d_j}\|X\|_2$, so that

$$\|\Phi(Z) - \Phi(Z')\|_2 \leq 2Cq^p d_a \sqrt{d_p d_{p+a} d_{n-a} d_i d_j}\|X\|_2\|Y\|_2$$
$$\leq 2Cq^{-a} q^{-(i+j+n)/2}\|X\|_2\|Y\|_2.$$

Again this is better than the estimate we are trying to prove.

Now Lemma 1.6 shows that $P_j' \simeq \lambda(\mathrm{id}_{j-2p-2a}\otimes\mathrm{id}_{p+2a})$ in $B(H_{j-2p-2a}\otimes H_{p+2a})$, up to $Cq^{\alpha p}d_p$ in operator norm, for some constant λ depending on all parameters (and $\alpha>0$ depending only on q). In HS norm we can control this error by $Cq^{\alpha p}d_p\sqrt{d_{j-2p-2a}d_{p+2a}}$. This yields

$$Z' \simeq Z'' := \lambda[(P_{i+k+p}\otimes\mathrm{id}_{j-3p-2a})(\mathrm{id}_i\otimes X\otimes\mathrm{id}_{j-2p-2a})(P_{i+k+p}\otimes\mathrm{id}_{j-3p-2a})]\otimes\mathrm{id}_{p+2a},$$

and we have the control

$$\begin{aligned}\|\Phi(Z')-\Phi(Z'')\|_2 &\le Cq^{\alpha p}d_ad_p\sqrt{d_id_{j-2p-2a}d_{p+2a}d_{p+a}d_{n-a}}\|X\|_2\|Y\|_2\\ &\le Cq^{\alpha p}q^{-a}q^{-p}q^{-(i+j+n)/2}\|X\|_2\|Y\|_2,\end{aligned}$$

which corresponds to the estimate in the statement.

We finally arrived at

$$\begin{aligned}\Phi(Z'') = \lambda(P_{i+k+p}\otimes\mathrm{id}_{j-3p-2a})(\mathrm{id}_i\otimes X\otimes\mathrm{id}_{j-2p-2a})(P_{i+k+p}\otimes\mathrm{id}_{j-3p-2a})\times\\ \times(\mathrm{Tr}_{p+a}\otimes\mathrm{Tr}_{n-a})[(\mathrm{id}_{p+a}\otimes t_a^*\otimes\mathrm{id}_{n-a})(\mathrm{id}_{p+2a}\otimes Y)(\mathrm{id}_{p+a}\otimes t_a\otimes\mathrm{id}_{n-a})P_{p+n}].\end{aligned}$$

We claim that the second line above vanishes. Indeed $(\mathrm{Tr}_{p+a}\otimes\mathrm{id}_{n-a})(P_{p+n})$ is a multiple of id_{n-a}, since it is an intertwiner of H_{n-a}. We are then left with

$$\mathrm{Tr}_{n-a}[(t_a^*\otimes\mathrm{id}_{n-a})(\mathrm{id}_a\otimes Y)(t_a\otimes\mathrm{id}_{n-a})] = (\mathrm{Tr}_a\otimes\mathrm{Tr}_{n-a})(Y),$$

which vanishes because $y\in p_nH^\circ$. Hence $\Phi(Z'')=0$ and the result is proved. ☐

Lemma 4.4. *For $r\le m/2$, $Z\in B(H_{m-2r})$, $S=P_m(Z\otimes\mathrm{id}_{2r})P_m$ and $T\in B(H_m)$ we have*

$$|(S\mid T)| \le \sqrt{d_r}\|(\mathrm{id}\otimes\mathrm{Tr}_r)(T)\|_2\|Z\|_2 + C\|Z\|_2\|T\|_2,$$

for some constant C depending only on q.

Proof. Recall once again from Lemma 1.3 that $P_m\simeq(P_{m-r}\otimes\mathrm{id}_r)(\mathrm{id}_{m-2r}\otimes P_{2r})$ up to Cq^r, where C is a constant depending only on q. Since $T(\mathrm{id}_{m-2r}\otimes P_{2r}) = T = P_mT$ we have

$$\begin{aligned}(S\mid T) &= \mathrm{Tr}_m(P_m(Z^*\otimes\mathrm{id}_{2r})P_mT)\\ &\simeq (\mathrm{Tr}_{m-2r}\otimes\mathrm{Tr}_{2r})((\mathrm{id}_{m-2r}\otimes P_{2r})(P_{m-r}\otimes\mathrm{id}_r)(Z^*\otimes\mathrm{id}_{2r})T)\\ &= (\mathrm{Tr}_{m-2r}\otimes\mathrm{Tr}_r\otimes\mathrm{Tr}_r)((P_{m-r}\otimes\mathrm{id}_r)(Z^*\otimes\mathrm{id}_r\otimes\mathrm{id}_r)T)\\ &= \mathrm{Tr}_{m-r}[P_{m-r}(Z^*\otimes\mathrm{id}_r)P_{m-r}(\mathrm{id}\otimes\mathrm{Tr}_r)(T)].\end{aligned}$$

By Cauchy-Schwarz the last quantity is dominated by $\sqrt{d_r}\|Z\|_2\|(\mathrm{id}\otimes\mathrm{Tr}_r)(T)\|_2$. Moreover the error term in the second line is similarly bounded by $Cq^r\|Z^*\otimes\mathrm{id}_r\otimes\mathrm{id}_r\|_2\|T\|_2 = Cq^r\sqrt{d_rd_r}\|Z\|_2$ $\|T\|_2\le C'\|Z\|_2\|T\|_2$. ☐

Theorem 4.5. *Fix k, k', $n\in\mathbb{N}$ and $X\in B(H_k)^{\circ\circ}$, $X'\in B(H_{k'})^{\circ\circ}$, $Y\in B(H_n)^\circ$. Then for all i, j, i', $j'\ge 10n$ and $m=n+i+k+j-2a=n+i'+k'+j'-2a'$ with $0\le a,a'\le n$, we have*

$$|(X_{i,j}*_mY\mid Y*_mX'_{i',j'})| \le Cd_m(q^{\alpha(i'+j')}+q^{\alpha\min(j,j')})q^{(k+k')/2}\|X\|_2\|X'\|_2\|Y\|_2^2,$$

where $\alpha>0$ is a constant depending only on q, and C is a constant depending on q and n.

Proof. We put $p=\lfloor\min(j,j')/10\rfloor-n$ and $r=n+2p$. Thanks to the assumption on j, j' we have $m\ge 2r$. We first apply Lemma 4.2 to find $Z\in B(H_{m-2r})$ such that $\|Y*_mX'_{i',j'}-S\|_2\le Cq^{i'+k'+j'-a'-2r}d_{a'}\sqrt{d_{i'}d_{j'}}\|X'\|_2\|Y\|_2$ with $S=P_m(Z\otimes\mathrm{id}_{2r})P_m$. The condition $j'\ge 2r$ is satisfied since $p\le\frac{1}{10}j'-n$. We have then

$$|(X_{i,j}*_mY\mid Y*_mX'_{i',j'})| \le |(X_{i,j}*_mY\mid S)| + \|X_{i,j}*_mY\|_2\|Y*_mX'_{i',j'}-S\|_2.$$

Note that $\|X_{i,j}*_mY\|_2\le d_a\sqrt{d_id_j}\|X\|_2\|Y\|_2$, and since $2r\le j'/2$ we have

$$\begin{aligned}\|X_{i,j}*_mY\|_2\|Y*_mX'_{i',j'}-S\|_2 &\le Cq^{i'+k'+j'-a'-2r}d_ad_{a'}\sqrt{d_id_jd_{i'}d_{j'}}\|X\|_2\|X'\|_2\|Y\|_2^2\\ &\le Cq^{i'/2}q^{-(i+j)/2}q^{k'}q^{-a-2a'}\|X\|_2\|X'\|_2\|Y\|_2^2\\ &\le C_nd_mq^{i'+j'/2}q^{(k+3k')/2}\|X\|_2\|X'\|_2\|Y\|_2^2,\end{aligned}$$

were C_n is a constant depending on n and q. We apply then Lemma 4.4 to $T = X_{i,j} *_m Y$ and our S. This yields

$$|(X_{i,j} *_m Y \mid S)| \leq \sqrt{d_r}\|(\mathrm{id}\otimes\mathrm{Tr}_r)(X_{i,j} *_m Y)\|_2\|Z\|_2 + C\|Z\|_2\|X_{i,j} *_m Y\|_2.$$

Lemma 4.2 also provides a bound on $\|Z\|_2$, in particular the second term on the right-hand side above is bounded by

$$\begin{aligned} Cd_{a'}d_a\sqrt{d_id_jd_{i'}d_{j'-2r}}\|X\|_2\|X'\|_2\|Y\|_2^2 &\leq Cq^{-a-a'}q^{-(i+j+i'+j')/2}q^r\|X\|_2\|X'\|_2\|Y\|_2^2 \\ &\leq C'_nd_mq^{\frac{1}{5}\min(j,j')}q^{(k+k')/2}\|X\|_2\|X'\|_2\|Y\|_2^2, \end{aligned}$$

since we have $r \geq \frac{1}{5}\min(j,j') - n - 2$.

We finally apply Lemma 4.3. Again the condition $j \geq 2n+3p$ is satisfied because $p \leq \frac{1}{10}j - n$. This yields constants $\alpha_0 \in]0,1[$, $C > 0$ depending only on q such that

$$\begin{aligned} \sqrt{d_r}\|(\mathrm{id}\otimes\mathrm{Tr}_r)(T)\|_2\|Z\|_2 &\leq C\sqrt{d_r}q^{\alpha_0p}q^{-p}q^{-a}q^{-(i+j+n)/2}d_{a'}\sqrt{d_{i'}d_{j'-2r}}\|X\|_2\|X'\|_2\|Y\|_2^2 \\ &\leq Cq^{\alpha_0p}q^{-p+r/2}q^{-a-a'}q^{-(i+j+n)/2}q^{-(i'+j')/2}\|X\|_2\|X'\|_2\|Y\|_2^2 \\ &\leq C''_nd_mq^{\alpha_0p}q^{(k+k')/2}\|X\|_2\|X'\|_2\|Y\|_2^2. \end{aligned}$$

Since $p \geq \frac{1}{10}\min(j,j') - n - 1$, this yields the result, with $\alpha = \min(\alpha_0/10, 1/5)$. □

Corollary 4.6. *Fix k, k', $n \in \mathbb{N}$ and $x \in p_kH^{\circ\circ}$, $x' \in p_{k'}H^{\circ\circ}$, $y \in p_nH^\circ$. Then for i, i', j, $j' \geq 10n$ we have*

$$|(x_{i,j}y \mid yx'_{i',j'})| \leq C(q^{\alpha(i+j)} + q^{\alpha(i'+j')} + q^{\alpha\max(\min(i,i'),\min(j,j'))})\|x\|_2\|x'\|_2\|y\|_2^2,$$

where $\alpha > 0$ is a constant depending only on q, and C is a constant depending on q and n.

Proof. We have $x = u_k(X)$, $x' = u_{k'}(X')$, $y = u_n(Y)$ with $X \in B(H_k)^{\circ\circ}$, $X' \in B(H_{k'})^{\circ\circ}$, $y \in B(H_n)^\circ$. Recall from Remark 2.8 that we have then $x_{i,j} = u_{i+k+j}(X_{i,j})$. Following the reminder in Section 1 — specifically Equation (1.4) and Notation 1.1 — we obtain $x_{i,j}y = \sum_{a=0}^n(\kappa_m^{i+k+j,n})^2u_m(X_{i,j} *_m Y)$, where $m = i+k+j+n-2a$ as usual. The same holds for $yx'_{i',j'}$, and the Peter–Weyl–Woronowicz Equation (1.1) yields

$$(x_{i,j}y \mid yx'_{i',j'}) = \sum_{a=0}^n \frac{1}{d_m}\big(\kappa_m^{i+k+j,n}\kappa_m^{n,i'+k'+j'}\big)^2(X_{i,j} *_m Y \mid Y *_m X'_{i',j'}).$$

According to Lemma 1.4, the constants κ are uniformly bounded by a constant depending only on q. Applying Theorem 4.5 and noticing that $q^{k/2}\|X\|_2 = q^{k/2}\sqrt{d_k}\|x\|_2 \leq C\|x\|_2$ we obtain

$$|(x_{i,j}y \mid yx'_{i',j'})| \leq C(q^{\alpha(i'+j')} + q^{\alpha\min(j,j')})\|x\|_2\|x'\|_2\|y\|_2^2,$$

where C is a constant depending only on q and n.

The estimate in the statement follows from this one by symmetry, by switching left and right in Lemmata 4.2, 4.3 and 4.4. More precisely, recall that the antipode S is isometric on $\ell^2(\mathbb{F})$ in the Kac case, and observe that $S(\chi_i) = \chi_i$, so that $S(x_{i,j}) = S(x)_{j,i}$. Applying the first part of this proof we thus get

$$\begin{aligned} |(x_{i,j}y \mid yx'_{i',j'})| &= |(yx'_{i',j'} \mid x_{i,j}y)| = |(S(x')_{j',i'}S(y) \mid S(y)S(x)_{j,i})| \\ &\leq C(q^{\alpha(i+j)} + q^{\alpha\min(i',i)})\|x\|_2\|x'\|_2\|y\|_2^2. \end{aligned}$$

Taking the best of this estimate and the previous one yields the result. □

To pass from the "local" result of Corollary 4.6 to the "global" results of Proposition 4.8 and Theorem 4.9 we will need to analyze the kernel appearing on the right-hand side in Corollary 4.6. We state separately the following elementary lemma which will be useful for this purpose.

Lemma 4.7. *Let A, $B \in \ell^2(\mathbb{N}\times\mathbb{N})$ and put $q_{p;i,k} = q^{\max(\min(i,k),\min(p-i,p-k))}$. Then there exists a constant $C > 0$ depending only on q such that*

$$\textstyle\sum_{i\geq 0}\sum_{k\geq 0}\sum_{p\geq i,k} q_{p;i,k}|A_{i,p-i}B_{k,p-k}| \leq C\|A\|_2\|B\|_2.$$

Proof. Denote $T = \{(i,k,p) \in \mathbb{N}^3 \mid p \geq i, p \geq k\}$. We start by applying Cauchy-Schwarz:

$$\left(\sum_T q_{p;i,k}|A_{i,p-i}B_{k,p-k}|\right)^2 \leq \sum_T q_{p;i,k}|A_{i,p-i}|^2 \times \sum_T q_{p;i,k}|B_{k,p-k}|^2.$$

By the symmetry in i and k it suffices to prove that $\sum_T q_{p;i,k}|A_{i,p-i}|^2 \leq C\|A\|_2^2$, which we can also write $\sum_{i=0}^\infty \sum_{p=i}^\infty s_{p;i}|A_{i,p-i}|^2 \leq C\|A\|_2^2$ with $s_{p;i} := \sum_{k=0}^p q_{p;i,k}$. This holds for all A if and only if $s_{p;i}$ is bounded independently of i and p. Since $q_{p;i,k} = q_{p;p-i,p-k}$ we have $s_{p;i} = s_{p;p-i}$, thus we can assume $0 \leq i \leq p-i$. We write then

$$\begin{aligned} s_{p;i} = \sum_{k=0}^p q_{p;i,k} &= \Big(\textstyle\sum_{k=0}^{i-1} + \sum_{k=i}^{p-i} + \sum_{k=p-i+1}^{p}\Big) q_{p;i,k} \\ &= \textstyle\sum_{k=0}^{i-1} q^{\max(k,p-i)} + \Big(\sum_{k=i}^{p-i} + \sum_{k=p-i+1}^{p}\Big) q^{\max(i,p-k)} \\ &= iq^{p-i} + \Big(\textstyle\sum_{k=i}^{p-i} q^{p-k}\Big) + iq^i \leq 2\sup_i(iq^i) + \frac{1}{1-q}. \end{aligned}$$ □

Recall from Notation 2.11 that for $w \in W$, $k \in \mathbb{N}^*$ we denote $H(w)$ resp. $H(k)$ the closure of AwA resp. AW_kA in H°, where W is our privileged basis of $H^{\circ\circ}$. Recall from Notation 3.1 that we denote $G(w)$ the Gram matrix of the family of vectors $w_{i,j}$, for $w \in W$.

Proposition 4.8. *Fix k, k', $n \in \mathbb{N}^*$ and $y \in p_nH^\circ$. Assume that we have a common upper bound $\|G(w)^{-1}\| \leq D\|w\|_2^{-2}$, $\|G(w)\| \leq D\|w\|_2^2$ for all $w \in W_k \cup W_{k'}$. Then for any $m \geq 10n$ and $\zeta \in V_m \cap H(k)$, $\zeta' \in V_m \cap H(k')$ we have $|(\zeta y \mid y\zeta')| \leq CDq^{\alpha(m-|k-k'|)}\|\zeta\|\|\zeta'\|$, where $\alpha > 0$ is a constant depending only on q, and C is a constant depending on q, n and y.*

Proof. By assumption the map $(w,i,j) \mapsto w_{ij}$ induces a bicontinuous isomorphism between $p_kH^{\circ\circ} \otimes \ell^2(\mathbb{N}\times\mathbb{N})$ and $H(k)$. More precisely, since $AwA \perp Aw'A$ for $w \neq w'$ in W_k, the Gram matrix $G(k)$ of $(w_{i,j})_{i,j,w}$, with $w \in W_k$, i, $j \in \mathbb{N}$, is block diagonal with $G(w)$, $w \in W_k$, as diagonal blocks, and thus it is bounded with bounded inverse by hypothesis. We can in particular decompose $\zeta = \sum_{i,j} x(i,j)_{i,j}$ with $x(i,j) \in p_kH^{\circ\circ}$ and, denoting $x = (x(i,j))_{i,j}$, we have $\|x\|_2^2 = \sum_{i,j}\|x(i,j)\|^2 \leq D\|\zeta\|^2$. Similarly we write $\zeta' = \sum_{i,j} x'(i,j)_{i,j}$ with $x'(i,j) \in p_{k'}H^{\circ\circ}$ and $\|x'\|_2^2 \leq D\|\zeta'\|^2$. We have then by Corollary 4.6:

$$\begin{aligned} |(\zeta y \mid y\zeta')| &\leq \textstyle\sum_{i,j}\sum_{i',j'}|(x(i,j)_{i,j}y \mid yx'(i',j')_{i',j'})| \\ &\leq C\,\textstyle\sum_{i,j}\sum_{i',j'}(q^{\alpha(i+j)} + q^{\alpha(i'+j')} + q^{\alpha\max(\min(i,i'),\min(j,j'))})\|x(i,j)\|\|x'(i',j')\| \end{aligned} \tag{4.1}$$

where C depends on q, n and y. Since ζ, $\zeta' \in V_m$ we have $x(i,j) = x'(i',j') = 0$ unless i, j, i', $j' \geq m$. Moreover the scalar product $(x(i,j)_{i,j}y \mid yx'(i',j')_{i',j'})$ vanishes unless $u_{i+k+j} \otimes u_n$ and $u_n \otimes u_{i'+k'+j'}$ have a common subobject, which entails $|i+k+j-i'-k'-j'| \leq 2n$. We remove from (4.1) the terms that do not satisfy these conditions. Moreover we regroup the three powers of q that appear in (4.1) into three distinct sums S_1, S_2, S_3 over i, i', j, j'.

We start with S_3. Denote $p = i+j-2m$, $p' = i'+j'-2m$, $l = p-p'$. This yields a bijection $(i,i',j,j') \mapsto (l,i,i',p)$ in $\mathbb{Z}^4$ and we shall compute S_3 by summing over (l,i,i',p) in the appropriate subset of $\mathbb{Z}^4$. If $l \geq 0$, we put further $\underline{i} = i-m \in \mathbb{N}$, $\underline{i}' = i'-m+l \in \mathbb{N}_{\geq l}$. Note that $j-m = p-\underline{i}$ and $j'-m = p-\underline{i}'$, so that

$$\begin{aligned} \max(\min(i,i'),\min(j,j')) &= \max(\min(\underline{i},\underline{i}'-l),\min(p-\underline{i},p-\underline{i}')) + m \\ &\geq \max(\min(\underline{i},\underline{i}'),\min(p-\underline{i},p-\underline{i}')) + m - |l| \\ \Rightarrow \quad q^{\alpha\max(\min(i,i'),\min(j,j'))} &\leq q^{\alpha(m-|l|)} q^\alpha_{p;\underline{i},\underline{i}'}, \end{aligned}$$

using the notation of Lemma 4.7. If $l < 0$, we put rather $\underline{i} = i-m-l \in \mathbb{N}_{\geq -l}$, $\underline{i}' = i'-m \in \mathbb{N}$ so that $j-m = p'-\underline{i}$ and $j'-m = p'-\underline{i}'$, and we obtain:

$$q^{\alpha\max(\min(i,i'),\min(j,j'))} \leq q^{\alpha(m-|l|)} q^\alpha_{p';\underline{i},\underline{i}'}.$$

The constraints j, $j' \geq m$ translate to $p \geq \underline{i}, \underline{i}'$ when $l \geq 0$, and to $p' \geq \underline{i}, \underline{i}'$ when $l < 0$. Re-organizing S_3 we thus obtain

$$S_3 \leq \sum_{l\geq 0} C q^{\alpha(m-|l|)} \sum_{\underline{i}=0}^{\infty} \sum_{\underline{i}'=l}^{\infty} \sum_{p \geq \underline{i},\underline{i}'} \|x(\underline{i}+m, p-\underline{i}+m)\| \|x'(\underline{i}'+m-l, p-\underline{i}'+m)\| \; q^{\alpha}_{p;\underline{i},\underline{i}'}$$
$$+ \sum_{l<0} C q^{\alpha(m-|l|)} \sum_{\underline{i}=-l}^{\infty} \sum_{\underline{i}'=0}^{\infty} \sum_{p' \geq \underline{i},\underline{i}'} \|x(\underline{i}+m+l, p'-\underline{i}+m)\| \|x'(\underline{i}'+m, p'-\underline{i}'+m)\| \; q^{\alpha}_{p';\underline{i},\underline{i}'}.$$

For the terms $l \geq 0$ we apply Lemma 4.7 with $A_{r,s} = \|x(r+m, s+m)\|$, $B_{r,s} = \|x'(r+m-l, s+m)\|$, which satisfy $\|A\|_2 = \|x\|_2$, $\|B\|_2 = \|x'\|_2$. By adding vanishing terms to the sum we can assume that the sum over $\underline{i}'$ starts at $\underline{i}' = 0$ to apply this Lemma. We apply Lemma 4.7 similarly to each term $l < 0$. Observe finally that $|l - (k' - k)| = |i + k + j - i' - k' - j'| \leq 2n$, so that l takes at most $4n + 1$ values and $|l| \leq |k - k'| + 2n$. Lemma 4.7 thus yields the following upper bound:

$$S_3 \leq CC'(4n+1) q^{\alpha(m-2n-|k-k'|)} \|x\|_2 \|x'\|_2 \leq C'' D q^{\alpha(m-|k-k'|)} \|\zeta\| \|\zeta'\|$$

with C'' depending on q, n and y.

The case of S_1 (and of S_2) is similar but the counterpart of Lemma 4.7 is simpler. We put $\underline{i} = i - m$, $\underline{j} = j - m$, $\underline{i}' = i' - m$, $\underline{j}' = j' - m$. Observe that for non-vanishing terms in the sum we have $i + j - 2m = \frac{1}{2}(p + p' + l) = \frac{1}{2}(\underline{i} + \underline{j}) + \frac{1}{2}(\underline{i}' + \underline{j}') + \frac{1}{2} l$, and still $|l| \leq |k - k'| + 2n$. This yields, using again Cauchy-Schwarz:

$$\begin{aligned} S_1 &= C \sum_{\underline{i},\underline{j}\geq 0} \sum_{\underline{i}',\underline{j}'\geq 0} q^{\alpha(2m+l/2)} (q^{\frac{\alpha}{2}(\underline{i}+\underline{j})} \|x(\underline{i}+m, \underline{j}+m)\|)(q^{\frac{\alpha}{2}(\underline{i}'+\underline{j}')} \|x'(\underline{i}'+m, \underline{j}'+m)\|) \\ &\leq C q^{\alpha(2m - \frac{1}{2}|k-k'| - n)} \sum_{\underline{i},\underline{j}\geq 0} q^{\frac{\alpha}{2}(\underline{i}+\underline{j})} \|x(\underline{i}+m, \underline{j}+m)\| \sum_{\underline{i}',\underline{j}'\geq 0} q^{\frac{\alpha}{2}(\underline{i}'+\underline{j}')} \|x'(\underline{i}'+m, \underline{j}'+m)\| \\ &\leq C q^{\alpha(2m - \frac{1}{2}|k-k'| - n)} \|x\|_2 \|x'\|_2 \textstyle\sum_{\underline{i},\underline{j}} q^{\alpha(\underline{i}+\underline{j})} \leq C''' D q^{\alpha(m-|k-k'|)} \|\zeta\| \|\zeta'\|, \end{aligned}$$

with C''' depending on q, n and y. □

Taking into account the finite propagation result established at the end of Section 2 we can finally prove the following global estimate.

Theorem 4.9. *Fix $n \in \mathbb{N}$ and $y \in p_n H^\circ \subset M$. Take the constant q_1 given by Theorem 3.10 and assume $q \leq q_1$. Then for any $m \geq 10n$ and $\zeta \in V_m$ we have $|(\zeta y \mid y\zeta)| \leq C q^{\alpha m} \|\zeta\|^2$, where $\alpha > 0$ is a constant depending only on q, and C is a constant depending on q, n and y.*

Proof. We have the orthogonal decomposition $\zeta = \sum_{k\in\mathbb{N}^*} \zeta_k$ with $\zeta_k \in \overline{AW_k A} = H(k)$. Proposition 2.16 shows that $y\zeta_{k'}$ decomposes into subspaces $H(l)$ with $|k' - l| \leq n$, and similarly $\zeta_k y$ decomposes into subspaces $H(l)$ with $|k - l| \leq n$, so that we have $\zeta_k y \perp y\zeta_{k'}$ if $|k - k'| > 2n$. Proposition 4.8 applies thank to Theorem 3.10 and the assumption on q. Thus we can write, using Cauchy-Schwarz:

$$\begin{aligned} |(\zeta y \mid y\zeta)| &\leq \textstyle\sum_{|k'-k|\leq 2n} |(\zeta_k y \mid y\zeta_{k'})| \leq C q^{\alpha(m-2n)} \textstyle\sum_{|k'-k|\leq 2n} \|\zeta_k\| \|\zeta_{k'}\| \\ &\leq C q^{\alpha(m-2n)} \textstyle\sum_{|k'-k|\leq 2n} \|\zeta_k\|^2 \leq C q^{\alpha m} q^{-2\alpha n} (4n+1) \|\zeta\|^2. \end{aligned}$$ □

5. Support localization

In this section we will show that for any $m \in \mathbb{N}$, elements $z \in A^\perp \cap M$ which almost commute to the generator $\chi_1 \in A$ have a small component in the subspace spanned by vectors $w_{i,j}$ with $i \leq m$ or $j \leq m$, and from this we deduce the Asymptotic Orthogonality Property for the MASA $A \subset M$.

Our strategy starts with algebraic arguments, using the constant structures for the left multiplication by $\chi_1 \in A$ on the basis $(w_{i,j})$, obtained at Proposition 5.1, to deduce relations between a vector $z \in H(w)$ and its commutator $[\chi_1, z]$, cf. Proposition 5.5. The main analytical input is then an estimate on coefficients appearing in these relations that we establish at Lemma 5.7, and which allows to establish the main result of this section, Theorem 5.9. We can then prove Theorem A by assembling the results of the article, following Popa's classical strategy.

The following computation of the structure constants is mainly a reformulation of Lemma 3.8 that we already used for the study of the Gram matrix. Recall Notation 2.11 for the orthonormal family $W = \bigsqcup_{k\geq 1} W_k$ which spans the A, A-bimodule H°. For $w \in W$ we have the associated vectors $w_{i,j} \in \mathring{A}w\mathring{A}$, where i, $j \in \mathbb{N}$. We agree to denote moreover $w_{i,j} = 0$ if $i < 0$ or $j < 0$. Finally, let us recall the definition of the coefficients A, B, C from the statement of Theorem 3.7:

$$A_p^n = \frac{d_{p+k}d_{p+k-1}}{d_n d_{n-1}}, \quad B_p^n = 2(-1)^k \operatorname{Re}(\mu)\frac{d_{p+k-1}d_{p-1}}{d_n d_{n-1}}, \quad C_p^n = -\frac{d_{p-1}d_{p-2}}{d_n d_{n-1}}.$$

They depend on $p \in \mathbb{N}$ and $n \in \mathbb{N}^*$, but also on $k \in \mathbb{N}$ and $\mu \in \mathbb{C}$ which will be fixed most of the time. Recall moreover that we are using the convention $d_p = 0$ for $p < 0$.

Proposition 5.1. *Let $w \in W_k$ with associated eigenvalue μ of the rotation map, and consider the associated coefficients A, B, C. Then for any i, $j \in \mathbb{N}$ we have, with $n = i + k + j$:*

$$\chi_1 w_{i,j} = w_{i+1,j} + (1 - A_j^n)w_{i-1,j} + B_j^n w_{i,j-1} + C_j^n w_{i+1,j-2}.$$

Proof. According to the fusion rules we have $\chi_1 w_{i,j} = p_{n+1}(\chi_1 w_{i,j}) + p_{n-1}(\chi_1 w_{i,j})$, and moreover $p_{n+1}(\chi_1 w_{i,j}) = p_{n+1}(\chi_1\chi_i w \chi_j) = w_{i+1,j}$ because $p_{n+1}(\chi_1 p_l(\chi_i w\chi_j)) = 0$ if $l < n$. We compute the second term $p_{n-1}(\chi_1 w_{i,j})$ in the Tannaka-Krein picture: putting $w = u_k(X)$ with $X \in B(H_k)^{\circ\circ}$, we have by (1.4):

$$p_{n-1}(\chi_1 w_{i,j}) = (\kappa_{n-1}^{1,n})^2\ u_{n-1}(\mathrm{id}_1 *_{n-1} X_{i,j}).$$

Recall the basic intertwiner $V_{n-1}^{1,n} = (P_1 \otimes P_n)(t \otimes \mathrm{id}_{n-1})P_{n-1}$. We have $V_{n-1}^{1,n*}V_{n-1}^{1,n} = (t^* \otimes \mathrm{id}_{n-1})$ $(\mathrm{id}_1 \otimes P_n)(t \otimes \mathrm{id}_{n-1}) = (\mathrm{Tr}_1 \otimes \mathrm{id})(P_n) = (d_n/d_{n-1})\mathrm{id}_{n-1}$, so that $(\kappa_{n-1}^{1,n})^2 = d_{n-1}/d_n$. Moreover we have by definition

$$\mathrm{id}_1 *_{n-1} X_{i,j} = (t^* \otimes \mathrm{id}_{n-1})(\mathrm{id}_1 \otimes X_{i,j})(t \otimes \mathrm{id}_{n-1}) = (\mathrm{Tr}_1 \otimes \mathrm{id})(X_{i,j}).$$

Switching left and right in Lemma 3.8 (or applying the antipode as in the proof of Corollary 4.6) we thus obtain

$$(\kappa_{n-1}^{1,n})^2(\mathrm{id}_1 *_{n-1} X_{i,j}) = \delta_{i>0}(1 - A_j^n)X_{i-1,j} + \delta_{j>0}B_j^n X_{i,j-1} + \delta_{j>1}C_j^n X_{i+1,j-2}.$$

This yields the formula in the statement. ◻

Corollary 5.2. *Fix $w \in W_k$, assume that $\{w_{i,j} \mid i, j \in \mathbb{N}\}$ is a Riesz basis, and take an element $z = \sum_{i,j} z_{i,j}w_{i,j}$ in $H(w)$. We put moreover $z_{i,j} = 0$ if $i < 0$ or $j < 0$. Writing similarly $[\chi_1, z] = \sum_{i,j}[\chi_1, z]_{i,j}w_{i,j}$ in $H(w)$, we have*

$$[\chi_1, z]_{i,j} = z_{i-1,j} - z_{i,j-1} + D_j^{n+1}z_{i+1,j} - D_i^{n+1}z_{i,j+1} + C_{j+2}^{n+1}z_{i-1,j+2} - C_{i+2}^{n+1}z_{i+2,j-1},$$

where we take $n = i + k + j$ and denote $D_j^n = 1 - A_j^n - B_{n-k-j}^n$.

Proof. The proposition gives, by summing in $H(w)$ over i, $j \in \mathbb{N}$:

$$\begin{aligned}\chi_1 z &= \textstyle\sum_{i,j} z_{i,j}w_{i+1,j} + \sum_{i\geq 1,j}(1 - A_j^{i+k+j})z_{i,j}w_{i-1,j} \\ &\qquad + \textstyle\sum_{j\geq 1,i} B_j^{i+k+j}z_{i,j}w_{i,j-1} + \sum_{j\geq 2,i} C_j^{i+k+j}z_{i,j}w_{i+1,j-2} \\ &= \textstyle\sum_{i\geq 1,j} z_{i-1,j}w_{i,j} + \sum_{i,j}(1 - A_j^{i+k+j+1})z_{i+1,j}w_{i,j} \\ &\qquad + \textstyle\sum_{i,j} B_{j+1}^{i+k+j+1}z_{i,j+1}w_{i,j} + \sum_{i\geq 1,j} C_{j+2}^{i+k+j+1}z_{i-1,j+2}w_{i,j}.\end{aligned}$$

With our convention we can add the terms $i = 0$ in the first and last sum, and for fixed i, $j \in \mathbb{N}$ this yields $(\chi_1 z)_{i,j} = z_{i-1,j} + (1 - A_j^{n+1})z_{i+1,j} + B_{j+1}^{n+1}z_{i,j+1} + C_{j+2}^{n+1}z_{i-1,j+2}$, where $n = i + k + j$. We also have $(z\chi_1)_{j,i} = z_{j,i-1} + (1 - A_j^{n+1})z_{j,i+1} + B_{j+1}^{n+1}z_{j+1,i} + C_{j+2}^{n+1}z_{j+2,i-1}$ by symmetry (or by applying the antipode). Switching i and j this reads $(z\chi_1)_{i,j} = z_{i,j-1} + (1 - A_i^{n+1})z_{i,j+1} + B_{i+1}^{n+1}z_{i+1,j} + C_{i+2}^{n+1}z_{i+2,j-1}$ and a substraction yields the result, since $D_j^{n+1} = 1 - A_j^{n+1} - B_{i+1}^{n+1}$. ◻

Iterating Corollary 5.2, we shall obtain more relations between a vector z and the commutator $[\chi_1, z]$: cf Equation (5.1) below where the case $p = 1$ corresponds in fact to Corollary 5.2. For fixed m, l, the *collection* of relations (5.1) for varying p will yield the crucial estimate of Theorem 5.9. The coefficients appearing in (5.1) are introduced inductively as follows.

Notation 5.3. We fix $k \in \mathbb{N}^*$, $|\mu| = 1$, and $m \in \mathbb{N}$. We define families of coefficients $f_{i,j}^{l,p}$, $g_{i,j}^{l,p}$ for i, j, l, $p \in \mathbb{N}$, and $\phi_i^{l,p}$ for $l, p \in \mathbb{N}$, $i \in \mathbb{Z}$, by induction on p, as follows. For $p = 0$ we first put $f_{i,j}^{l,0} = \delta_{(i,j)=(m,l)}$ and $g_{i,j}^{l,0} = 0$ for all l, i, $j \in \mathbb{N}$. Then assuming that $f_{i,j}^{l,p}$, $g_{i,j}^{l,p}$ are constructed for a given p and all l, i, $j \in \mathbb{N}$ we first put

$$\phi_i^{l,p} = -\sum_{s=-p}^{i} f_{m+p-s,l+p+s}^{l,p}$$

for $-p \le i \le m+p$ and $\phi_i^{l,p} = 0$ for the other values of $i \in \mathbb{Z}$. Then we define $g_{i,j}^{l,p+1} = g_{i,j}^{l,p}$ if $i + j \le m + l + 2p - 1$, $g_{i,j}^{l,p+1} = \phi_{m+p-i}^{l,p}$ if $i + j = m + l + 2p + 1$ and $g_{i,j}^{l,p+1} = 0$ else. Finally we put $f_{i,j}^{l,p+1} = 0$ if $i + j \neq m + l + 2p + 2$, and

$$f_{i,j}^{l,p+1} = D_i^n \phi_{m+p-i}^{l,p} - D_j^n \phi_{m+p-i+1}^{l,p} + C_i^n \phi_{m+p-i+2}^{l,p} - C_j^n \phi_{m+p-i-1}^{l,p}$$

if $i + j = m + l + 2p + 2$, with $n = i + k + j$.

Remark 5.4. The last relation in fact implies also $f_{i,j}^{l,p+1} = 0$ if $i + j = m + l + 2p + 2$ and $i > m + 2p + 2$, because then $m + p - i < -p - 2$. Hence $f_{i,j}^{l,p} = 0$ unless $i + j = m + l + 2p$ and $i \le m + 2p$. By definition, one can recover the coefficients f from ϕ as follows: $f_{i,m+l+2p-i}^{l,p} = \phi_{m+p-i-1}^{l,p} - \phi_{m+p-i}^{l,p}$ for $i = 0, \dots, m + 2p$ (which for $i = m + 2p$ also reads $f_{m+2p,l}^{l,p} = -\phi_{-p}^{l,p}$), and $f_{i,j}^{l,p} = 0$ if $i+j \neq m+l+2p$ or $i > m+2p$. Also, we have $\phi_i^{l,0} = -1$ for $0 \le i \le m$ and 0 otherwise. On the other hand, we also record the fact that $g_{i,j}^{l,p} = 0$ unless $m+l+1 \le i+j \le m+l+2p-1$ and $i + j$, $m + l + 1$ have the same parity.

Proposition 5.5. *Fix $w \in W_k$ with associated ρ-eigenvalue μ, and $m \in \mathbb{N}$. Assume that $\{w_{i,j} \mid i, j \in \mathbb{N}\}$ is a Riesz basis. For any $z = \sum_{i,j} z_{i,j} w_{i,j} \in H(w)$ we have, using the coefficients of Notation 5.3:*

$$\forall p \in \mathbb{N}, l \in \mathbb{N} \quad z_{m,l} = \sum_{i,j\in\mathbb{N}} f_{i,j}^{l,p} z_{i,j} + \sum_{i,j\in\mathbb{N}} g_{i,j}^{l,p} [\chi_1, z]_{i,j}. \tag{5.1}$$

Proof. We proceed by induction over p, noting that (5.1) holds trivially for $p = 0$. Assume now that it is satisfied for some fixed $p \in \mathbb{N}$. Using Corollary 5.2 we then write, with the convention $i' = m + l + 2p - i$ in each term of the sums:

$$\begin{aligned} z_{m,l} = {} & \textstyle\sum_{i=0}^{m+2p} f_{i,i'}^{l,p} z_{i,i'} - \sum_{i=0}^{m+2p} \phi_{m+p-i}^{l,p} (z_{i-1,i'+1} - z_{i,i'}) \\ & + \textstyle\sum_{i,j} g_{i,j}^{l,p} [\chi_1, z]_{i,j} + \sum_{i=0}^{m+2p} \phi_{m+p-i}^{l,p} [\chi_1, z]_{i,i'+1} \\ & - \textstyle\sum_{i=0}^{m+2p} \phi_{m+p-i}^{l,p} (D_{i'+1}^n z_{i+1,i'+1} - D_i^n z_{i,i'+2} + C_{i'+3}^n z_{i-1,i'+3} - C_{i+2}^n z_{i+2,i'}), \end{aligned}$$

where $n = k + m + l + 2p + 2 = i + k + i' + 2$. By definition of the coefficients ϕ, the first two sums cancel each other: indeed $z_{i,i'}$, for $0 \le i \le m + 2p$, appears with the factor $\phi_{m+p-i-1}^{l,p} - \phi_{m+p-i}^{l,p} = f_{i,m+l+2p-i}^{l,p}$ in the second one. The fourth sum contains exactly the terms missing to the third one to pass from $g_{i,j}^{l,p}$ to $g_{i,j}^{l,p+1}$, so that we have

$$\begin{aligned} z_{m,l} & = \textstyle\sum_{i,j} g_{i,j}^{l,p+1} [\chi_1, z]_{i,j} - \sum_{i=0}^{m+2p} \phi_{m+p-i}^{l,p} (D_{i'+1}^n z_{i+1,i'+1} - D_i^n z_{i,i'+2} \\ & \qquad\qquad + C_{i'+3}^n z_{i-1,i'+3} - C_{i+2}^n z_{i+2,i'}) \\ & = \textstyle\sum_{i,j} g_{i,j}^{l,p+1} [\chi_1, z]_{i,j} + \sum_{i=0}^{m+2p+2} z_{i,i'+2} (\phi_{m+p-i}^{l,p} D_i^n - \phi_{m+p-i+1}^{l,p} D_{i'+2}^n \\ & \qquad\qquad + \phi_{m+p-i+2}^{l,p} C_i^n - \phi_{m+p-i-1}^{l,p} C_{i'+2}^n), \end{aligned}$$

still with $n = k + m + l + 2p + 2$, and using $\phi_i^{l,p} = 0$ for $i < -p$ or $i > m + p$. We recognize in the last sum the definition of $f_{i,j}^{l,p+1}$, with $j = m + l + 2p - i + 2 = i' + 2$, so that (5.1) holds for $p + 1$. □

Now the main tool to obtain Theorem 5.9, together with the relations (5.1), is an estimate on the coefficients ϕ that we establish at the elementary but technical Lemma 5.7 below. First we prove the following easy estimates on the coefficients C and D.

Lemma 5.6. *For any $a \in \mathbb{N}$, $b \in \mathbb{N} \cup \{-1\}$ and $k \in \mathbb{N}^*$ we have, putting $n = a + b + k + 2$:*

$$|D_{b+1}^n - D_a^n| \le \frac{3q^{a+b+3}q^{-|b-a+1|}}{(1-q^{2n})(1-q^{2n+2})},$$

$$|D_{b+1}^n| \le 1 + \frac{2q^{n-a+b+1}}{(1-q^{2n})(1-q^{2n+2})}, \qquad |C_{b+1}^n| \le \frac{q^{2n-2b}}{(1-q^{2n})(1-q^{2n+2})}.$$

Proof. We start from the identity

$$d_{n-b-2}d_{n-b-3} - d_{n-a-1}d_{n-a-2} = \pm d_{2n-a-b-3}d_{|b-a+1|-1}.$$

This can be seen by a direct computation using q-numbers, or using the fusion rules as follows. Both sides vanish if $a = b + 1$, according to our convention $d_{-1} = 0$ (and otherwise all indices are in $\mathbb{N}$). Assume for instance $a < b + 1$. Then $d_{n-b-2}d_{n-b-3}$ (resp. $d_{n-a-1}d_{n-a-2}$) is the sum of the dimensions d_c where c is odd and ranges from 1 to $2n - 2b - 5$ (resp. $2n - 2a - 3$). On the other hand the right-hand side is the sum of dimensions d_c where c is odd and ranges from $2n - 2b - 3$ to $2n - 2a - 3$, so that the relation holds with a negative sign. The other case follows (with a positive sign) by exchanging a and $b+1$. Dividing out by $d_n d_{n-1}$ and using the estimate $q^{-c}(1 - q^{2c+2}) = (1 - q^2)d_c \le q^{-c}$ we obtain

$$|(1 - A_{b+1}^n) - (1 - A_a^n)| = \frac{d_{2n-a-b-3}d_{|b-a+1|-1}}{d_n d_{n-1}} \le \frac{q^{a+b+3-|b-a+1|}}{(1-q^{2n+2})(1-q^{2n})}.$$

One can proceed in the same way with the constants B. By the same reasoning as above, or by a direct computation, we have

$$d_a d_{n-b-2} - d_{b+1}d_{n-a-1} = \pm d_n d_{|b-a+1|-1},$$

indeed the products on the left are equal to the sum of the dimensions d_c where c has the same parity as k and ranges from k to $n+a-b-2$ (resp. from k to $n-a+b$), so that in the difference c ranges from $n + a - b$ to $n - a + b$ if $b > a - 1$ (resp. from $n - a + b + 2$ to $n + a - b - 2$ if $b < a - 1$), and we find exactly the same terms on the right. Dividing out by $d_n d_{n-1}$ this yields

$$|B_{a+1}^n - B_{b+2}^n| = 2|\operatorname{Re}\mu|\frac{d_n d_{|b-a+1|-1}}{d_n d_{n-1}} \le 2\frac{q^{n-|b-a+1|}}{(1-q^{2n+2})(1-q^{2n})}.$$

Since $n \ge a + b + 3$, adding this estimate and the previous one yields the first estimate of the statement.

The other estimates are easier. We have clearly $0 \le A_{b+1}^n \le 1$, hence $D_{b+1}^n \le 1 + |B_{a+1}^n|$, and using again the estimates $(1 - q^2)d_c \le q^{-c}$ we obtain

$$|B_{a+1}^n| = 2|\operatorname{Re}\mu|\frac{d_a d_{n-b-2}}{d_n d_{n-1}} \le \frac{2q^{n-a+b+1}}{(1-q^{2n+2})(1-q^{2n})}.$$

Finally the estimate for C_{b+1}^n follows exactly like the one for B_{a+1}^n above. □

Lemma 5.7. *Assume that one can choose $3.4 < R < 0.995/(2q^2)$. Take $k \in \mathbb{N}^*$, $|\mu| = 1$, and $m \in \mathbb{N}$. Then there exists a constant K, depending only on q and m, such that $|\phi_i^{l,p}| \le Kq^{2|i|}(2R)^i$ for all $l \ge m$, $p \in \mathbb{N}$ and $-p \le i \le m + p$.*

Proof. Note that the assumption implies the inequalities $6q^2 < 2Rq^2 < 1$, and in particular $q^2 < 1/6$, which we will use frequently in this proof. The recursive construction of the coefficients

f yields the following recursion relation over p for the coefficients ϕ. For $-p-1 \le i \le m+p+1$ we have, putting $n = m+l+2p+k+2$:

$$\begin{aligned}\phi_i^{l,p+1} &= -\sum_{s=-p-1}^{i} f^{l,p+1}_{m+p+1-s,l+p+1+s} \\ &= \sum_{s=-p-1}^{i} (\phi_s^{l,p} D^n_{l+p+1+s} - \phi_{s-1}^{l,p} D^n_{m+p+1-s} + \phi_{s-2}^{l,p} C^n_{l+p+1+s} - \phi_{s+1}^{l,p} C^n_{m+p+1-s}) \\ &= \phi_i^{l,p}(D^n_{l+p+1+i} - C^n_{m+p-i+2}) - \phi_{i+1}^{l,p} C^n_{m+p-i+1} \\ &\qquad + \sum_{s=-p}^{i-1} \phi_s^{l,p}\Big(D^n_{l+p+1+s} - D^n_{m+p-s} + \delta_{s\le i-2} C^n_{l+p+s+3} - C^n_{m+p-s+2}\Big),\end{aligned}$$

since $\phi_s^{l,p} = 0$ if $s < -p$.

We now combine this recursion relation with the estimates of Lemma 5.6. We still take $n = k+m+l+2p+2$ and we denote $\rho_n = (1-q^{2n})^{-1}(1-q^{2n+2})^{-1}$. Note that we have $\rho_n \le \rho_t$ if $t \le n$, and $\rho_3 < 1.005$ (by comparing with the value at $q^2 = 0.995/6.8$). Since $k \ge 1$, we have $n \ge 3$, hence the estimate $\rho_n \le 1.005$ that we will use later in the proof. Now Lemma 5.6 gives, for $-p-1 \le i \le m+p+1$:

$$\begin{aligned}|\phi_i^{l,p+1}| \le |\phi_i^{l,p}|\Big(1 + 2\rho_n q^{n-m+l+2i+1} + \rho_n q^{2n-2m-2p+2i-2}\Big) + |\phi_{i+1}^{l,p}|\rho_n q^{2n-2m-2p+2i} \\ + \sum_{s=-p}^{i-1} |\phi_s^{l,p}| \times \big(3\rho_n q^{m+l+2p+3-|l-m+2s+1|} + \rho_n q^{2n-2l-2p-2s-4} + \rho_n q^{2n-2m-2p+2s-2}\big).\end{aligned}$$

Observe that the sum vanishes for $i = -p-1$, as well as $\phi^{l,p}_{-p-1}$, $\phi^{l,p}_{m+p+1}$ and $\phi^{l,p}_{m+p+2}$ by convention. Since $k \ge 1$ and $l \ge m$, and using the value of n, we have the following lower bounds for $-p \le s \le i-1$:

- $n - m + l + 2i + 1 \ge 2(p+i+m+2)$,
- $2n - 2m - 2p + 2i \ge 2(p+i+m+3)$,
- $m + l + 2p + 3 - |l-m+2s+1| \ge 2(p-|s|+m+1)$,
- $2n - 2l - 2p - 2s - 4 \ge 2(p-|s|+m+1)$,
- $2n - 2m - 2p + 2s - 2 \ge 2(p-|s|+m+2)$.

Applying these bounds and factoring $3+1+q^2 \le \frac{25}{6}$ in the sum we arrive at the slightly simpler estimate:

$$|\phi_i^{l,p+1}| \le |\phi_i^{l,p}|\big(1 + 3\rho_n q^{2(p+i+m+2)}\big) + |\phi_{i+1}^{l,p}|\rho_n q^{2(p+i+m+3)} + \tfrac{25}{6}\rho_n \sum_{s=-p}^{i-1} |\phi_s^{l,p}| q^{2(p-|s|+m+1)}.$$

Then we denote $\psi_i^{l,p} = |\phi_i^{l,p}| q^{-2|i|}(2R)^{-i}$, so that our aim is now to find K such that $\psi_i^{l,p} \le K$ for all l, $p \in \mathbb{N}$, $-p \le i \le m+p$. For ψ the previous estimate becomes

$$\psi_i^{l,p+1} \le \psi_i^{l,p}\big(1 + 3\rho_n q^{2(p+i+m+2)}\big) + \psi_{i+1}^{l,p}\rho_n 2R q^{2(p+i+m+2)} + \tfrac{25}{6}\rho_n q^{2(p-|i|+m+1)} \sum_{s=-p}^{i-1} (2R)^{s-i}\psi_s^{l,p},$$

where we have used $|i+1| - |i| \ge -1$ in the second term. Let us denote $K_r = (6q^2)^{-m} \times \prod_{t=0}^{r-1}(1+q^{2t})$, which is increasing with r and starts with $K_0 = (6q^2)^{-m}$. We will prove, for each $l \ge m$ by induction on p, the following estimate:

$$(H_p) \quad \forall i \in \{-p, \dots, p+m\} \quad \psi_i^{l,p} \le K_{p+m-|i|}.$$

For $p = 0$ and $0 \le i \le m$ we have indeed $\phi_i^{l,p} = -1$, hence $\psi_i^{l,p} = (2Rq^2)^{-i} \le (6q^2)^{-m} \le K_{m-|i|}$. Assume now that the estimates hold for a fixed $p \in \mathbb{N}$, and let us establish them at $p+1$.

Together with (H_p), our recursive estimate on ψ yields, for $-p-1 \leq i \leq m+p+1$:

$$\psi_i^{l,p+1} \leq \delta_{-p\leq i\leq p+m} K_{p+m-|i|}\big(1+3\rho_n q^{2(p+i+m+2)}\big) + \delta_{i\leq p+m-1} K_{p+m-|i+1|}\rho_n 2Rq^{2(p+i+m+2)}$$
$$+ \tfrac{25}{6}\rho_n q^{2(p-|i|+m+1)} \sum_{s=-p}^{i-1} (2R)^{s-i} K_{p+m-|s|}. \tag{5.2}$$

Consider first $i = -p-1$. The above estimate reads in this case $\psi_{-p-1}^{l,p+1} \leq K_m\rho_n 2Rq^{2(m+1)}$. Since $2Rq^2 < 0.995$ and $\rho_n \leq 1.005$, we obtain $\psi_{-p-1}^{l,p+1} \leq K_m$ as needed.

Then we consider the case $-p \leq i \leq -1$. For $s \leq i$ we have $K_{p+m-|s|} = K_{p+m+s} \leq K_{p+m-|i|}$. Moreover we have $K_{p+m-|i+1|}2Rq^4 = K_{p+m-|i|}(1+q^{2(p+m+i)})2Rq^4 \leq \frac{1}{3}K_{p+m-|i|}$ because $2Rq^2 < 1$ and $q^2(1+q^{2(p+m+i)}) \leq 2q^2 \leq \frac{1}{3}$. Hence (5.2) yields

$$\begin{aligned}\psi_i^{l,p+1} &\leq K_{p+m-|i|}\big(1+\tfrac{3}{36}\rho_n q^{2(p+i+m)} + \tfrac{1}{3}\rho_n q^{2(p+i+m)} + \tfrac{25}{36}\rho_n q^{2(p-|i|+m)} \textstyle\sum_{s=-\infty}^{i-1}(2R)^{s-i}\big)\\ &\leq K_{p+m-|i|}\big(1+\tfrac{5}{12}\rho_n q^{2(p+i+m)} + \tfrac{25}{36}\rho_n q^{2(p+m-|i|)}/(2R-1)\big)\\ &\leq K_{p+m-|i|}\big(1+\tfrac{5}{9}\rho_n q^{2(p+m-|i|)}\big) \leq K_{p+m-|i|}\big(1+q^{2(p+m-|i|)}\big) = K_{p+1+m-|i|},\end{aligned}$$

where we used $2R-1 \geq 5$ and $\rho_n \leq \frac{9}{5}$.

Now we consider the case $0 \leq i \leq p+m$. Let us observe that for any $t \in \mathbb{N}$ we have $(1+q^{2t})/2 \leq 1$, hence $2^{-t}\prod_{r=0}^{t-1}(1+q^{2r}) \leq 1$. Then for $|s| \leq i$ we can write $K_{p+m-|s|}2^{s-i} = K_{p+m-i}2^{s-i}\prod_{t=p+m-i}^{p+m-|s|-1}(1+q^{2t}) \leq K_{p+m-i}2^{|s|-i}\prod_{r=0}^{i-|s|-1}(1+q^{2r}) \leq K_{p+m-i}$. On the other hand for $s \leq -i$ we clearly have $K_{p+m-|s|} \leq K_{p+m-i}$ (and $2^{s-i} \leq 1$). Using this, our estimate (5.2) thus yields

$$\begin{aligned}\psi_i^{l,p+1} &\leq K_{p+m-i}\big(1+3\rho_n q^{2(p+i+m+2)} + \rho_n 2Rq^{2(p+i+m+2)} + \tfrac{25}{6}\rho_n q^{2(p-i+m+1)} \textstyle\sum_{s=-p}^{i-1} R^{s-i}\big)\\ &\leq K_{p+m-i}\big(1+q^{2(p-i+m)}\big(\tfrac{3}{36}\rho_n + \tfrac{1}{6}\rho_n + \tfrac{25}{36}\rho_n/(R-1)\big)\big)\\ &\leq K_{p+m-i}(1+\tfrac{43}{72}\rho_n q^{2(p+m-i)}) \leq K_{p+m-i}(1+q^{2(p+m-i)}) = K_{p+1+m-i},\end{aligned}$$

where we used $2Rq^2 < 1$, $q^2 \leq \frac{1}{6}$, then $R-1 \geq 2$ and $\rho_n \leq \frac{72}{43}$.

Finally when $i = p+m+1$ the first two terms in the estimate (5.2) vanish and we are left with the sum which can be dealt with as before:

$$\begin{aligned}\psi_{p+m+1}^{l,p+1} &\leq \tfrac{25}{6}\rho_n \textstyle\sum_{s=-p}^{p+m}(2R)^{s-p-m-1}K_{p+m-|s|}\\ &\leq \tfrac{25}{6}K_0\rho_n \textstyle\sum_{s=-p}^{p+m} R^{s-p-m-1}2^{-1} \leq \tfrac{25}{6}K_0\rho_n/(2(R-1)) \leq K_0,\end{aligned}$$

since $R \geq 3.4$ and $\rho_n \leq 1.1$. This is the required estimate to conclude the proof of (H_{p+1}).

We have now proved by induction that (H_p) holds for all $p \in \mathbb{N}$. Moreover we have $K_r \leq K := \lim_{s\to\infty} K_s$ for all $r \in \mathbb{N}$, with $K < +\infty$ because $q < 1$. Hence the lemma is proved. □

Notation 5.8. We consider the following *non-orthogonal* projections E_m, Q_m defined as follows: for all $w \in W$, i, $j \in \mathbb{N}$

$$\begin{aligned}E_m(w_{i,j}) &= w_{i,j} \quad \text{if } i \geq m \text{ and } j \geq m, \quad 0 \quad \text{otherwise ;}\\ Q_m(w_{i,j}) &= w_{i,j} \quad \text{if } j \geq i = m, \quad 0 \quad \text{otherwise.}\end{aligned}$$

Observe that if $\{w_{i,j} \mid w \in W, i, j \in \mathbb{N}\}$ is a Riesz basis, these projections extend to idempotents in $B(H^\circ)$, and the range of E_m is the subspace V_m from Notation 4.1.

Theorem 5.9. *Assume that $\{w_{i,j} \mid w \in W, i, j \in \mathbb{N}\}$ is a Riesz basis and that $N \geq 3$. Then there exist constants L_m such that we have, for any $p \in \mathbb{N}^*$, $m \in \mathbb{N}$ and $z \in H^\circ$:*

$$\|(1-E_m)z\|_2^2 \leq L_m p^{-1}\|z\|_2^2 + L_m p\|[\chi_1, z]\|_2^2.$$

Proof. Note that for $N = 3$ we have $q^{-2} = \frac{1}{2}(N^2 + N\sqrt{N^2-4} - 2) \simeq 6.854$, so that for $N \geq 3$ we have $q^{-2} > 6.85$ and $0.995/(2q^2) > 3.4$. As a result we can find a number R such as in the hypothesis of Lemma 5.7. To start with, we deduce from that lemma estimates on various sums of coefficients f and g. For each m we denote K_m the constant provided by the lemma.

Recall that $f^{l,p}_{i,j}$ vanishes except for the following entries: $f^{l,p}_{m+p-i,l+p+i} = \phi^{l,p}_{i-1} - \phi^{l,p}_i$ with $-p \le i \le p+m$, and the convention $\phi^{l,p}_{-p-1} = 0$. Thus for fixed $l \ge m$ and p we have $\sum_{i,j} |f^{l,p}_{i,j}| \le 2\sum_{i=-p}^{p+m} |\phi^{l,p}_i|$. As a result Lemma 5.7 yields $\sum_{i,j} |f^{l,p}_{i,j}| \le 2K_m S$ with $S = \sum_{i=-\infty}^{+\infty} q^{2|i|}(2R)^i$, which is finite because $2Rq^2 < 1$ and $q^2/2R < q^2/6 < 1$ by choice of R. In the same way for fixed i, j we have $f^{l,p}_{i,j} = 0$ unless $i+j \ge m$ and $l \ge m$, $p \in \mathbb{N}$ satisfy $l = i+j-m-2p$. Thus we obtain, putting $r = p+m-i$, the same upper bound for the sum over l and p:

$$\sum_{p,l\ge m} |f^{l,p}_{i,j}| = \sum_{p=0}^{\lfloor (i+j)/2-m\rfloor} \left|\phi^{i+j-m-2p,p}_{m+p-i} - \phi^{i+j-m-2p,p}_{m+p-i-1}\right|$$
$$= \sum_{r=m-i}^{\lfloor (j-i)/2\rfloor} \left|\phi^{m+j-i-2r,r+i-m}_{r} - \phi^{m+j-i-2r,r+i-m}_{r-1}\right| \le 2K_m S.$$

On the other hand, recall that if $g^{l,p}_{i,j} \ne 0$ then there exists $0 \le r \le p-1$ such that $i+j = m+l+2r+1$, and then $g^{l,p}_{i,j} = \phi^{l,r}_{r+m-i}$. Hence for fixed $l \ge m$ and p we have

$$\sum_{i,j} |g^{l,p}_{i,j}| \le \sum_{r=0}^{p-1}\sum_{i=0}^{m+l+2r+1} |\phi^{l,r}_{r+m-i}| \le \sum_{r=0}^{p-1}\sum_{i=-\infty}^{+\infty} |\phi^{l,r}_i| \le pK_m S.$$

Similarly, for fixed i, j, p we have $\sum_{l\ge m} |g^{l,p}_{i,j}| = \sum_{r=0}^{\rho} |\phi^{i+j-m-2r-1,r}_{r+m-i}|$ where $\rho = \min(p-1, \lfloor (i+j-1)/2\rfloor - m)$, hence once again $\sum_{l\ge m} |g^{l,p}_{i,j}| \le K_m S$.

Now we can proceed to the main part of the proof. We start by Q_m instead of $1-E_m$. By decomposing H° into the pairwise orthogonal sub-bimodules $H(w)$ we can assume that z belongs to $H(w)$ for some w — indeed Q_m and the commutator with χ_1 commute with the projections onto these submodules. Since $(w_{i,j})$ is a Riesz basis we can replace $\|Q_m z\|_2^2$ with $\sum_{l\ge m} |z_{m,l}|^2$, $\|z\|_2^2$ with $\sum_{i,j} |z_{i,j}|^2$ and $\|[\chi_1, z]\|_2^2$ with $\sum_{i,j} |[\chi_1, z]_{i,j}|^2$. Then for any fixed p we have the following estimates, using Proposition 5.5 and Cauchy-Schwartz:

$$\sum_{l\ge m} |z_{m,l}|^2 = \sum_{l\ge m}\left|\sum_{i,j} f^{l,p}_{i,j} z_{i,j} + \sum_{i,j} g^{l,p}_{i,j}[\chi_1, z]_{i,j}\right|^2$$
$$\le 2\sum_{l\ge m}\left|\sum_{i,j} f^{l,p}_{i,j} z_{i,j}\right|^2 + 2\sum_{l\ge m}\left|\sum_{i,j} g^{l,p}_{i,j}[\chi_1, z]_{i,j}\right|^2$$
$$\le 2\sum_{l\ge m}\sum_{i,j} |f^{l,p}_{i,j}| \sum_{i,j} |f^{l,p}_{i,j} z^2_{i,j}| + 2\sum_{l\ge m}\sum_{i,j} |g^{l,p}_{i,j}| \sum_{i,j} |g^{l,p}_{i,j}[\chi_1, z]^2_{i,j}|$$
$$\le 4K_m S \sum_{l\ge m, i,j} |f^{l,p}_{i,j} z^2_{i,j}| + 2pK_m S \sum_{i,j} |[\chi_1, z]_{i,j}|^2 \sum_{l\ge m} |g^{l,p}_{i,j}|$$
$$\le 4K_m S \sum_{l\ge m, i,j} |f^{l,p}_{i,j} z^2_{i,j}| + 2pK_m^2 S^2 \sum_{i,j} |[\chi_1, z]_{i,j}|^2.$$

Then we take the average of these inequalities over $p = 0, \dots, r-1$:

$$\sum_{l\ge m} |z_{m,l}|^2 \le \frac{4K_m S}{r}\sum_{i,j} |z^2_{i,j}| \sum_{p<r, l\ge m} |f^{l,p}_{i,j}| + \frac{2K_m^2 S^2}{r}\sum_{p<r} p \sum_{i,j} |[\chi_1, z]_{i,j}|^2,$$
$$\le \frac{8K_m^2 S^2}{r}\sum_{i,j} |z_{i,j}|^2 + rK_m^2 S^2 \sum_{i,j} |[\chi_1, z]_{i,j}|^2.$$

It is then easy to upgrade this estimate from Q_m to $1-E_n$. First of all by symmetry we have the same estimate for the sum of $|z_{l,m}|^2$ over $l > m$, for fixed m. Then for fixed n we put $L_n = \sum_{m=0}^{n-1} K_m^2$, and by summing over $0 \le m \le n-1$ we obtain for all r:

$$\sum_{l<n \text{ or } m<n} |z_{m,l}|^2 \le 16L_n S^2 r^{-1} \sum_{i,j} |z_{i,j}|^2 + 2L_n S^2 r \sum_{i,j} |[\chi_1, z]_{i,j}|^2.$$

This gives the required estimate by the Riesz basis property. □

Proof of Theorem A. Take the constant q_1 provided by Theorem 3.10, and $N_0 \in \mathbb{N}$, $N_0 \geq 3$, such that the associated constant q_0 satisfies $q_0 < q_1$. Then for $N \geq N_0$ we have $q < q_1$, so that $\{w_{i,j} \mid w \in W, i, j \in \mathbb{N}\}$ is a Riesz basis.

To prove the AOP, take elements $z_r \in A^\perp \cap M$ such that $\|z_r\| \leq 1$ and $\|[\chi_1, z_r]\|_2 \to_\omega 0$. We want to prove that $(yz_r \mid z_r y) \to_\omega 0$ for any $y \in A^\perp \cap M$. By Kaplansky's density theorem and linearity, we can assume that $y \in p_n H^\circ \cap M$ for some fixed $n \in \mathbb{N}^*$, with $\|y\| \leq 1$. Now for any $m \in \mathbb{N}$ we can write

$$|(yz_r \mid z_r y)| \leq |(yE_m(z_r) \mid E_m(z_r)y)| + \|(1 - E_m)(z_r)\|_2(\|z_r\|_2 + \|E_m(z_r)\|_2).$$

We apply our Theorem 4.9 to $\zeta = E_m(z_r)$, obtaining $|(yE_m(z_r) \mid E_m(z_r)y)| \leq Cq^{\alpha m}\|E_m(z_r)\|_2^2$ for $m \geq 10n$. The projections E_m are not orthogonal, but since $(w_{i,j})_{w,i,j}$ is a Riesz basis they are bounded independently of m. Thus we get

$$|(yz_r \mid z_r y)| \leq Cq^{\alpha m} + C\|(1 - E_m)(z_r)\|_2.$$

for some new constant C independent of m and r. We now take $\epsilon > 0$ and choose a fixed $m \geq 10n$ such that $Cq^{\alpha m} \leq \epsilon/2$. Then we apply Theorem 5.9: for all $p \in \mathbb{N}^*$ and r we have

$$\|(1 - E_m)(z_r)\|_2^2 \leq L_m p^{-1} + L_m p\|[\chi_1, z_r]\|_2^2.$$

We choose p such that $L_m p^{-1} \leq \epsilon^2/8C^2$. Finally by assumption for ω-almost all r we have $\|[\chi_1, z_r]\|_2^2 \leq \epsilon^2/8C^2 L_m p$. The above estimates then show that for the same r's we have $|(yz_r \mid z_r y)| \leq \epsilon$, and this concludes the proof of the AOP. □

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